Ch. 6
Torsion

6.1 Introduction

→ In this chapter, we shall consider the problem of twisting, or torsion. A slender element subjected primarily to twist is usually called a shaft.

▶ Use of shafts under torsion

i) A twisted shaft can be used to provide a spring with prescribed stiffness with respect to rotation; examples of this are the torsion-bar spring system on automobiles.

ii) On a different scale, the measurement of extremely small forces by an instrument which uses a very fine wire in torsion as the basic spring.

→ We are interested primarily in,

① The twisting moment, torque, which can be transmitted by the shaft without damage to the material.

② The components of stresses in the materials under this torque.

③ The stresses in the shaft. In the use of a shaft as a torsional spring, we are interested primarily in the relation between the applied twisting moment and the resulting angular twist of the shaft.

▶ Procedure of the analysis of torsion problem

i) Geometric behavior of a twisted shaft

ii) Stress - strain relations

iii) Conditions of equilibrium

6.2 Geometry of Deformation of a Twisted Circular Shaft

→ Let us start our consideration of possible modes of deformation by isolating from the shaft a slice Δz in length with faces originally plane and normal to the axis of the shaft.

cf. We take this slice from somewhere near the middle of the shaft so that we are away from any possible end effects.

▶ Analysis of deformation

→ In case that material is isotropic and the slice has full geometric circular symmetry about the *z* axis,

1 ▷ When a circular shaft is twisted, its cross-sections must remain plane.

cf. Skip the proof. Refer to Fig. 6.4

2 ▷ Straight diameters are carried into straight diameters by the twisting deformation.

<<Proof>>

<Case.1>

A deformation would violate geometric compatibility since the curvatures of $A_1'O_1'H_1'$ and $B'C'J'$ have opposite sense. It would be impossible to fit these two elements together when deformed as shown.

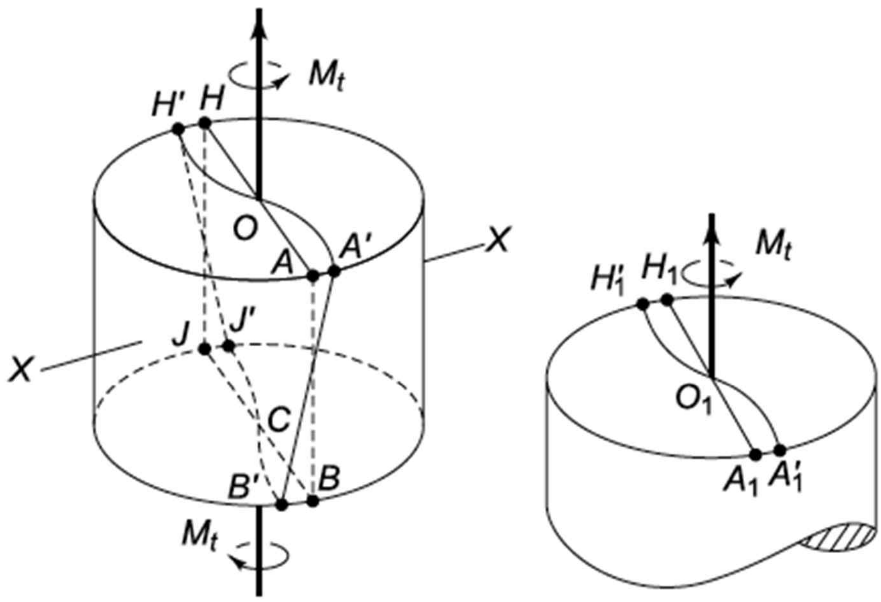

Fig. 6.5 *The assumed shape of B′CJ′ does not match $A_1'O_1H_1'$*

<Case.2>

We rotate the element in Fig. 6.6a about the axis XX which is perpendicular to the element $HOABCJ$. After a rotation of 180° the element is upside down, as shown in Fig. 6.6 (b). Now we compare Fig. 6.6 (a) with 6.6 (b). The elements are of identical shape and material and are subjected to identical loadings. Therefore, they should have identical deformations. The curvatures of diameters in the two elements, however, are of opposite sense.

