Mechanics and Design

Chapter 3. Energy Methods

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CONTENTS

Work

Work

$$\mathbf{F} \cdot \mathbf{ds} = F \cos \theta ds$$

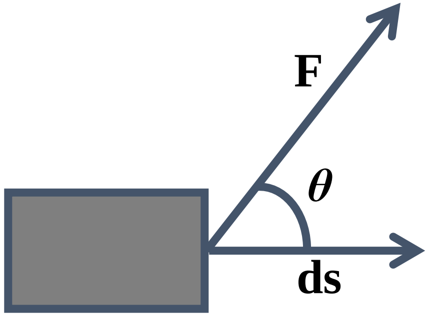


Fig. 3.1 Work

- Force vector : $\mathbf{F}^T = \begin{pmatrix} F_x & F_y & F_z \end{pmatrix}$
- Displacement vector : $\mathbf{ds}^T = \begin{pmatrix} dx & dy & dz \end{pmatrix}$
- Angle : θ = angle between **F** and d**s**.

Inner product means

- Inner product of two vectors results in a scalar, that is, the work is a scalar quantity.
- No work is done when the direction of the displacement is perpendicular to that of the force.

Work

General Work

$$\int \mathbf{F}(\mathbf{s}) \cdot \mathbf{ds}$$

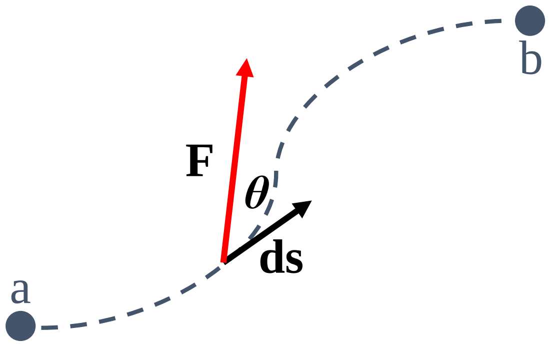


Fig. 3.2 General Work

We use general work when force varies with a point of application.

There are two kinds of work.

- Conservative: work done by external force is stored in the form of potential energy, and recoverable.

  (ex. gravitational potential energy, elastic potential energy)

- Non-conservative: work done in system is not recoverable.

  (ex. sliding block with friction)

Work

General Work (Elastic Spring)

$$\int \mathbf{F} \cdot \mathbf{ds} = \int_0^{\delta^*} F d\delta = U$$

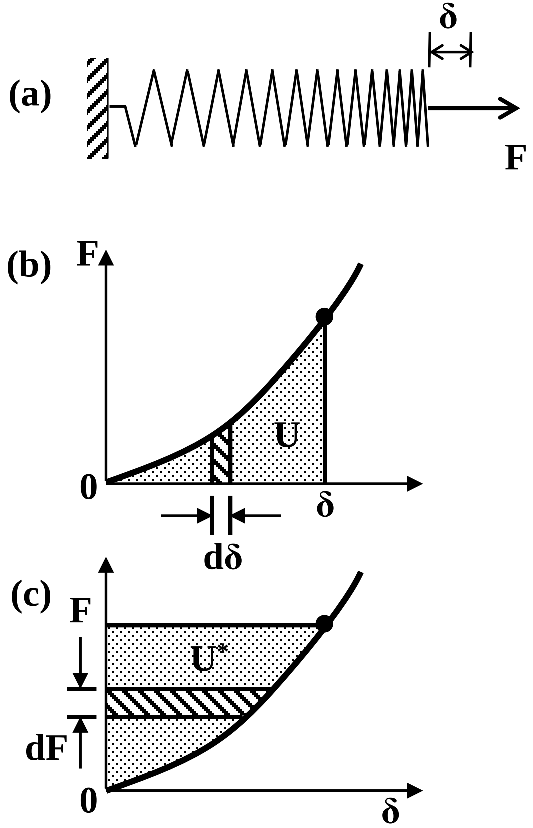


Fig. 3.1 Nonlinear spring undergoes a gradual elongation.

F (external force) remains in equilibrium with the internal tension (spring force = kδ).

- The general work is stored in the form of a potential energy.
- The potential energy appears as the shaded area in Fig. 3.1 (b).
- U (potential energy) is a function of elongation δ.

