# 5.4 Elastic Stress-strain Relations

## ▶ Assumptions in this section

i) We shall generalize the elastic behavior in the tension test to arrive at relations which connect all six components of stress with all six components of elastic strain.

ii) We shall restrict ourselves to materials which are linearly elastic. (linear elasticity)

iii) We also restrict ourselves to strains small compared to unity. (small strain)

iv) We shall consider the materials that are independent of orientation which is assumed to be isotropic. (isotropic)

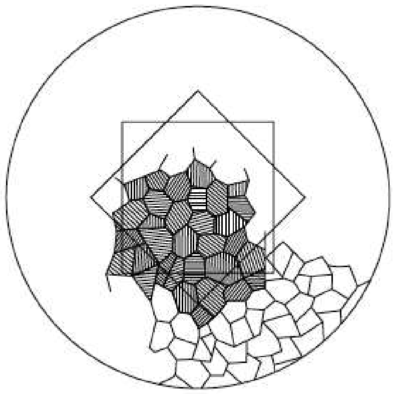


**Fig. 5.12** *Statistically isotropic material*

## ▶ Definitions

$$\sigma_x = E\epsilon_x, \qquad \epsilon_x = \frac{\sigma_x}{E}$$

### 1. Young's modulus (or modulus of elasticity)

i) The modulus of elasticity $E$ is numerically equal to the slope of the linear-elastic region in stress-strain curve and it is the material property.

ii) The modulus of elasticity at compression and extension is same.

iii) Unit: Because $\epsilon$ is a dimensionless number, it is homogeneous to stress σ.

$$\tau_{xy} = G\gamma_{xy}, \qquad \gamma_{xy} = \tau_{xy}/G$$

### 2. Shear modulus of elasticity $G$

i) Unit: $[G] = [E] = [\sigma] = [\tau]$

ii) The relation between $G$ and $E$

$$G = \frac{E}{2(1+\nu)} \tag{5.3}$$

→ $E, G,$ and $\nu$ are dependent each other.

→ In common materials, $0 < \nu < 0.5$, so $\frac{E}{3} < G < \frac{E}{2}$.

## 3. Poisson's ratio

→ Tests in uniaxial compression show a lateral extensional strain which has the same fixed fraction to the longitudinal compressive strain.

$$\nu = -\frac{Lateral \quad Strain}{Axial \quad Strain}$$

i) For isotropic, linear-elastic material

$$\epsilon_y = \epsilon_z = -\nu\epsilon_x = -\nu\sigma_x/E$$

*cf.* The conditions that lateral strain is proportional to axial strain in linear-elastic region

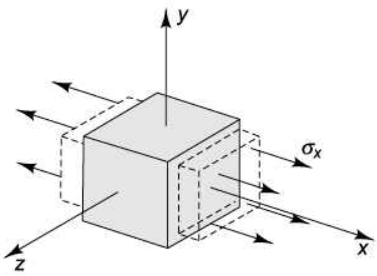


Fig.5.13 Uniaxial normal stress

① Material has the same components in all regions.

→ Homogeneous

② Material properties are independent of orientation.

→ Isotropic

Meanwhile, the lumbers are not isotropic but homogeneous.

In general, the structural materials (i.e., steel) is satisfied with the above requirements.

▶ The conclusions obtained under the assumption that the material is isotropic

i) No shear-strain due to normal stress components.

ii) The principal axes of strain at a point of a stressed body coincide with the principal axes of stress at that point.

iii) Each shear stress component produces only its corresponding shear-strain component.

iv) No strain components other than $\gamma_{zx}$, can exist, singly or in combination, as a result of the shear-stress component $\tau_{zx}$.

v) The thermal strain cannot produce the shear strain.

