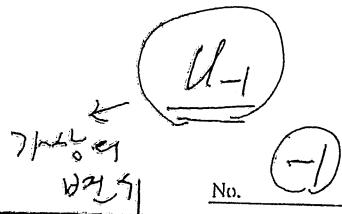


"upload
toHP"



5.3 central difference method

$$\ddot{u}_c = \frac{u_{i+1} - u_{i-1}}{2\Delta t} ; \quad \ddot{u}_o = \frac{u_i - u_{i-1}}{2\Delta t} \quad \dots \quad (1)$$

$$\ddot{u}_c = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta t)^2} ; \quad \ddot{u}_o = \frac{u_i - 2u_o + u_{i-1}}{(\Delta t)^2} \quad \dots \quad (2)$$

$$u_1 = \ddot{u}_o (2\Delta t) + u_{i-1} \quad \dots \quad (3) \quad \leftarrow \text{from (1)}$$

$$(3) \rightarrow (2) \quad \ddot{u}_o (2\Delta t) + u_{i-1} - 2u_o + u_{i-1}$$

$$u_o = \frac{\ddot{u}_o (2\Delta t) + u_{i-1} - 2u_o + u_{i-1}}{(\Delta t)^2}$$

$$\ddot{u}_o (\Delta t)^2 = \ddot{u}_o (2\Delta t) - 2u_o + 2u_{i-1}$$

$$u_{i-1} = u_o - \Delta t (\ddot{u}_o) + \frac{(\ddot{u}_o) (\Delta t)^2}{2} \quad \dots \quad (4)$$

u_o, u_{i-1}

Eg. of motion at time 0,

$$m \ddot{u}_o + c \dot{u}_o + k u_o = p_o$$

$$\text{or} \quad \ddot{u}_o = \frac{1}{m} (p_o - c \dot{u}_o - k u_o) \quad \dots \quad (5)$$

(5) \rightarrow (4)

$$u_{i-1} = u_o - (\Delta t) (\ddot{u}_o) + \frac{(\Delta t)^2}{2} \times \frac{1}{m} (p_o - c \dot{u}_o - k u_o) \quad \checkmark$$

To Initialize

Initial condition
 $t = -\Delta t$ initial value

Direc of Integration Approximation and Load Operators

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = r \quad \dots \quad (1)$$

$$\hat{x}_{t+\Delta t} = A \hat{x}_t + L r_t \quad \leftarrow \text{Recursive relationship}$$

where $\hat{x}_{t+\Delta t}, \hat{x}_t$ = vectors storing the solution quantities
(e.g., displ, velocity, ...etc)

A = matrix for integration approximation operators

L = vector for load operators

note) $\hat{x}_{t+2\Delta t} = A \hat{x}_{t+\Delta t} + L r_{t+\Delta t}$

$$= A [A \hat{x}_t + L r_t] + L \cdot r_{t+\Delta t} \quad \boxed{\text{+ } L \cdot r_{t+\Delta t}}$$

$$= A^2 \hat{x}_t + A L r_t + L \cdot r_{t+\Delta t}$$

... by induction,

$$\hat{x}_{t+n\Delta t} = A^n \hat{x}_t + A^{n-1} L r_{t+\Delta t} + \dots \quad \dots \quad (3)$$

"Central Difference Method" e1 73

... (4)

$$\ddot{x}_t + 2\zeta\omega\dot{x}_t + \omega^2 x_t = r_t$$

$$\ddot{x}_t = \frac{1}{\Delta t^2} (x_{t-\Delta t} - 2x_t + x_{t+\Delta t})$$

$$\dot{x}_t = \frac{1}{\Delta t} (-x_{t-\Delta t} + x_{t+\Delta t})$$

$$x_{t+\Delta t} = \frac{2 - \omega^2 \Delta t^2}{1 + \xi \omega \Delta t} x_t - \frac{1 - \xi \omega \Delta t}{1 + \xi \omega \Delta t} x_{t-\Delta t} + \frac{\Delta t^2}{1 + \xi \omega \Delta t} x_t \quad \dots \quad (5)$$