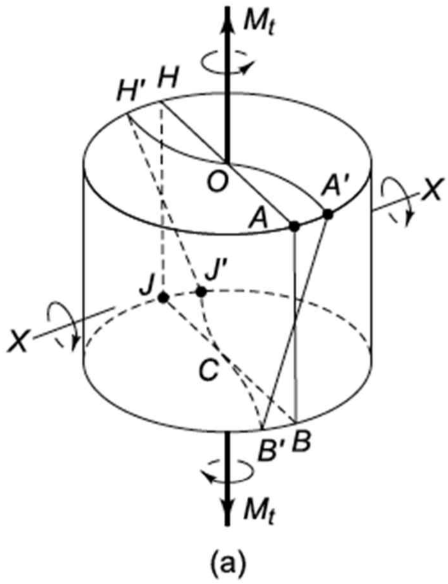

Fig. 6.6 *Rotating (a) about X-X through 180° yields (b), which has undergone different deformation even though the twisting moment and geometry are the same*

<Conclusion>

Thus the assumption that diametral lines deform into curved lines is ruled out by symmetry, and we are forced to the conclusion that the deformation pattern must be as indicated in Fig. 6.7.

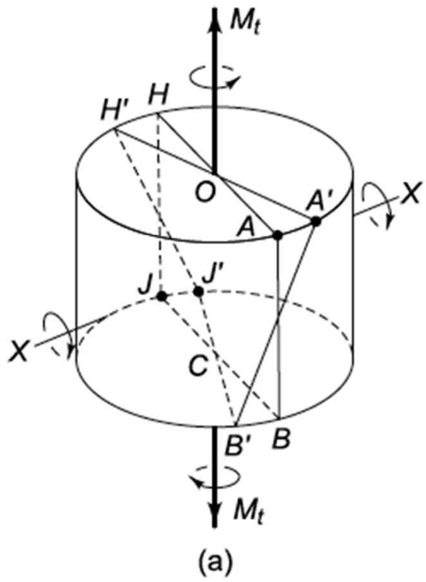

Fig. 6.7 *If the diameter HA remains straight during deformation, then rotation of (a) about X-X produces (b) which is identical in terms of deformation*

3 ▷ Symmetry of deformation has not ruled out a symmetrical expansion or contraction of the circular cross section or a lengthening or shortening of the cylinder. It does not seem plausible, however, that such dilational deformations would be an important part of the deformation due to a twisting moment.

cf. We assume that $\epsilon_r = \epsilon_\theta = \epsilon_z = 0$ in this section.

4 ▷ On the basis of this assumption we shall arrive at a consistent theory which meets all the requirements of the theory of elasticity, providing the amount of twist is small.