▶ The stress-strain relations of a linear-elastic isotropic material with all components of stress present

$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\epsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right] \qquad \gamma_{yz} = \frac{\tau_{yz}}{G} \qquad (5.2)$$

$$\epsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] \qquad \gamma_{zx} = \frac{\tau_{zx}}{G}$$

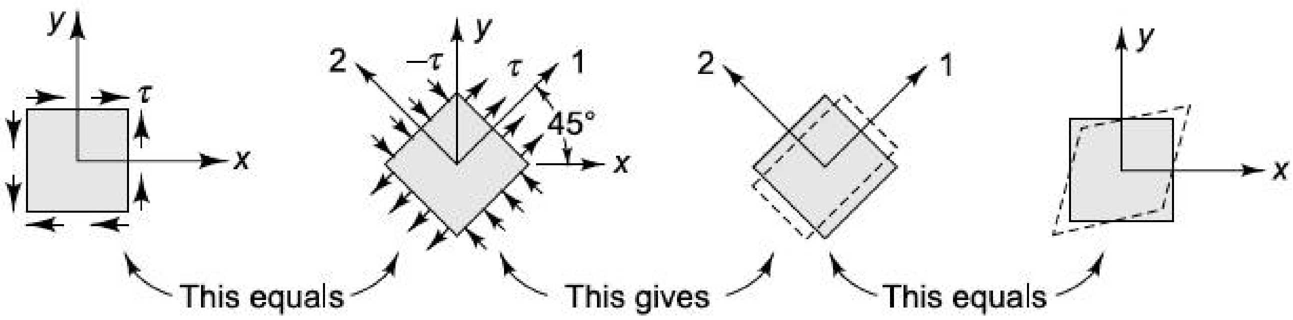



**Fig. 5.16** *Equivalent states of stress and strain*

→ From Fig. 5.16,

$$\epsilon_1 = \frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E} = \frac{\tau(1+\nu)}{E}\,, \quad \epsilon_2 = \frac{\sigma_2}{E} - \nu\frac{\sigma_1}{E} = -\frac{\tau(1+\nu)}{E}$$

Meanwhile, upon use of the strain transformation formulas

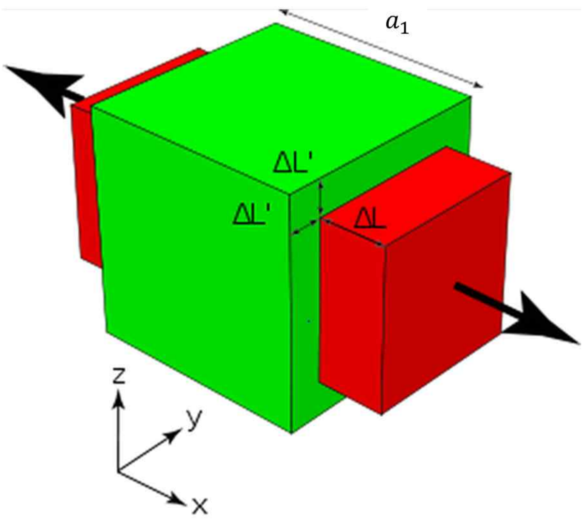
$$\gamma_{xy} = \epsilon_1 - \epsilon_2 = \frac{2(1+\nu)}{E}\tau$$

This equation and $\gamma_{xy} = \frac{\tau}{G}$ must be equal, so

$$G = \frac{E}{2(1+\nu)} \tag{5.3}$$

→ It is true, although it will not be proved here, that no other choice of coordinate axes gives any added information about the elastic constants, and thus for an isotropic material there are just two independent elastic constants.

▶ Volume change of the isotropic, linear-elastic material at extension




$$\Delta L = a_1\epsilon$$

$$\Delta L' = b_1\nu\epsilon = c_1\nu\epsilon$$

The lengths of each side after deformation are

$$\begin{cases} a_1(1+\epsilon) \\ b_1(1-\nu\epsilon) \\ c_1(1-\nu\epsilon) \end{cases}$$

$$\therefore\ v_f = a_1 b_1 c_1 (1+\epsilon)(1-\nu\epsilon)^2$$

$$= a_1 b_1 c_1 (1 - 2\nu\epsilon + \nu^2\epsilon^2 + \epsilon - 2\nu\epsilon^2 + \nu^2\epsilon^3)$$

$$v_f = a_1 b_1 c_1 (1 + \epsilon - 2\nu\epsilon)$$

$$\therefore\ e = \frac{\Delta V}{V_0} = \frac{V_f - V_0}{V_0} = \frac{a_1 b_1 c_1(\epsilon - 2\nu\epsilon)}{a_1 b_1 c_1}$$

$$= \epsilon(1-2\nu) = \frac{\sigma}{E}(1-2\nu)$$

→ Volume increase of a slender member in tensile test can be obtained when $\epsilon, \nu$ are known.