$$\text{*) } \lambda_1, \lambda_2 = 1 - 2\pi^2 \left(\frac{\omega t}{T}\right)^2 \pm \sqrt{\left(1 - 2\pi^2 \left(\frac{\omega t}{T}\right)^2\right)^2 - 1}$$

$$\Rightarrow \frac{\omega t}{T} = \frac{1}{\pi} \text{ or } -\frac{1}{\pi}, |\lambda_1, \lambda_2|_{\max} = 1.0, \text{ ii) } \frac{\omega t}{T} > \frac{1}{\pi} \text{ or } \frac{-1}{\pi}, |\lambda_1, \lambda_2|_{\min} > 1.0$$

No. ③

$$\underbrace{\begin{Bmatrix} \hat{x}_{t+\Delta t} \\ x_t \end{Bmatrix}}_{\hat{x}_{t+\Delta t}} = \underbrace{\begin{bmatrix} \frac{2-\omega^2 \Delta t^2}{1+5\omega \Delta t} & \frac{-(1-5\omega \Delta t)}{1+5\omega \Delta t} \\ 1 & 0 \end{bmatrix}}_A \underbrace{\begin{Bmatrix} x_t \\ x_{t-\Delta t} \end{Bmatrix}}_{\hat{x}_t} + \underbrace{\begin{bmatrix} \Delta t^2 \\ \frac{1+5\omega \Delta t}{1-5\omega \Delta t} \end{bmatrix}}_L \times r_t \quad \dots (6)$$

The stability of integration \leftarrow
(error blow-up or not)

Examine
the behavior of the
numerical solution
for arbitrary initial
conditions.

($r = 0$)

$$\hat{x}_{t+\Delta t} = A^n \hat{x}_t \quad \dots (6) \quad A \underline{P} = \underline{P} \underline{J}$$

$$A [\underline{v}_1 \dots \underline{v}_n] = [\underline{v}_1 \dots \underline{v}_n] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} [1 - \lambda_i \underline{I}] \times \underline{V}_C = 0$$

spectral decomposition of A ;

$$A^n = \underline{P} \underline{J}^n \underline{P}^{-1}$$

matrix of
eigenvectors
of A

Jordan form of A
with eigenvalues of
 A on its diagonal

$$\therefore A = \underline{P} \underline{J} \underline{P}^{-1}$$

$$\begin{aligned} A^2 &= \underline{P} \underline{J} (\underline{P}^{-1} \cdot \underline{P}) \underline{J} \underline{P}^{-1} \\ &= \underline{P} \underline{J}^2 \underline{P}^{-1} \end{aligned}$$

matrix norm
of \underline{J}^n

$\rightarrow \rho(A) = \text{the spectral radius of } A$
 $= \max |\lambda_i| ; i=1, 2, \dots$

$$A^n = \underline{P} \underline{J}^n \underline{P}^{-1}$$

Then, \underline{J}^n is bounded for $n \rightarrow \infty$ if and only if $\rho(A) \leq 1$.
(our stability criterion)