Work

Application

Total work done by the external loads (at each point A_i, load is $\boldsymbol{P_i}$, and displacement is $\boldsymbol{A_i}$)

=

Total Potential Energy U

$$\sum_i \int_0^{\mathbf{s}_i} \mathbf{P}_i \cdot \mathbf{ds}_i = U$$

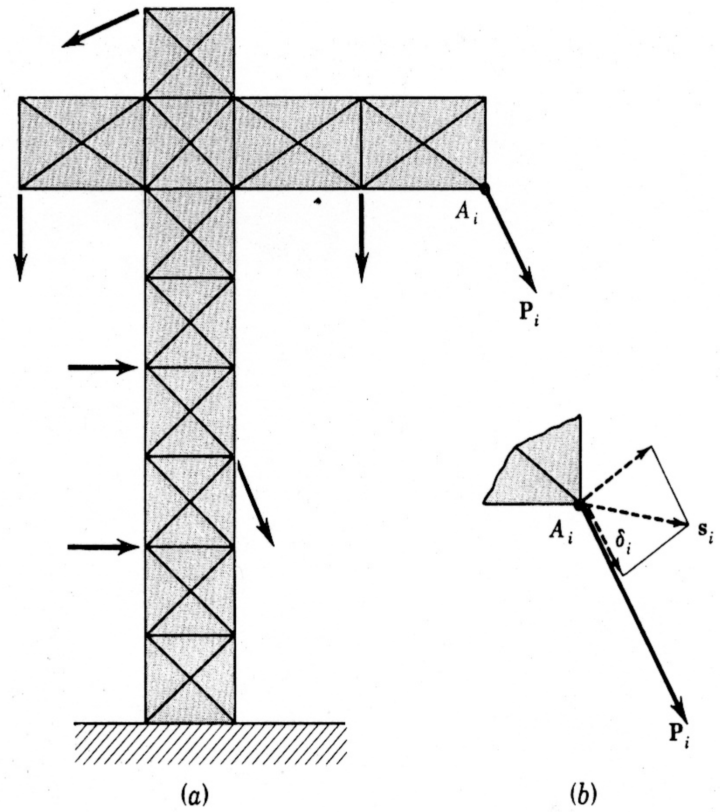


Fig. 3.2 General elastic structure.

(a) P_i at A_i

(b) s_i at A_i

Work

Complementary Work

$$\int \mathbf{s}\cdot\mathbf{dF} = \int_0^F \delta dF = U *$$

When complementary work is done on this system, their internal force states alter in such a way that they are capable of giving up equal amounts of complementary work when they are returned to their original force states. Under these circumstances the complementary work done on such system is said to be stored as complementary energy.

- This energy appears as the shaded area in Fig. 3.1 (c)
- U* (complementary energy) is a function of the force F.

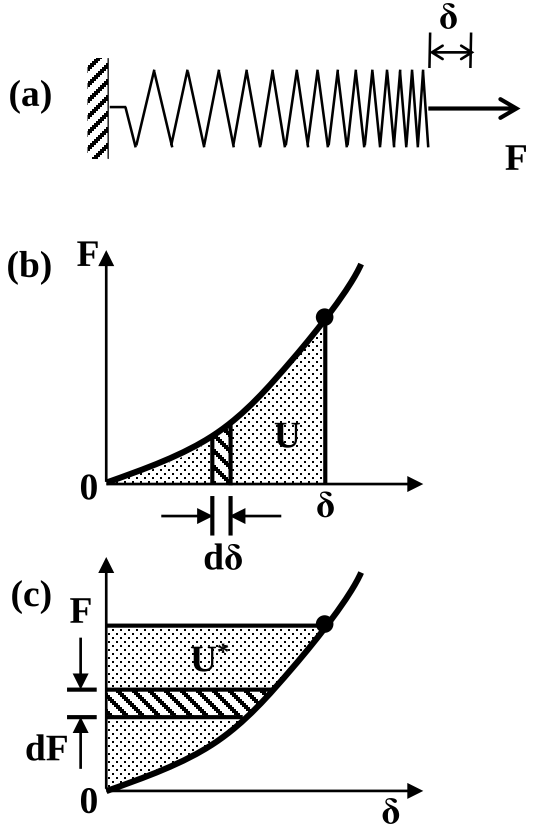


Fig. 3.1 Nonlinear spring undergoes a gradual elongation.

Work

Application

Total complementary work done by the external loads
(at each point A_i, load is $\boldsymbol{P_i}$, and displacement is $\boldsymbol{A_i}$)

=

Total Complementary Energy U^*

$$\sum_i \int_0^{\mathbf{P}_i} \mathbf{s}_i \cdot \mathbf{dP}_i = \sum_i \int_0^{P_i} \delta_i dP_i = U^*$$

where $\mathbf{s}_i$ can be decomposed into parallel and perpendicular to $\mathbf{P}_i$. The parallel component is δ_i. (Fig. 3.2b)