*cf.* If $\nu > 0.5$, there is a contradiction that volume decreases when material is extended, so $\nu_{max} = 0.5$.

i) In linear-elastic region: $\frac{1}{4} \sim \frac{1}{3} \rightarrow \therefore 0.3\epsilon < e < 0.5\epsilon$

ii) In plastic region: in general, $\Delta V = 0$, so it is fine that $\nu = 0.5$.

▶ <u>Unit volume change in three-axial stresses</u>

→ Having unit length and $V_0 = 1$,

$$V_f = (1+\epsilon_x)(1+\epsilon_y)(1+\epsilon_z)$$

$$e = \frac{\Delta V}{V_0} = \frac{V_f - V_0}{V_0} = \frac{V_f}{V_0} - 1 \fallingdotseq \epsilon_x + \epsilon_y + \epsilon_z$$
$$= \frac{1-2\nu}{E}(\sigma_x + \sigma_y + \sigma_z)$$

*cf.* The shear-stress components cannot have an effect on the volume change.

$$e = -\frac{3(1-2\nu)}{E} p \qquad \frac{1}{\kappa} = \frac{E}{3(1-2\nu)}$$

$\kappa$: Bulk modulus or modulus of compression

## 5.5 Thermal strain

▶ In the elastic region the effect of temperature on strain appears in two ways.

i) By causing a modification in the values of the elastic constants

ii) By directly producing a strain even in the absence of stress

*cf.* For an isotropic material, symmetry arguments show that the thermal strain must be a pure expansion or contraction with no shear-strain components referred to any set of axes.

$$\begin{cases} \epsilon_x^t = \epsilon_y^t = \epsilon_z^t = \alpha(T - T_0) \\ \quad \gamma_{xy}^t = \gamma_{yz}^t = \gamma_{zx}^t = 0 \end{cases} \tag{5.4}$$

▶ Total strain $\epsilon$

$$\epsilon = \epsilon^t + \epsilon^e \tag{5.5}$$

## 5.6 Complete equations of elasticity

→ The problem was outlined previously in broad generality by the three steps given in (2.1). For convenience we summarize below, under the three steps of (2.1), explicit equations which must be satisfied at each point of a nonaccelerating, isotropic, linear-elastic body subject to small strains.

▶ Equilibrium (3 equations; 6 unknowns)

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y = 0 \tag{5.6}$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0$$

## ▶ Geometric Compatibility (6 equations and 9 unknowns)

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} & \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\ \epsilon_y &= \frac{\partial v}{\partial y} & \gamma_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial y} \\ \epsilon_z &= \frac{\partial w}{\partial z} & \gamma_{zx} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \end{aligned} \tag{5.7}$$

## ▶ Stress-strain-temperature relation (6 equations)

$$\begin{aligned} \epsilon_x &= \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] + \alpha(T - T_0) & \gamma_{xy} &= \frac{\tau_{xy}}{G} \\ \epsilon_y &= \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right] + \alpha(T - T_0) & \gamma_{yz} &= \frac{\tau_{yz}}{G} \\ \epsilon_z &= \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] + \alpha(T - T_0) & \gamma_{zx} &= \frac{\tau_{zx}}{G} \end{aligned} \tag{5.8}$$

→ The equilibrium equations (5.6), the strain-displacement equations (5.7), and the strain-stress-temperature relations (5.8) provide 15 equations for the six components of stress, the six components of strain, and the three components of displacement.

*cf.* The complete equations (5.6), (5.7), and (5.8) apply to deformations of isotropic, linearly elastic solids which involve small strains and for which it is acceptable to apply the equilibrium requirements in the undeformed configuration.

*cf.* We shall be primarily concerned with the three steps of (2.1), expressed not in the infinitesimal formulation of (5.6), (5.7), and (5.8) but expressed, instead, on a macroscopic level in terms of rods, shafts, and beams.

▶ Example 5.2 A long, thin plate of width $b$, thickness $t$, and length $L$ is placed between two rigid walls a distance $b$ apart and is acted on by an axial force $P$, as shown in Fig. 5.17 (a). We wish to find the deflection of the plate parallel to the force $P$. We idealize the situation in Fig 5.17 (b).