determinant

Ex) For the CDM with $\xi = 0.0$

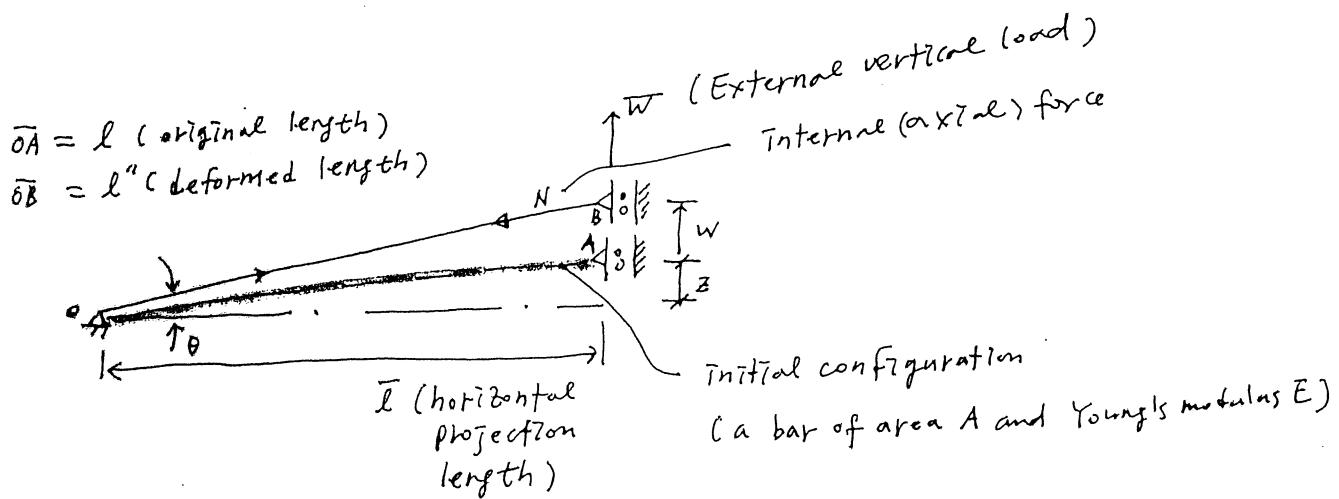
$$\begin{bmatrix} 2-\omega^2 \Delta t^2 & -1 \\ 1 & 0 \end{bmatrix} \times \underline{V} = \lambda \times \underline{V} ; \begin{vmatrix} 2-\omega^2 \Delta t^2 & -1 \\ -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda_{1,2} = \frac{2-\omega^2 \Delta t^2}{2} \pm \sqrt{\frac{(2-\omega^2 \Delta t^2)^2}{4} - 1} \quad \begin{aligned} &\text{L} (2-\omega^2 \Delta t^2 - \lambda)(-\lambda) \\ &+ 1 = 0 \\ &\lambda_1 \leq 1 \quad \Leftrightarrow \frac{\Delta t}{T} \leq \frac{1}{\pi}, \quad \text{so } \Delta t_{\text{tot}} < \frac{T}{\pi} \end{aligned}$$

↳ to (*) ↳ function of $(\frac{\Delta t}{T})$ ↳ $\omega = \frac{2\pi}{T}$

"5.72 23"

A Simple Example for (Geometric) Nonlinearity Analysis



(i) Equilibrium Equation at "Deformed" configuration \leftarrow to induce geometric nonlinearity

$$\sum_{\text{ext.}}^{\uparrow} \tau_c = 0 ; \frac{w - N \sin \theta}{\bar{l}} = 0 \quad \text{or} \quad w = N \sin \theta = N \times \left(\frac{z+w}{\bar{l}''} \right) \simeq \frac{N(z+w)}{\bar{l}} \quad \text{--- (1)}$$