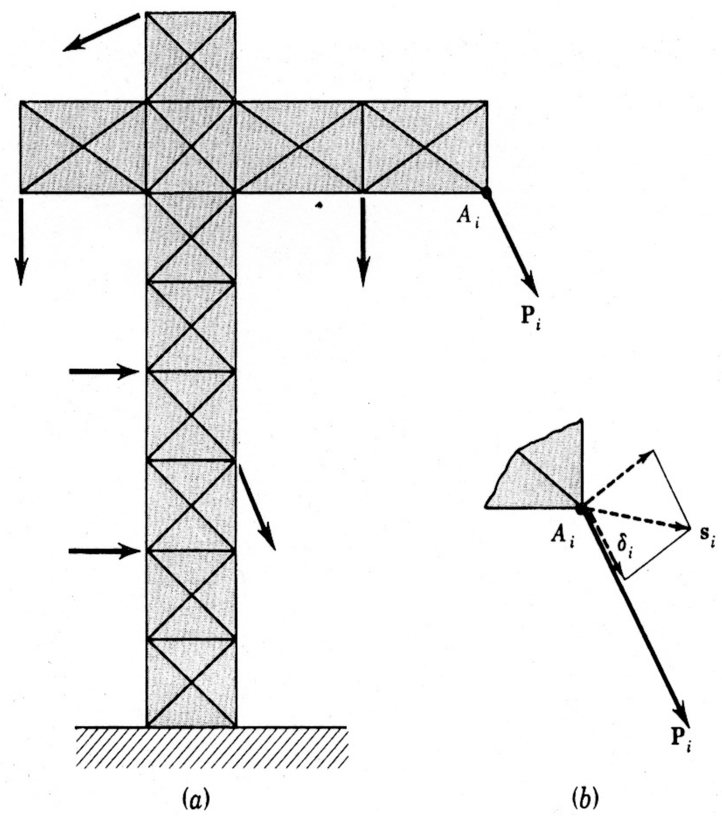


Fig. 3.2 General elastic structure.

(a) P_i at A_i
(b) s_i at A_i

Castigliano's Theorem

Now if the loads in Fig. 3.2 (a) are gradually increased from zero so that the system passes through a succession of equilibrium states, the total complementary work done by all the external loads will equal the total complementary energy U* stored in all the internal elastic members.

Let's consider a small increment ΔP_i

$$\delta_i \Delta P_i = \Delta U^*$$

or

$$\frac{\Delta U^*}{\Delta P_i} = \delta_i$$

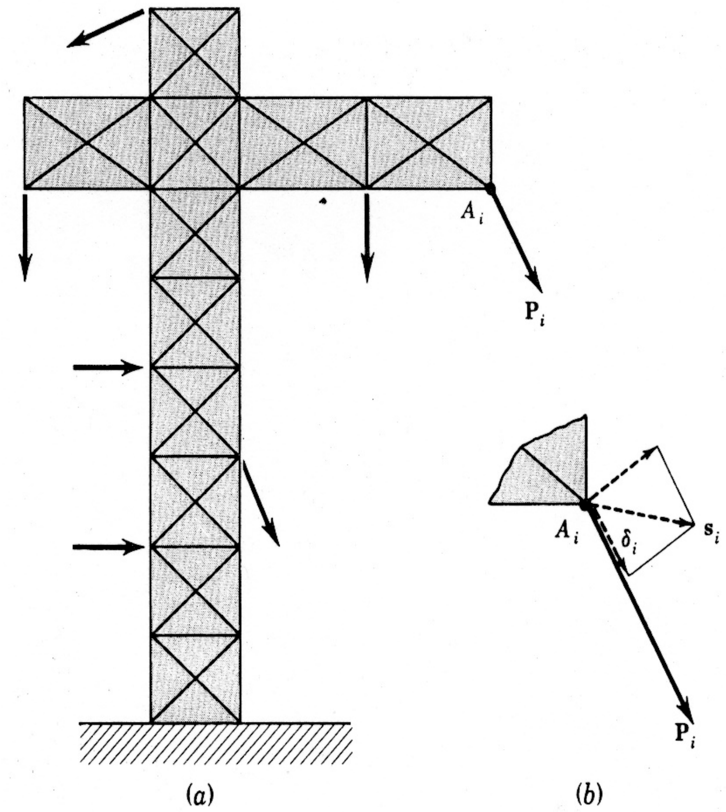


Fig. 3.2 General elastic structure.

(a) P_i at A_i
(b) s_i at A_i

Castigliano's Theorem

In the limit as $\Delta P_i \rightarrow 0$ this approaches a derivative which we indicates as a partial derivative since all other loads were held fixed. δ_i is a in-line deflection.

$$\frac{\partial U^*}{\partial P_i} = \delta_i$$

This result is a form of Castigliano's theorem.

The theorem can be extended to include moment loads.