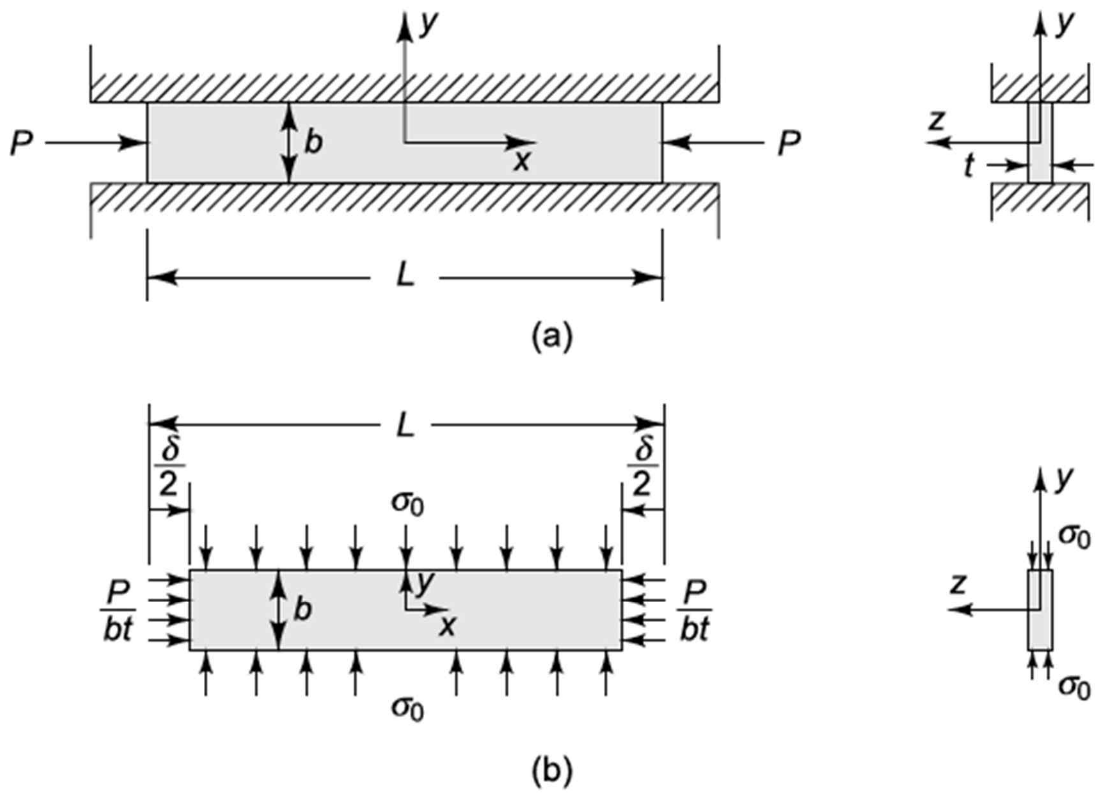



**Fig. 5.17** *Example 5.2. (a) Actual problem; (b) idealized model*

▷ Assumptions

i) The axial force $P$ results in an axial normal stress uniformly distributed over the plate area, including the end areas.

ii) There is no normal stress in the thin direction. (Note that this implies a case of plane stress in the $xy$ plane.)

iii) There is no deformation in the $y$ direction, that is, $\epsilon_y = 0$. (Note that this implies a case of plane strain in the $xz$ plane.)

iv) There is no friction force at the walls (or, alternatively, it is small enough to be negligible).

v) The normal stress of contact between the plate and wall is uniform over the length and width of the plate. We now satisfy the requirements (2.1) for the idealized model of Fig. 5.17 (b).

▷ Equilibrium

$$\sigma_x = -\frac{P}{bt}, \quad \sigma_y = -\sigma_0, \qquad \sigma_z = 0 \tag{a}$$

$$\tau_{xy} = \tau_{yz} = \tau_{zx} = 0$$

These stresses also satisfy the equilibrium equations (5.6).

▷ Geometric compatibility

$$\epsilon_y = 0 \tag{b}$$

$$\epsilon_x = -\frac{\delta}{L} \tag{c}$$

▷ Stress-strain relation

→ eq. (5.8) is

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y), \qquad \epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x), \qquad \epsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

$$\gamma_{xy} = \gamma_{yz} = \gamma_{zx} = 0 \tag{d}$$

→ Solving the system of equations (a), (b), (c), and (d)

$$\sigma_y = \nu\sigma_x = -\frac{\nu P}{bt}, \qquad \delta = \frac{(1-\nu^2)PL}{Ebt}$$

$$\epsilon_z = \frac{\nu(1+\nu)P}{Ebt} = \frac{\nu}{1-\nu}\frac{\delta}{L}$$

*cf.* We note that the presence of the rigid walls reduces the axial deflection of the plate by the factor $(1-\nu^2)$.