(ii) The strain in the bar \nwarrow assuming that θ is small,

$$\begin{aligned} \varepsilon &= \frac{\bar{l}'' - \bar{l}}{\bar{l}} = \frac{\sqrt{(z+w)^2 + \bar{l}^2}^{1/2} - (z^2 + \bar{l}^2)^{1/2}}{(z^2 + \bar{l}^2)^{1/2}} \\ &= \frac{\bar{l} \left\{ 1 + \left(\frac{z+w}{\bar{l}} \right)^2 \right\}^{1/2}}{\bar{l} \left\{ 1 + \left(\frac{z}{\bar{l}} \right)^2 \right\}^{1/2}} - 1 = \left\{ 1 + \left(\frac{z+w}{\bar{l}} \right)^2 \right\}^{1/2} \left\{ 1 + \left(\frac{z}{\bar{l}} \right)^2 \right\}^{-1/2} - 1 \\ &\simeq \left\{ 1 + \frac{1}{2} \left(\frac{z+w}{\bar{l}} \right)^2 \right\} \left\{ 1 - \frac{1}{2} \left(\frac{z}{\bar{l}} \right)^2 \right\} - 1 \\ &= \left\{ 1 + \frac{1}{2} \left(\frac{z}{\bar{l}} \right)^2 + \left(\frac{z}{\bar{l}} \right) \left(\frac{w}{\bar{l}} \right) + \frac{1}{2} \left(\frac{w}{\bar{l}} \right)^2 \right\} \left\{ 1 - \frac{1}{2} \left(\frac{z}{\bar{l}} \right)^2 \right\} - 1 \\ &= \cancel{1 + \frac{1}{2} \left(\frac{z}{\bar{l}} \right)^2 + \left(\frac{z}{\bar{l}} \right) \left(\frac{w}{\bar{l}} \right) + \frac{1}{2} \left(\frac{w}{\bar{l}} \right)^2} - \cancel{\frac{1}{2} \left(\frac{z}{\bar{l}} \right)^2 - \frac{1}{4} \left(\frac{z}{\bar{l}} \right)^4 - \frac{1}{2} \left(\frac{z}{\bar{l}} \right)^3 \left(\frac{w}{\bar{l}} \right) - \frac{1}{4} \left(\frac{z}{\bar{l}} \right)^2 \left(\frac{w}{\bar{l}} \right)^2} \cancel{+} \\ &= \left(\frac{z}{\bar{l}} \right) \left(\frac{w}{\bar{l}} \right) + \frac{1}{2} \left(\frac{w}{\bar{l}} \right)^2 \end{aligned}$$

higher order

$$\therefore \varepsilon \simeq \left(\frac{z}{\bar{l}} \right) \left(\frac{w}{\bar{l}} \right) + \frac{1}{2} \left(\frac{w}{\bar{l}} \right)^2 \quad \text{--- (2)}$$

\nwarrow approximate but valid in relation to
"a shallow truss theory"
"nonlinear function of w "

(iii) The axial force in the bar

$$N = A(E\varepsilon) = AE \left\{ \left(\frac{z}{l}\right) \left(\frac{w}{l}\right) + \sum \left(\frac{w}{l}\right)^2 \right\} \quad \dots \dots (3)$$

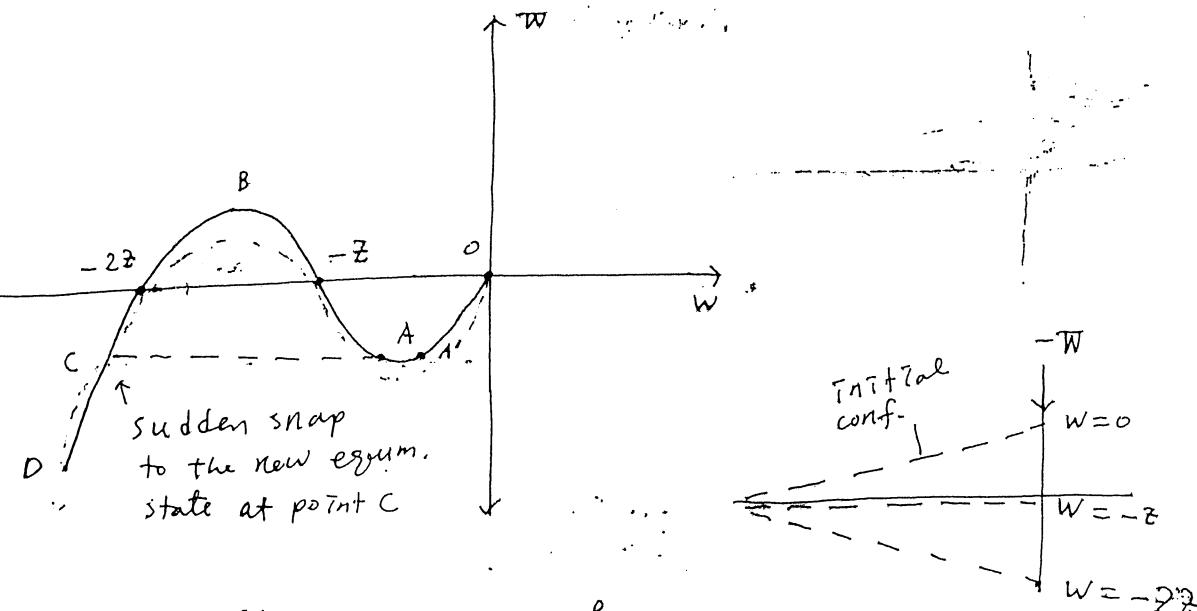
(iv) The load W versus the displacement w relationship