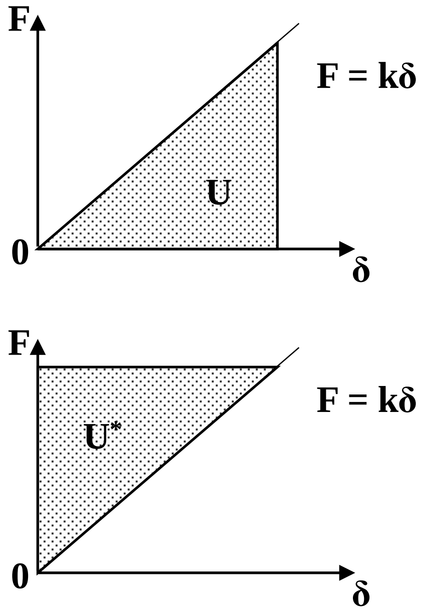
$$\frac{\partial U^*}{\partial M_i} = \phi_i$$

where M_i is moment loads, and ϕ_i is an angle of rotation.

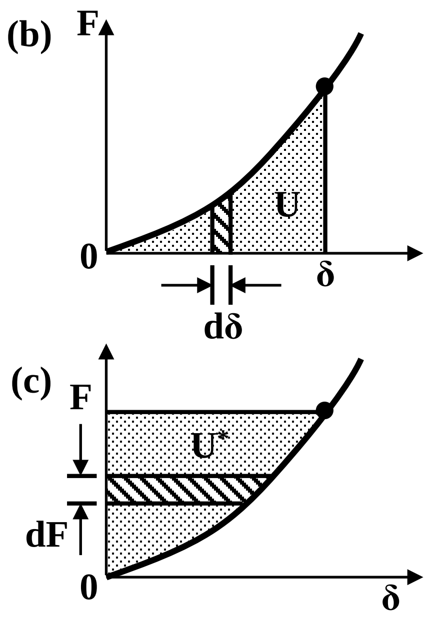
Castigliano's Theorem

Castigliano's theorem in linear system

In linear system $U^* = U$



In nonlinear system $U^* \neq U$



Castigliano's Theorem

Example : Linear spring (1)

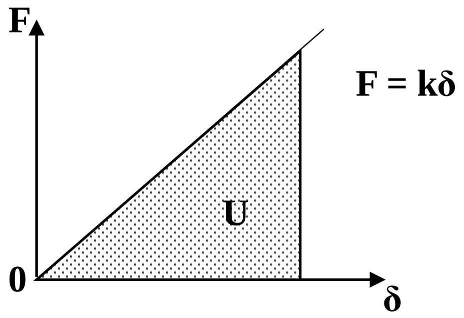
$$F = k\delta$$

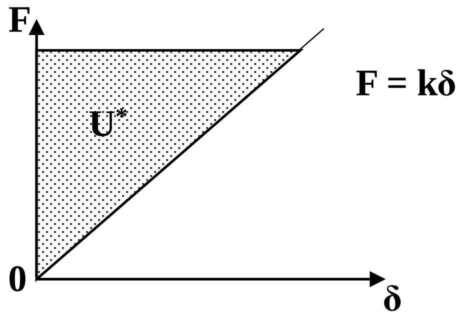
$$U = \frac{1}{2}k\delta^2, U^* = \frac{F^2}{2k}$$

$$U = U^*$$

$$U = \frac{1}{2}k\delta^2 = \frac{1}{2}F\delta = \frac{F^2}{2k}$$

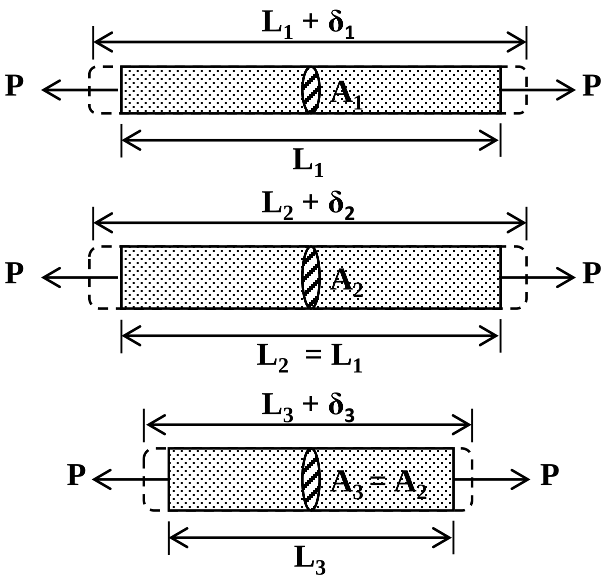
where *k* is spring constant





Castigliano’s Theorem

Example : Linear spring (2)