▷ Strain-displacement relation

$$u = -\frac{\delta}{L}x\,, \quad v = 0\,, \quad w = \frac{\nu}{1-\nu}\frac{\delta}{L}z$$

*cf.* It is relatively easy to get an exact or nearly exact solution to an idealized approximation of the real problem.

## 5.7 Complete Elastic Solution for a Thick-walled Cylinder

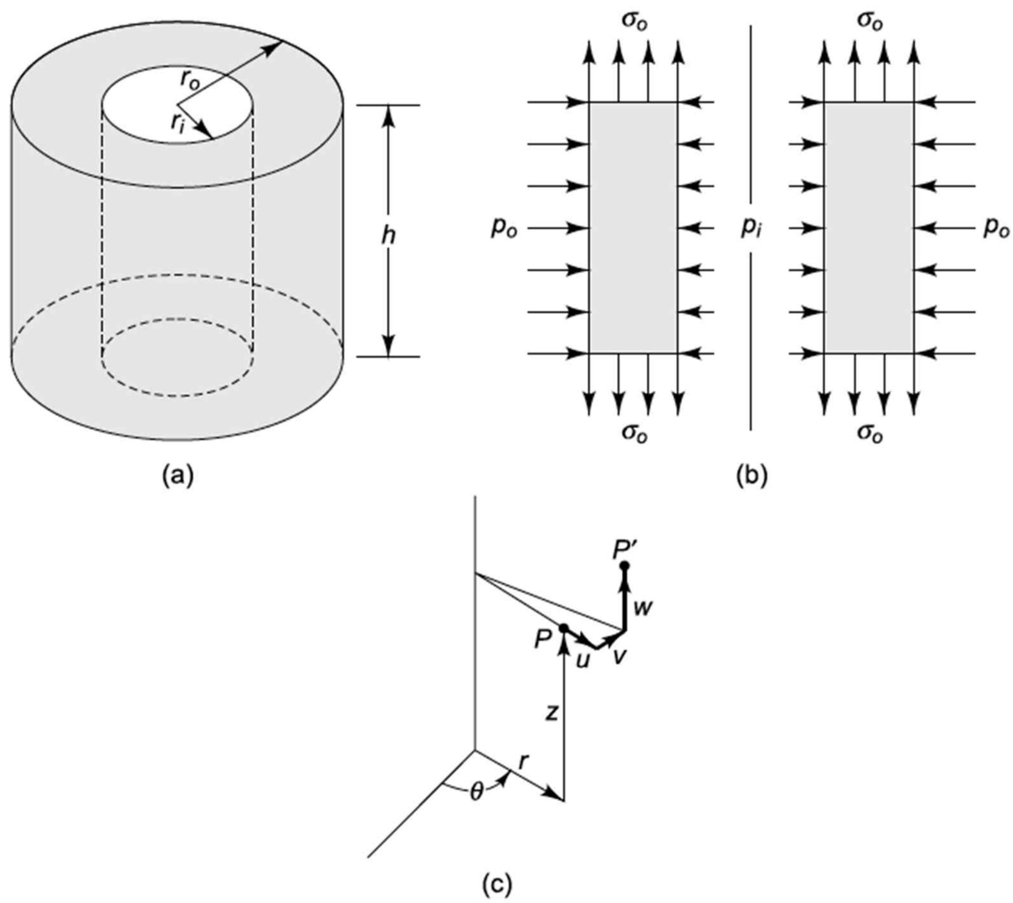



**Fig. 5.18** *Thick-walled cylinder (a) subjected to inner and outer pressures and axial tension (b). Cylindrical coordinates and displacement components (c).*

→ There is uniform inner pressure $p_i$, uniform outer pressure $p_o$, and uniform axial tensile stress $\sigma_o$.