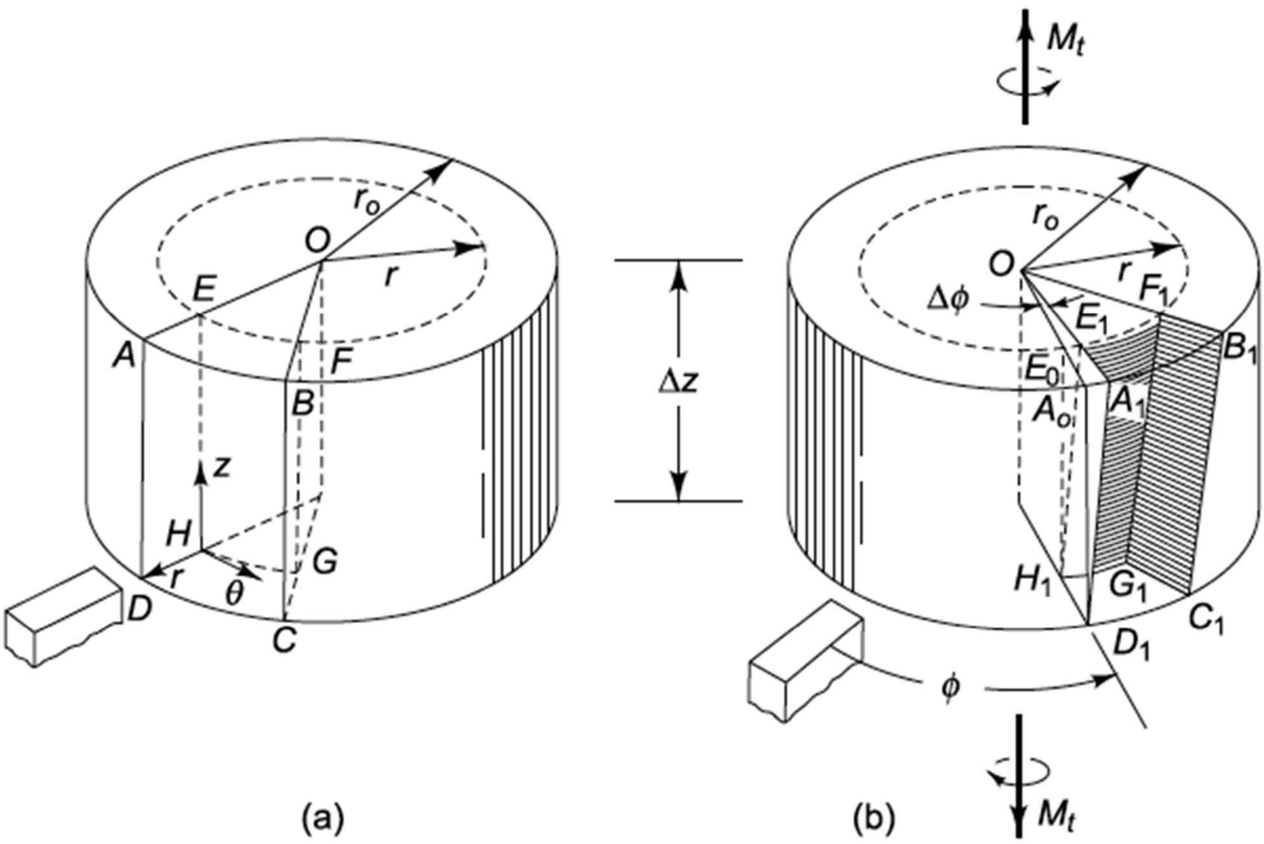

Fig. 6.8 *Analysis of deformation of a slice of circular shaft subjected to torsion*

$$\gamma_{\theta z} = \lim_{\Delta z \to 0} \frac{E_0 E_1}{H_1 E_0} = \lim_{\Delta z \to 0} \frac{r\Delta\phi}{\Delta z} = r\frac{d\phi}{dz} \tag{6.1}$$

→ It is important to emphasize that this states that the shear strain varies in direct proportion to the radius, from no shear at the center to a greatest shear at the outside, where $r = r_0$ (i.e., the element $A_1B_1C_1D_1$ in Fig. 6.8 has this greatest shear strain)

5 ▷ Twist per unit length (or Rate of twist)

We call $d\phi/dz$ the twist per unit length and it is a constant along a uniform section of shaft subjected to twisting moments at the ends.

6 ▷ Strains in the shaft

→ Thus, from symmetry and the plausible assumption that the extensional strains are zero, we have arrived at the following distribution of strains.

$$\epsilon_r = \epsilon_\theta = \epsilon_z = \gamma_{r\theta} = \gamma_{rz} = 0$$

$$\gamma_{\theta z} = r\frac{d\phi}{dz} \tag{6.2}$$

cf. These strains were derived from the geometrically compatible deformation of Fig 6.8 by simple geometry. We next turn to a consideration of the force-deformation relations of the shaft material.