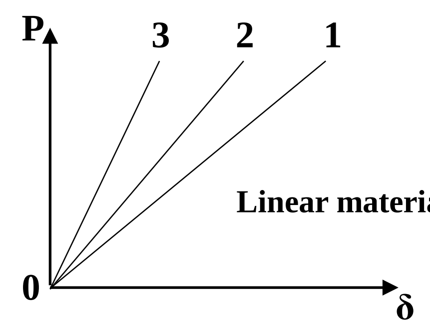
For the linear uniaxial member in Figs. 3.3 and 3.4

$$k = \frac{EA}{L}$$

$$U = \frac{EA}{2L}\delta^2 = \frac{P^2 L}{2EA}$$

Finally, the in-line deflection δ_i at any loading point A_i is obtained by differentiation of the potential energy with respect to the load

$$\delta_i = \frac{\partial U}{\partial P_i} = \frac{\partial}{\partial P_i}\left(\frac{P^2 L}{2EA}\right) = \frac{PL}{EA}$$



Castigliano's Theorem

Example 1*

Consider the system of two springs shown in Fig. 3.5. We shall use Castigliano's theorem to obtain the deflections δ_1 and δ_2 which are due to the external loads P_1 and P_2 .

To satisfy the equilibrium requirements the internal spring forces must be

$$F_1 = P_1 + P_2$$

$$F_2 = P_2$$

The total elastic energy, using $U = \frac{1}{2} k\delta^2 = \frac{1}{2} F\delta = \frac{F^2}{2k}$, is

$$U = U_1 + U_2 = \frac{(P_1 + P_2)^2}{2k_1} + \frac{P_2{}^2}{2k_2}$$

The deflections then follow the form of $\delta_i = \frac{\partial U}{\partial P_i}$

$$\delta_1 = \frac{\partial U}{\partial P_1} = \frac{P_1 + P_2}{k_1}$$

$$\delta_2 = \frac{\partial U}{\partial P_2} = \frac{P_1 + P_2}{k_1} + \frac{P_2}{k_2}$$

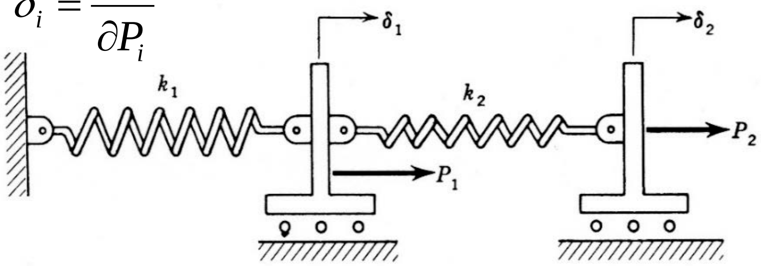


Fig. 3.5 Example 2.11*

* Lardner, Thomas J. *An introduction to the mechanics of solids*. McGraw-Hill College, 1972.

Castigliano's Theorem

Example 2*

Let us consider again Example 2.4* (also Example 1.3 *), and determine the deflections using Castigliano's theorem.

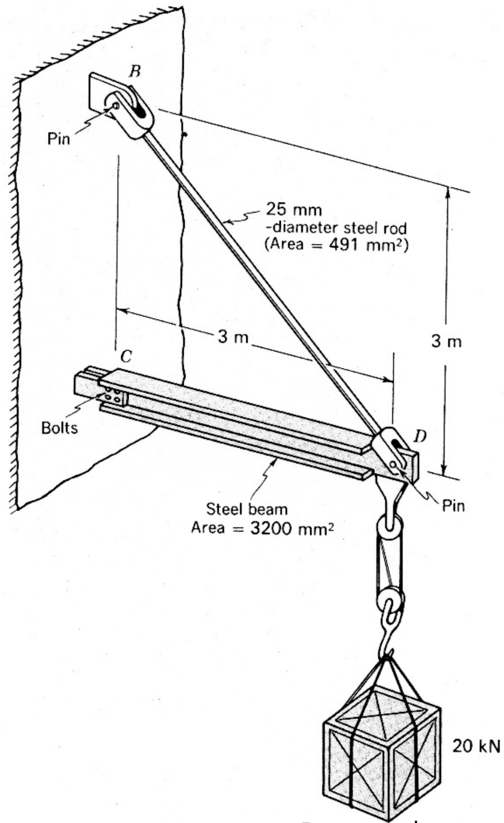
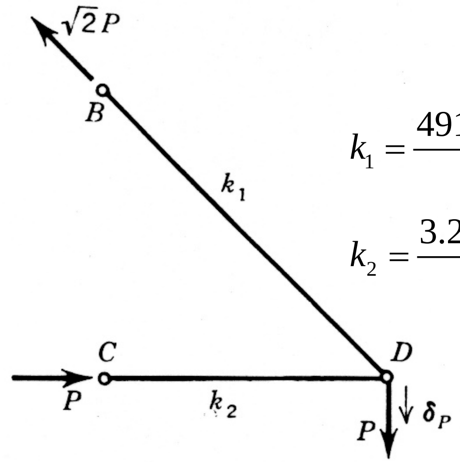