▶ Boundary condition

i) For $r = r_i$

$$\sigma_r = -p_i\,, \qquad \tau_{rz} = 0\,, \quad \tau_{r\theta} = 0 \tag{a}$$

ii) For $r = r_0$

$$\sigma_r = -p_0\,, \qquad \tau_{rz} = 0\,, \quad \tau_{r\theta} = 0 \tag{b}$$

iii) For $z = 0 \;\&\; z = h$

$$\sigma_z = \sigma_0, \qquad \tau_{rz} = 0\,, \qquad \tau_{\theta z} = 0 \tag{c}$$

▶ Geometric compatibility

→ Based on the uniformity of the axial loading, $\sigma_z = \sigma_0$ throughout the interior and that all stresses and strains are independent of $z$.

→ Based on symmetry, we shall look for a solution in which $\nu$ and the $\theta$ component of displacement vanish everywhere and in which all stresses, strains, and displacements are independent of $\theta$.

→ The shear stresses $\tau_{r\theta}, \tau_{\theta z}, \tau_{rz}$ and the corresponding strains $\gamma_{r\theta}, \gamma_{\theta z}, \gamma_{rz}$ vanish everywhere.

$$\tau_{r\theta} = \tau_{\theta z} = \tau_{rz} = \gamma_{r\theta} = \gamma_{\theta z} = \gamma_{rz} = 0$$

▶ Equilibrium equation for cylindrical coordinate system

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + 2\frac{\tau_{r\theta}}{r} = 0$$

$$\frac{\partial \tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{zr}}{r} = 0$$

Remaining equilibrium equation

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{d}$$

▶ Strain-displacement equations for cylindrical coordinate system

$$\epsilon_r = \frac{\partial u}{\partial r} \qquad \epsilon_\theta = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} \qquad \epsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{r\theta} = \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} \quad \gamma_{\theta z} = \frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z} \quad \gamma_{zr} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

Remaining strain-displacement equations

$$\epsilon_r = \frac{du}{dr}, \qquad \epsilon_\theta = \frac{u}{r}, \qquad \epsilon_z = \frac{dw}{dz} \tag{e}$$

- Stress-strain relation for cylindrical coordinate system

$$\sigma_r = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\epsilon_r + \nu(\epsilon_\theta + \epsilon_z)]$$

$$\sigma_\theta = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\epsilon_\theta + \nu(\epsilon_r + \epsilon_z)] \tag{f}$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\epsilon_z + \nu(\epsilon_r + \epsilon_\theta)]$$

Substitute (e) into (f) yields

$$\sigma_r = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\frac{du}{dr} + \nu\left(\frac{u}{r} + \epsilon_z\right)]$$

$$\sigma_\theta = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\frac{u}{r} + \nu\left(\frac{du}{dr} + \epsilon_z\right)]$$

Let $k$ indicate constant in equation (f) and from equilibrium,

$$k = \frac{E}{(1+\nu)(1-2\nu)}$$

$$\frac{d\sigma_r}{dr} = k\left[\frac{(1-\nu)\, d^2u}{dr^2} + \nu\left(\frac{1}{r}\frac{du}{dr} - u\frac{1}{r^2}\right)\right]$$

$$\frac{\sigma_r - \sigma_\theta}{r} = k[(1-2\nu)\frac{1}{r}\frac{du}{dr} + 2\nu\frac{u}{r^2} - \frac{u}{r^2}]$$

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = k(1-\nu)\left[\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2}\right] = 0 \ \rightarrow \text{Cauchy equation}$$

$$\therefore\ u = Ar + \frac{B}{r} \qquad \epsilon_r = \frac{du}{dr} = A - \frac{B}{r^2} \qquad \epsilon_\theta = A + \frac{B}{r^2}$$

$$\therefore\ \sigma_r = k\left[A + \nu\epsilon_z - (1-2\nu)\frac{B}{r^2}\right]$$

$$\sigma_\theta = k\left[A + \nu\epsilon_z + (1-2\nu)\frac{B}{r^2}\right]$$

## ▶ Boundary condition

$$\sigma_{r_i} = k\left[A + \nu\epsilon_z - (1-2\nu)\frac{B}{r_i^2}\right] = -p_i$$

$$\sigma_{r_0} = k\left[A + \nu\epsilon_z - (1-2\nu)\frac{B}{r_0^2}\right] = -p_0$$