(3) \rightarrow (4)

$$W = \frac{(z+w)}{l} \times AB \left(\frac{zw}{l^2} + \frac{1}{2} \frac{w^2}{l^2} \right)$$

$$= \left(\frac{AE}{l^3} \right) \left(\frac{1}{2} w^3 + \frac{3}{2} zw^2 + z^2 w \right) \quad \dots \dots (4)$$

$$\left\{ \begin{array}{l} = \left(\frac{AE}{l^3} \right) \left(\frac{1}{2} w(w^2 + 3zw + 2z^2) \right) \\ = \left(\frac{AE}{2l^3} \right) (w)(w+z)(w+2z) \quad ; \quad W=0 \rightarrow w=0, -z, -2z \end{array} \right.$$



(v) the tangent stiffness (matrix) k_t

$$\begin{aligned} k_t &= \frac{dW}{dw} = \frac{d}{dw} \left[\left(\frac{z+w}{l} \right) N \right] = \left(\frac{z+w}{l} \right) \frac{dN}{dw} + \frac{N}{l} \\ &= \frac{EA}{l} \left(\frac{z+w}{l} \right)^2 + \frac{N}{l} \quad \dots \dots (5) \end{aligned}$$

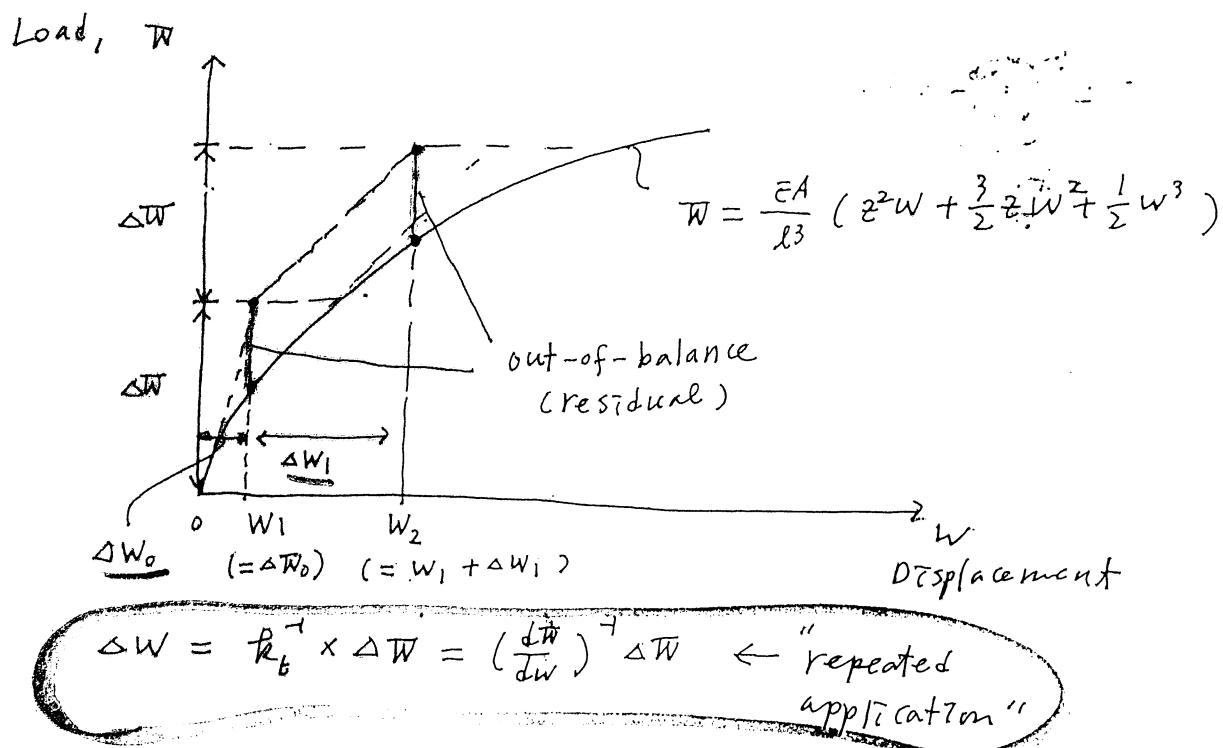
(vi) Computation of w for a given W . \leftarrow load control