6.3 Stresses obtained from Stress-Strain Relation

Using Hooke's law in cylindrical coordinates, we find that the stress components related to the strain components given by (6.2) are

$$\sigma_r = \sigma_\theta = \sigma_z = \tau_{r\theta} = \tau_{rz} = 0$$

$$\tau_{\theta z} = G\gamma_{\theta z} = Gr\frac{d\phi}{dz} \tag{6.3}$$

∴ The only component acting is the tangential shear stress component $\tau_{\theta z}$, whose magnitude varies linearly with radius as given by (6.3).

cf. These stress components are shown as follow (Fig. 6.9).

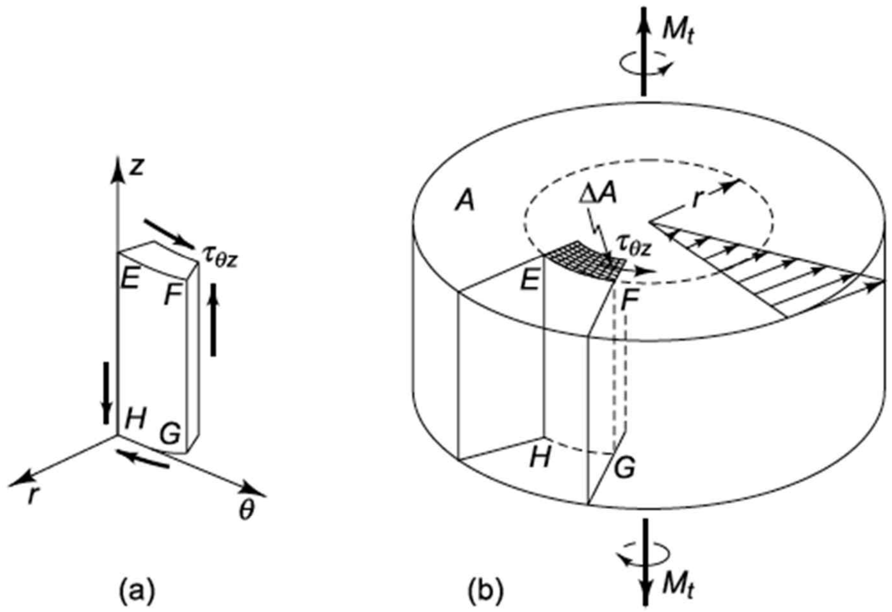

Fig. 6.9 *(a) Stress components acting on a small element; (b) distribution of shearing stress on cross section*

Inside each element, such as that shown in Fig.6.9a is in equilibrium because the shear stress $\tau_{\theta z}$ does not change in the θ direction (because of symmetry) nor in the z direction. (Because of the uniformity of the deformation and stress pattern along the length of the shaft.)

6.4 Equilibrium Requirement

▶ <u>From Fig. 6.9</u>

i) Both the shear strain $\gamma_{\theta z}$ and the shear stress $\tau_{\theta z}$ are proportional to the rate of twist $d\phi/dz$

ii) The stress distribution given by Eq. (6.3) and shown in Fig. 6.9b leaves the external cylindrical surface of the shaft free of stress, as it should.

iii) Both $\gamma_{\theta z}$ *and* $\tau_{\theta z}$ don't change in the θ nor in the *z* direction.

iv) The shearing stress is therefore the same on each *z* and θ face of the element in Fig.6.9a, and thus the element is in equilibrium.

▶ Equilibrium

$$\int_A r(\tau_{\theta z} dA) = M_t \tag{6.4}$$

6.5 Stress and Deformation in a Twisted Elastic Circular Shaft.

▶ $\tau_{\theta z} - M_t - \phi$ Relations

$$M_t = \int_A r(\tau_{\theta z}\, dA) = \int_A r\left[Gr\frac{d\phi}{dz}\, dA\right]$$

$$= G\frac{d\phi}{dz}\int_A r^2\, dA = I_z\frac{d\phi}{dz}G \tag{6.5}$$

where $I_z = \pi r_o^4/2 = \pi d^4/32$ (polar moment of inertia) (6.6)