Fig. 3.6 Example 2.4*

In Fig. 3.7 the isolated system from Example 2.4* is shown together with the applied loads

Because we will treat the members of the frame as springs, their "constants" are given.



$$k_1 = \frac{491\times10^{-6}\times205\times10^{6}}{3\sqrt{2}} = 23.73\text{MN/m}$$

$$k_2 = \frac{3.2\times10^{-3}\times205\times10^{6}}{3} = 218.67\text{MN/m}$$

Fig. 3.7 Example 2.12*

* Lardner, Thomas J. *An introduction to the mechanics of solids*. McGraw-Hill College, 1972.

Castigliano's Theorem

Example 2* (Continued)

We use the equilibrium requirements to express the member forces F_1 and F_2 in terms of the load *P* so that the total energy is

$$U = U_1 + U_2 = \frac{P_1^2}{2k_1} + \frac{P_2^2}{2k_2} = \frac{P^2}{k_1} + \frac{P^2}{2k_2}$$

We can calculate directly the deflection of point *D* from $\delta_i = \frac{\partial U}{\partial P_i}$

$$\delta_1 = \frac{\partial U}{\partial P} = \frac{\partial}{\partial P}\left(\frac{P^2}{k_1} + \frac{P^2}{2k_2}\right) = 2P\left(\frac{1}{k_1} + \frac{1}{2k_2}\right)$$

$$\delta_P = 2 \times 20 \times 10^3 \times \left(0.0421 + 0.0023\right) \times 10^{-6} = 1.77\text{mm}$$

In order to calculate the horizontal deflection at point *D* using Castigliano's theorem, there must be a horizontal force at *D*. But the horizontal force at *D* is zero.

We can satisfy both requirements by applying a fictitious horizontal force Q and setting Q = 0.

* Lardner, Thomas J. *An introduction to the mechanics of solids*. McGraw-Hill College, 1972.

Castigliano's Theorem

Example 2* (Continued)

Figure 3.8 shows the frame isolated with both *P* and *Q* applied.

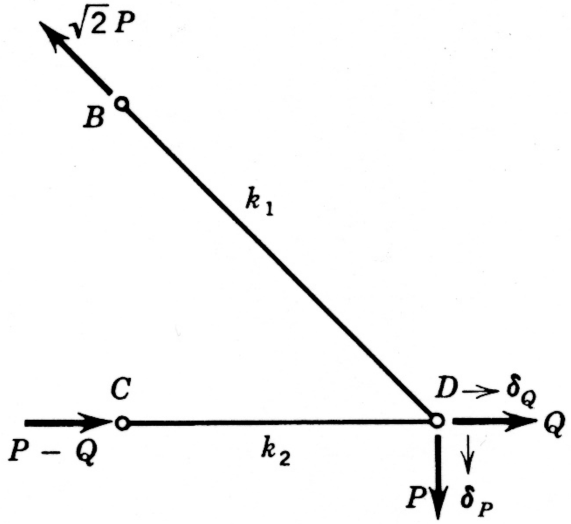


Fig. 3.8 Structure if Fig. 3.7 with fictitious load Q at D

The total energy in terms of the loads *P* and *Q* is

$$U = \frac{P^2}{k_1} + \frac{1}{2k_2}(P-Q)^2$$

$$\delta_Q = \frac{\partial U}{\partial Q} = 0 - \frac{P-Q}{k_2} = \frac{-P}{k_2} = -0.0915\text{mm}$$

* Lardner, Thomas J. *An introduction to the mechanics of solids*. McGraw-Hill College, 1972.

Castigliano's Theorem

Example 3*

Let us use Castigliano's theorem to determine deflections in the Truss problem that we considered in Example 2.5* and in the computer solution example of Sec. 2.5*.