OA' ? \therefore only
3 2 1 8 2 8 4 3 0 1
 \therefore 0 8 2 6 1 2 0 2 1

vs.
Disp. control

(3)

An Incremental solution Scheme (Euler Scheme)



$$EX) EA = 5 \times 10^7 N, z = 25 \text{ mm}, l = 2500 \text{ mm}, \Delta W = -7 \text{ N}$$

(i) For the 1st step, $w_0 = 0 \& N_0 = 0$

$$k_{t,0} = \frac{EA}{l} \left(\frac{z+0}{l} \right)^2 + \frac{0}{l} = \frac{EA}{l} \left(\frac{z}{l} \right)^2 = \frac{5 \times 10^7}{2500} \left(\frac{25}{2500} \right)^2 = 2.0$$

$$w_1 = \Delta W_0 = k_{t,0}^{-1} \times \Delta W = \frac{-7}{2.0} = -3.5$$

note: $W(w_1 = -3.5) = \frac{5 \times 10^7}{2500^3} (25^2 \times (-3.5) + \frac{3}{2} (25)(-3.5)^2 + \frac{1}{2} (-3.5)^3)$

$$= \frac{-4449}{-5160} \neq -7.0 = \Delta W$$

↑
(4) ↗
lack of equilibrium

$$\therefore N_1 = 5 \times 10^7 \left\{ \left(\frac{25}{2500} \right) \left(\frac{-3.5}{2500} \right) + \frac{1}{2} \left(\frac{-3.5}{2500} \right)^2 \right\} = -651$$

↖ (3) ↗

(ii) The 2nd step,

$$k_{t,1} = \frac{5 \times 10^7}{2500} \left(\frac{25 - 3.5}{2500} \right)^2 + \frac{(-651)}{2500} = 1.22 \text{ (soft by 26)}$$

$$\Delta W_1 = \frac{-7}{1.22} = -5.74 \rightarrow w_2 = w_1 + \Delta W_1 \approx -3.5 - 5.74 = -9.24$$