From Eq. (6.5), we obtain the rate of twist $d\phi/dz$ in terms of the applied twisting moment

$$\frac{d\phi}{dz} = \frac{M_t}{GI_z} \tag{6.7}$$

$$\phi = \int_0^L \frac{M_t}{GI_z}dz = \frac{M_t L}{GI_z}\ [rad] \tag{6.8}$$

When we substitute $d\phi/dz$ from (6.7) into (6.3), we obtain the stress in terms of the applied twisting moment.

$$\tau_{\theta z} = Gr\frac{d\phi}{dz} = Gr\frac{M_t}{GI_z} = \frac{M_t r}{I_z} \qquad (6.9)$$

▶ Confer

i) The conditions for Eqs. (6.2) and (6.3) satisfy

① The fundamental equations of elasticity

② The requirements of equilibrium for every small element

③ Geometric compatibility

④ Hooke's law

⑤ No stress on the outside cylindrical surface

ii) Edge effect

If the shaft is reasonably long, our estimate (6.8) of the total twist is probably not very much affected by the manner of loading at the ends. We cannot, however, use (6.9) to predict the local stresses at the ends.

iii) With respect to central axis,

$$I_z = \pi r_o^4 / 2 = \pi d^4/32 \qquad (6.6)$$

▶ Torsional Stiffness

$$k = \frac{M_t}{\phi} = \frac{GI_z}{L}$$

cf. It gives the twisting moment per radian of twist.

cf. This ratio is analogous to a spring constant which gives tensile force per unit length of stretch.

▶ Example 6.2 A couple of 70N·m is applied to a 25-mm-diameter 2024-0 aluminum-alloy shaft, as shown in Fig. 6.11 (a). The ends *A* and *C* of the shaft are built-in and prevented from rotating, and we wish to know the angle through which the center cross section *O* of the shaft rotates.