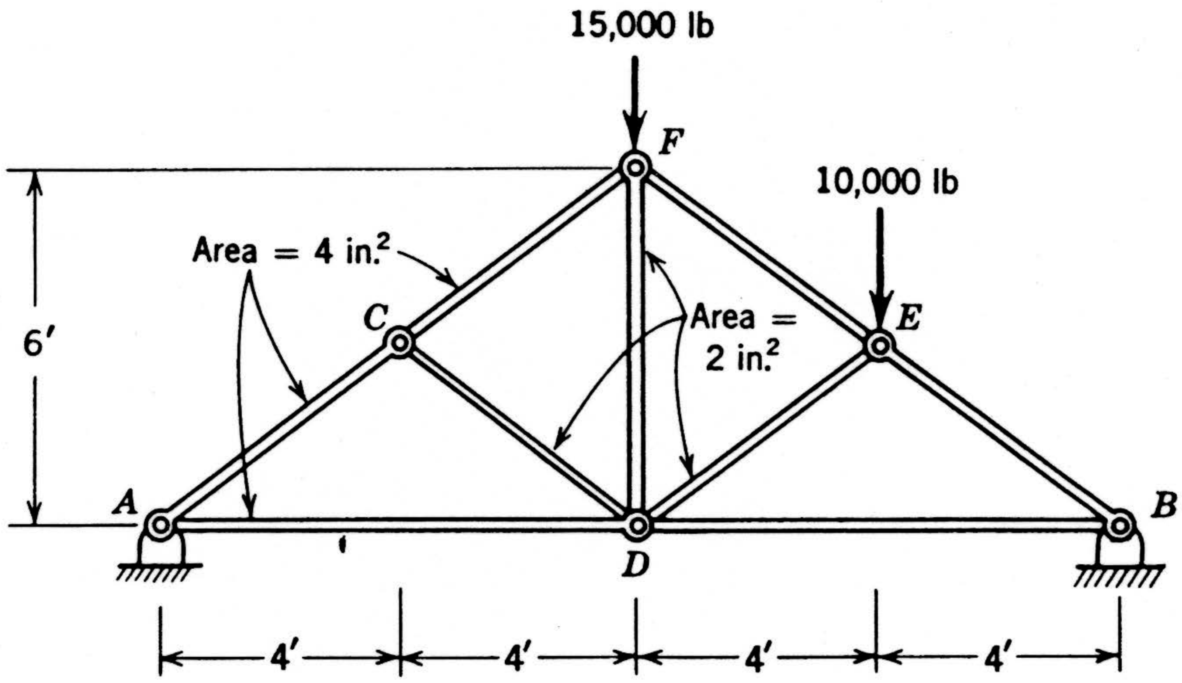


Fig. 3.9 Statically indeterminate version of trusses in Example 2.5*.

* Lardner, Thomas J. *An introduction to the mechanics of solids*. McGraw-Hill College, 1972.

Castigliano's Theorem

Example 3* (Continued)

If a truss is made of n axially loaded members,

$$U_i = \frac{F_i^{\,2} L_i}{2 A_i E_i} \qquad \text{(energy stored in the } i \text{ th member)}$$

$$U = \sum_{i=1}^{n} U_i \qquad \text{(total energy in the system of } n \text{ members)}$$

The deflection at any external load P in the direction of P, is

$$\delta_P = \frac{\partial U}{\partial P} = \frac{\partial}{\partial P} \sum_{i=1}^{n} \frac{F_i^{\,2} L_i}{2 A_i E_i} = \sum_{i=1}^{n} \frac{F_i L_i}{A_i E_i} \frac{\partial F_i}{\partial P} = \sum_{i=1}^{n} F_i \frac{L_i}{A_i E_i} \frac{\partial F_i}{\partial P} \qquad \text{(a)}$$

* Lardner, Thomas J. *An introduction to the mechanics of solids*. McGraw-Hill College, 1972.

Castigliano's Theorem

Example 3* (Continued)

We will number the members as shown in Fig. 3.10.

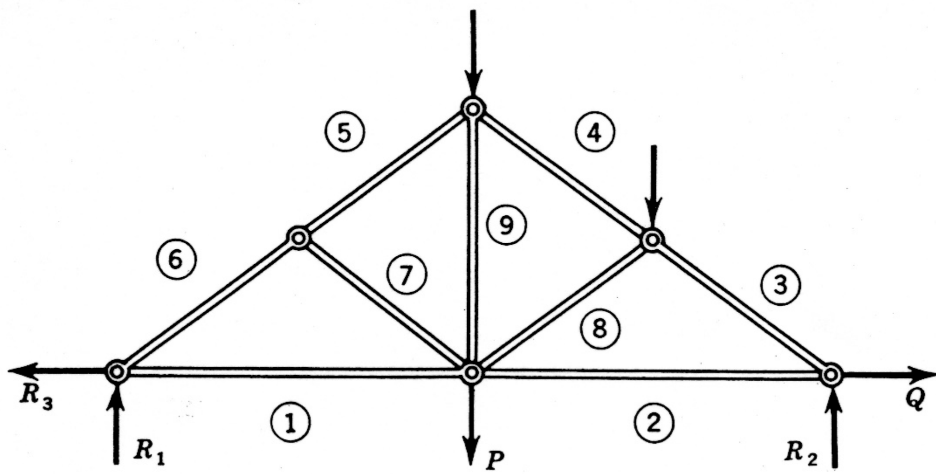


Fig. 3.10 Example 2.13*.