$$\begin{bmatrix} -\frac{p_i}{k} - \nu\epsilon_z \\ -\frac{p_0}{k} - \nu\epsilon_z \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1-2\nu}{r_i^2} \\ 1 & -\frac{1-2\nu}{r_0^2} \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \mathrm{X}\begin{bmatrix} A \\ B \end{bmatrix}$$

$$det\ (X) = \frac{(1-2\nu)(r_0^2 - r_i^2)}{(r_0 r_i)^2}$$

$$X^{-1} = \frac{1}{dex\,(X)}\begin{bmatrix} -\frac{1-2\nu}{r_0^2} & \frac{1-2\nu}{r_i^2} \\ -1 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} A \\ B \end{bmatrix} = \frac{(r_0 r_i)^2}{(1-2\nu)(r_0^2 - r_i^2)}\begin{bmatrix} -\frac{1-2\nu}{r_0^2} & \frac{1-2\nu}{r_i^2} \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -\frac{p_i}{k} - \nu\epsilon_z \\ -\frac{p_0}{k} - \nu\epsilon_z \end{bmatrix}$$

$$\therefore A = \frac{1}{k}\frac{-p_0 r_0^2 + p_i r_i^2}{r_0^2 - r_i^2} - \nu\epsilon_z$$

$$B = \frac{1}{(1-2\nu)k}(-p_0 + p_i)\frac{(r_0 r_i)^2}{r_0^2 - r_i^2}$$

$$\therefore \sigma_r = k\left[\frac{1}{k}\frac{-p_0 r_0^2 + p_i r_i^2}{(r_0^2 - r_i^2)} - (1-2\nu)\frac{1}{(1-2\nu)k}(-p_0 + p_i)\frac{\left(\frac{r_0 r_i}{r}\right)^2}{r_0^2 - r_i^2}\right]$$

$$= -\frac{p_i[(r_0/r)^2 - 1] + p_0[(r_0/r_i)^2 - (r_0/r)^2]}{(r_0/r_i)^2 - 1}$$

$$\sigma_\theta = k\left[\frac{1}{k}\frac{-p_0 r_0^2 + p_i r_i^2}{(r_0^2 - r_i^2)} + (1-2\nu)\frac{1}{(1-2\nu)k}(-p_0 + p_i)\frac{\left(\frac{r_0 r_i}{r}\right)^2}{r_0^2 - r_i^2}\right]$$

$$= \frac{p_i[(r_0/r)^2+1]-p_0[(r_0/r_i)^2+(r_0/r)^2]}{(r_0/r_i)^2-1}$$

$$\epsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_r + \sigma_\theta)]$$

$$= \frac{\sigma_0}{E} - \frac{2\nu}{E}\frac{p_i r_i^2 - p_0 r_0^2}{r_0^2 - r_i^2}$$

→ Note that $\epsilon_z$ is independent of position within the cylinder.

## ▶ Stress-strain equations (following textbook)

From generalized Hooke's law

$$\epsilon_r = \frac{1}{E}[\sigma_r - \nu(\sigma_\theta + \sigma_z)]$$

$$\epsilon_\theta = \frac{1}{E}[\sigma_\theta - \nu(\sigma_z + \sigma_r)] \qquad \text{(f)}$$

$$\epsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_r + \sigma_\theta)]$$

i) From the first two equations of (f) we solve for the transverse stresses $\sigma_r$ and $\sigma_\theta$, in terms of $\epsilon_r$ and $\epsilon_\theta$ and thus obtain the stresses also as functions of $u$.

ii) Finally, substituting the stresses into (d) leads to the following differential equation for $u(r)$

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0 \qquad \text{(h)}$$

$$\therefore\ r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - u = 0$$

$$u = Ar + \frac{B}{r} \ \rightarrow \text{general solution} \qquad \text{(i)}$$

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - u = r^2 m(m-1)r^{m-2} + m\,r^{m-1} - r^m$$

$$\therefore\ (m(m-1) + m - 1)r^m = 0$$

$$\therefore \; u = Ar + Br^{-1}$$

Apply the boundary conditions

$$\begin{cases} \sigma_r = -\dfrac{p_i[(r_0/r)^2-1]+p_0[(r_0/r_i)^2-(r_0/r)^2]}{(r_0/r_i)^2-1} \\ \sigma_\theta = \dfrac{p[(r_0/r)^2+1]-p_0[(r_0/r_i)^2+(r_0/r)^2]}{(r_0/r_i)^2-1} \end{cases} \tag{5.9}$$

→ The axial strain is obtained by substituting these stresses together with $\sigma_z = \sigma_o$ into the third equation of (f).

$$\therefore \; \epsilon_z = \frac{\sigma_0}{E} - \frac{2\nu}{E}\frac{p_i r_i^2 - p_0 r_0^2}{r_0^2 - r_i^2} \tag{5.10}$$

→ Note that $\epsilon_z$ is independent of position within the cylinder.