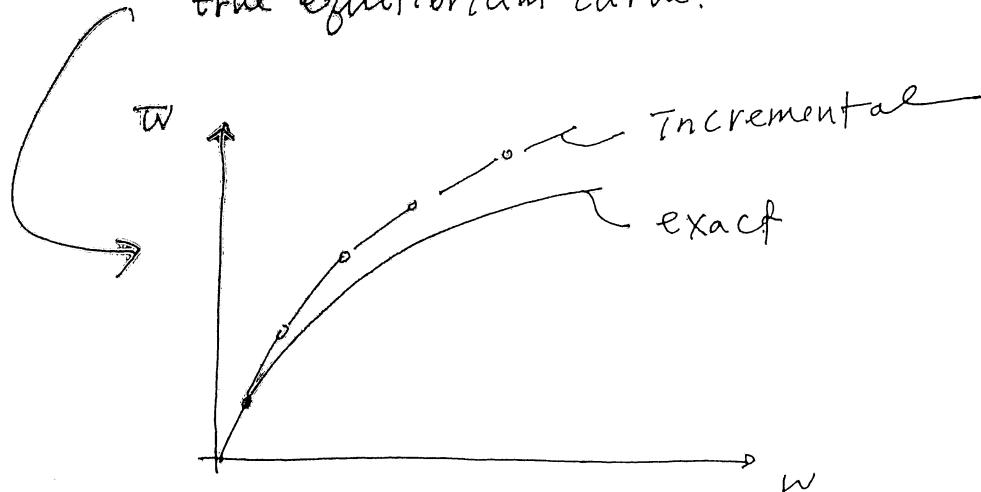
(4)

Note: \bar{w} ($w = w_2 = -9.24$)

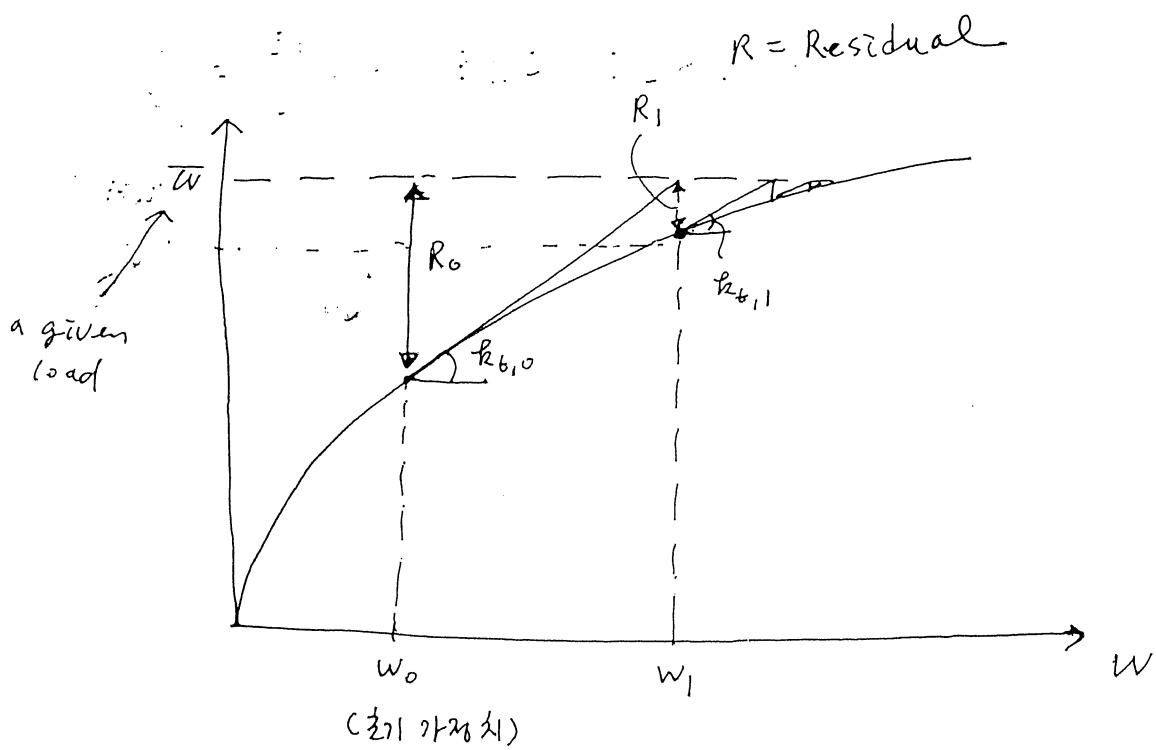
$$= \frac{5 \times 10^7}{2500^3} \left\{ 25^2 (-9.24) + \frac{3}{2} (25)(-9.24)^2 + \frac{1}{2} (-9.24)^3 \right\}$$

$$= -9.50 \neq 2\Delta\bar{w} = -14$$

Irreversibly, the solution will drift from the true equilibrium curve.