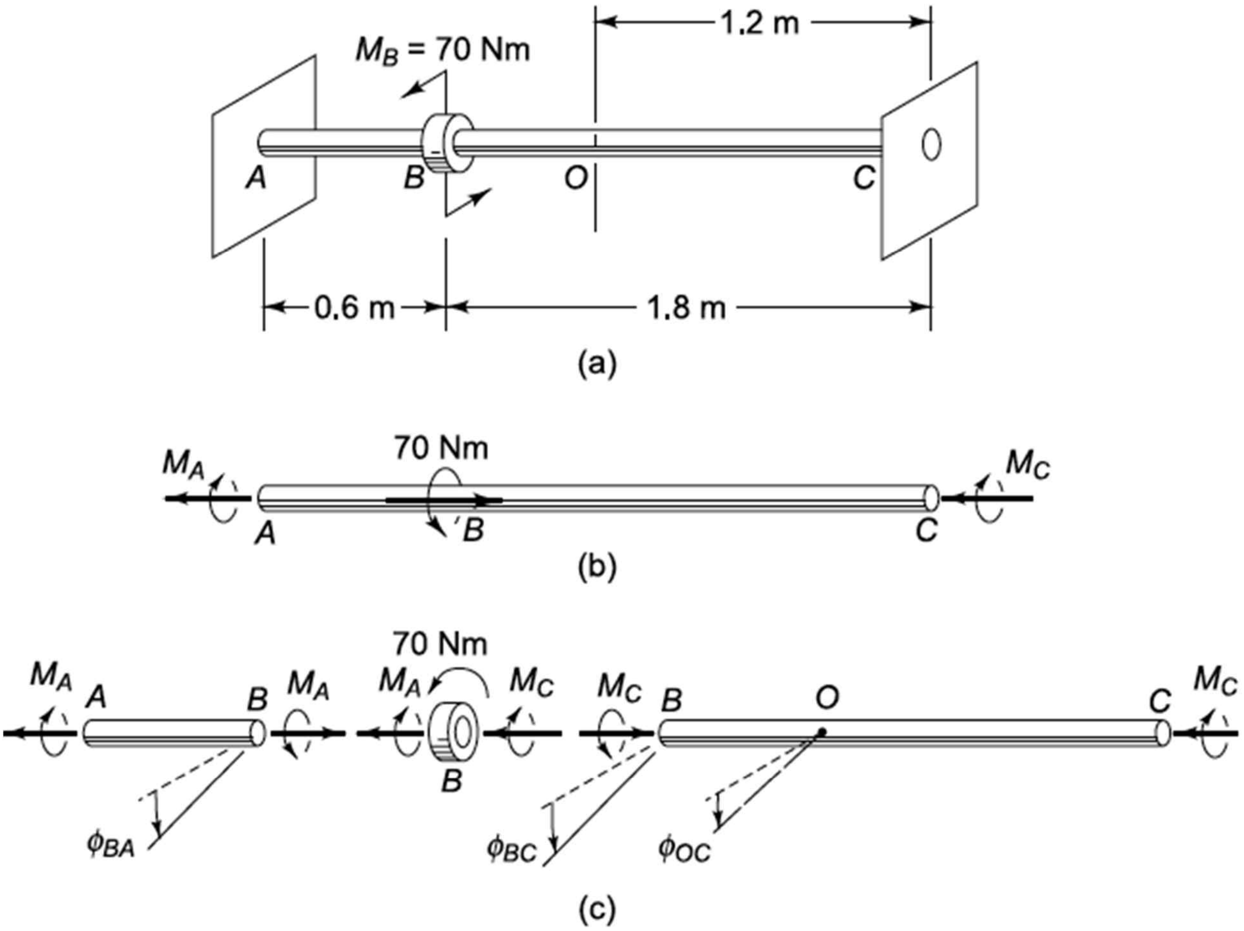

Fig. 6.11 *Example 6.2*

1 ▷ Equilibrium

From fig. (b)

$$M_A + M_C - 70 = 0 \tag{a}$$

2 ▷ Geometry

$$\phi_{BC} = \phi_{BA} \tag{b}$$

3 ▷ Load-Deformation

$$\phi_{BA} = \frac{M_A L_{AB}}{GI_z}, \qquad \phi_{BC} = \frac{M_C L_{BC}}{GI_z}, \qquad \phi_{OC} = \frac{M_C L_{OC}}{GI_z} \tag{c}$$

→ From Eq. (b)

$$\frac{M_A L_{AB}}{GI_z} = \frac{M_C L_{BC}}{GI_z}$$

$$\therefore \; M_C = \frac{L_{AB}}{L_{BC}} M_A$$

→ From Eq.(a)

$$M_A + \frac{L_{AB}}{L_{BC}} M_A - 70 = 0$$

$$M_A = \frac{70}{1+L_{AB}/L_{BC}} = 52.5\ N \cdot m$$

$$\therefore \; M_C = 70 - M_A = 17.5\ N \cdot m$$

$$\therefore \; \phi_{OC} = \frac{M_C L_{OC}}{GI_z} = \frac{17.5(1.2)}{[26(10)^9][\pi(0.025)^4/32]} = 0.021\ rad \; = 1.20°$$

▶ Summary

$$\gamma_{\theta z} = r\frac{d\phi}{dz} \tag{6.2}$$

$$\tau_{\theta z} = \gamma_{\theta z} G = rG\,\frac{d\phi}{dz} \tag{6.3}$$

$$M_t = \int_0^r \tau_{\theta z} r(2\pi r\ dr) = G\frac{d\phi}{dz}\int r^2\, dA = I_z G\frac{d\phi}{dz} \tag{6.5}$$

$$\frac{d\phi}{dz} = \frac{M_t}{GI_z}, \qquad \phi = \frac{d\phi}{dz}L = \frac{M_t L}{GI_z} \qquad (6.7)\ (6.8)$$

$$\tau_{\theta z} = Gr\left(\frac{M_t}{GI_z}\right) = \frac{M_t r}{I_z} \tag{6.9}$$