## ▶ Analysis

i) The axial displacement $w$ thus varies linearly with $z$.

ii) The transvers stresses $(\sigma_r, \sigma_\theta)$ are independent of $\sigma_o$ and $\epsilon_z$ depends on the axial loading $\sigma_o$.

iii) When the inner and outer pressures are both equal (that is, $p_i = p_o = p$), we find that $\sigma_r = \sigma_\theta = -p$ throughout the interior.

iv) When the outer pressure is absent ($p_o = 0$), we note that an inner pressure $p_i$ results in a compressive radial stress which varies from $\sigma_r = -p_i$ at the inner wall to $\sigma_r = 0$ at the outer wall.

v) Note that the numerically greatest stress in both Fig. 5.19 (a) and Fig. 5.19 (b) is the tangential stress $\sigma_\theta$ at the inner wall of the cylinder.

vi) When the cylinder wall-thickness $t = r_o - r_i$ becomes small in comparison with $r_i$, the solution (5.9) approaches the thin-walled-tube approximation of Prob. 4.10 (see also Prob. 5.47).

vii) When the axial stress vanishes $(\sigma_o = 0)$, the cylinder is said to be subject to a plane stress distribution. In this case the axial strain $\epsilon_z$ is generally not zero. ($\because$ plane stress distribution $\neq$ plane strain distribution)

viii) We can use the exact result (5.9) to illustrate the concept of stress concentration.

*cf.* A characteristic of the solution (5.9) is that, although it depends on the material's being homogeneous, isotropic, and linearly elastic, the stresses are independent of the actual magnitudes of the elastic parameters $E$ and $\nu$.

*cf.* Note that the results (5.9) and (5.10) involve the quite significance in engineering.

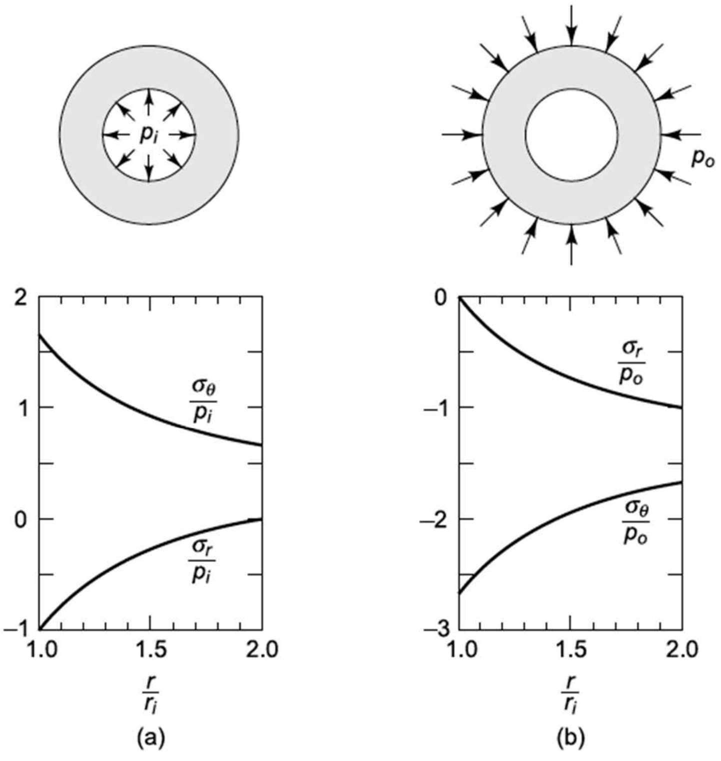



**Fig. 5.19** *Distribution of radial stress $\sigma_r(r)$ and tangential stress $\sigma_\theta(r)$ in cylinder with $r_o = 2r_i$ due to (a) inner pressure $p_i$ and (b) outer pressure $p_o$*