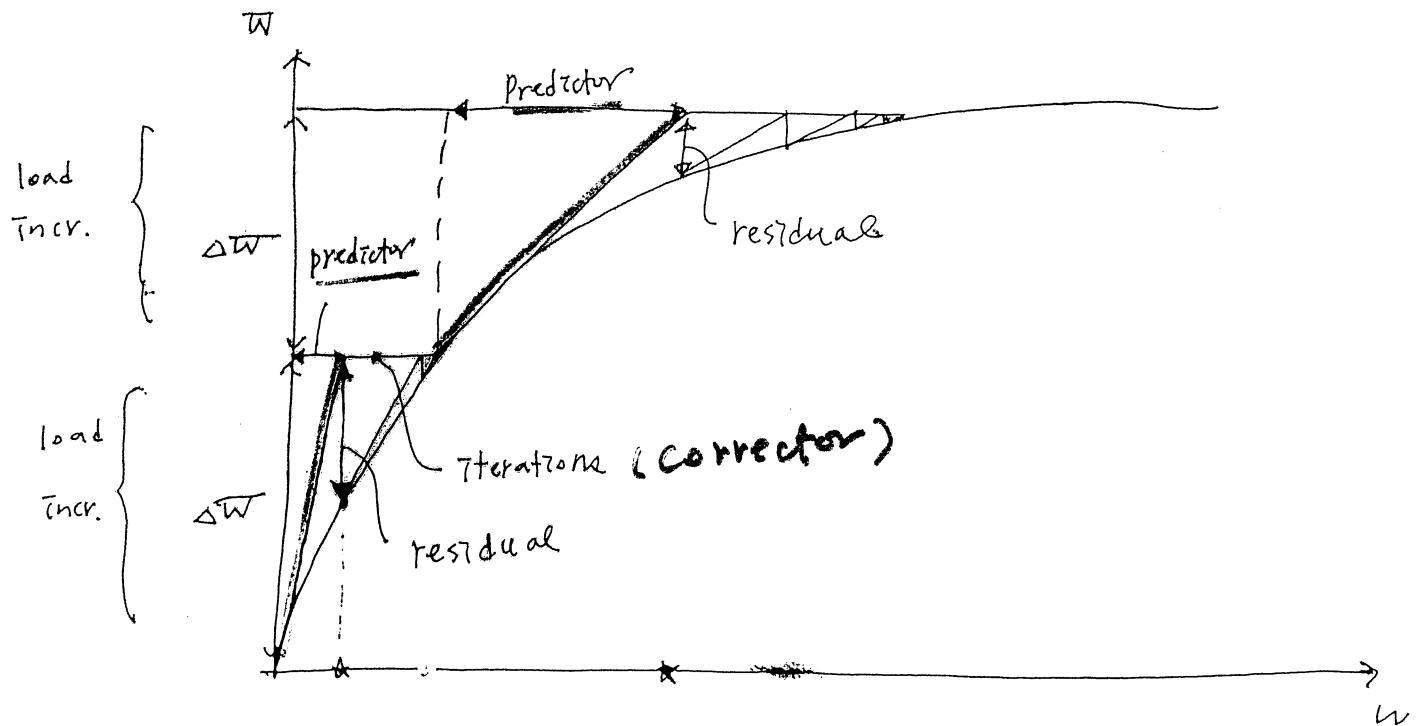
An iterative solution (the Newton-Raphson method)



(5)

A Combination of Incremental predictors with N-R

Re.

Iterations

* Modification to this solution procedure:

i) the retention of the original tangent stiffness at the beginning of each load incr.

ii) the initial stress method

↳ Use the stiffness from the very first incremental solution.