In example 2 we solved for the forces F_i due to the actual applied loads.
We can now set up a system for evaluating (a).
The deflection at the joint at which the fictitious load P is applied, it appears that we need to find the forces F_i in each members as a function of the actual applied loads and fictitious load P.
However, once the member forces are found, we set $P = 0$ in (a).
Therefore, we can use immediately the member forces F_i from the actual loads and the forces for a unit load at P to evaluate $\partial F_i / \partial P$.

* Lardner, Thomas J. *An introduction to the mechanics of solids*. McGraw-Hill College, 1972.

Castigliano's Theorem

Example 3* (Continued)

In the Table 3.1 we have tabulated the individual quantities in (a) as well as their products.

Table 3.1 Truss solution by energy methods

i	F_i 10^3 lb	$(L/AE)^*$ in./lb	$\frac{\partial F_i}{\partial P}$	$\frac{\partial F_i}{\partial Q}$	$\left(\frac{FL}{AE}\frac{\partial F}{\partial P}\right)_i^\dagger$	$\left(\frac{FL}{AE}\frac{\partial F}{\partial Q}\right)_i^\dagger$
1	$+13.33+Q$	2.4×10^{-6}	$+2/3$	$+1$	21.36×10^{-3}	32.0×10^{-3}
2	$+20.0+Q$	2.4×10^{-6}	$+2/3$	$+1$	31.95×10^{-3}	48.0×10^{-3}
3	-25.0	1.5×10^{-6}	$-5/6$	0	31.26×10^{-3}	
4	-16.67	1.5×10^{-6}	$-5/6$	0	20.85×10^{-3}	
5	-16.67	1.5×10^{-6}	$-5/6$	0	20.85×10^{-3}	
6	-16.67	1.5×10^{-6}	$-5/6$	0	20.85×10^{-3}	
7	0	1.5×10^{-6}	0	0	0	
8	-8.33	3.0×10^{-6}	0	0	0	
9	$+5.0$	3.6×10^{-6}	$+1$	0	18.00×10^{-3}	
					$\Sigma=0.1651$ in. $=\delta_y$	$\Sigma=0.080$ in. $=\delta_x$

* Calculated for $E=10\times10^6$ lb/in.^{2}
† $Q=0$.

* Lardner, Thomas J. *An introduction to the mechanics of solids*. McGraw-Hill College, 1972.

Castigliano's Theorem

Example 3* (Continued)

If we wish to solve for the deflection at *P*, we must reevaluate the products in row 1 and 2 of Table 3.1 with *Q*.

i	$\left(\frac{FL}{AE}\frac{\partial F}{\partial P}\right)_{i,\,Q\neq 0}$
1	$21.36 \times 10^{-3} + 1.6Q \times 10^{-6}$
2	$31.95 \times 10^{-3} + 1.6Q \times 10^{-6}$

The values for member 3 through 9 do not change since they carry no *Q*. Therefore

$$\delta_P = \frac{\partial U}{\partial P} = 0.1651 + 3.2Q \times 10^{-6} \qquad \text{here} \quad Q = -16.67 \times 10^3$$

$$\therefore \delta_P = 0.1651 - 0.0534 = 0.1117\text{in}$$

* Lardner, Thomas J. *An introduction to the mechanics of solids*. McGraw-Hill College, 1972.

Rotation of Axes, etc

It can be seen that the second order tensor map a vector to another vector, that is,

$$\begin{aligned}\mathbf{u} = T\cdot\mathbf{v} = \ &(T_{11}\ \mathbf{e}_1\otimes\mathbf{e}_1 + T_{12}\ \mathbf{e}_1\otimes\mathbf{e}_2 + T_{13}\mathbf{e}_1\otimes\mathbf{e}_3 \\ &+ T_{21}\ \mathbf{e}_2\otimes\mathbf{e}_1 + T_{22}\ \mathbf{e}_2\otimes\mathbf{e}_2 + T_{23}\ \mathbf{e}_2\otimes\mathbf{e}_3 \\ &+ T_{31}\ \mathbf{e}_3\otimes\mathbf{e}_1 + T_{32}\ \mathbf{e}_3\otimes\mathbf{e}_2 + T_{33}\ \mathbf{e}_3\otimes\mathbf{e}_3)\cdot(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3) \\ = \ &(T_{11}v_1 + T_{12}v_2 + T_{13}v_3)\mathbf{e}_1 + (T_{21}v_1 + T_{22}v_2 + T_{23}v_3)\mathbf{e}_2 \\ &+ (T_{31}v_1 + T_{32}v_2 + T_{33}v_3)\mathbf{e}_3 \\ = \ &T_{ij}v_j\mathbf{e}_i\end{aligned}$$

Symmetric tensors and skew tensors

Symmetric Tensor ➡ $T_{ij} = T_{ji}$

Skew or Antisymmetric Tensor ➡ $T_{ij} = -T_{ji}$

THANK YOU
FOR LISTENING