CH. 6
TORSION

6.1 Introduction

→ In this chapter, we shall consider the problem of twisting, or torsion. A slender element subjected primarily to twist is usually called a shaft.

▶ Use of shafts under torsion

i) A twisted shaft can be used to provide a spring with prescribed stiffness with respect to rotation; examples of this are the torsion-bar spring system on automobiles.

ii) On a different scale, the measurement of extremely small forces by an instrument which uses a very fine wire in torsion as the basic spring.

→ We are interested primarily in,

① The twisting moment, torque, which can be transmitted by the shaft without damage to the material.

② The components of stresses in the materials under this torque.

③ The stresses in the shaft. In the use of a shaft as a torsional spring, we are interested primarily in the relation between the applied twisting moment and the resulting angular twist of the shaft.

▶ Procedure of the analysis of torsion problem

i) Geometric behavior of a twisted shaft

ii) Stress - strain relations

iii) Conditions of equilibrium

6.2 Geometry of Deformation of a Twisted Circular Shaft

→ Let us start our consideration of possible modes of deformation by isolating from the shaft a slice Δz in length with faces originally plane and normal to the axis of the shaft.

cf. We take this slice from somewhere near the middle of the shaft so that we are away from any possible end effects.

▶ Analysis of deformation

→ In case that material is isotropic and the slice has full geometric circular symmetry about the *z* axis,

1 ▷ When a circular shaft is twisted, its cross-sections must remain plane.

cf. Skip the proof. Refer to Fig. 6.4

2 ▷ Straight diameters are carried into straight diameters by the twisting deformation.

<<Proof>>

<Case.1>

A deformation would violate geometric compatibility since the curvatures of $A_1'O_1'H_1'$ and $B'C'J'$ have opposite sense. It would be impossible to fit these two elements together when deformed as shown.

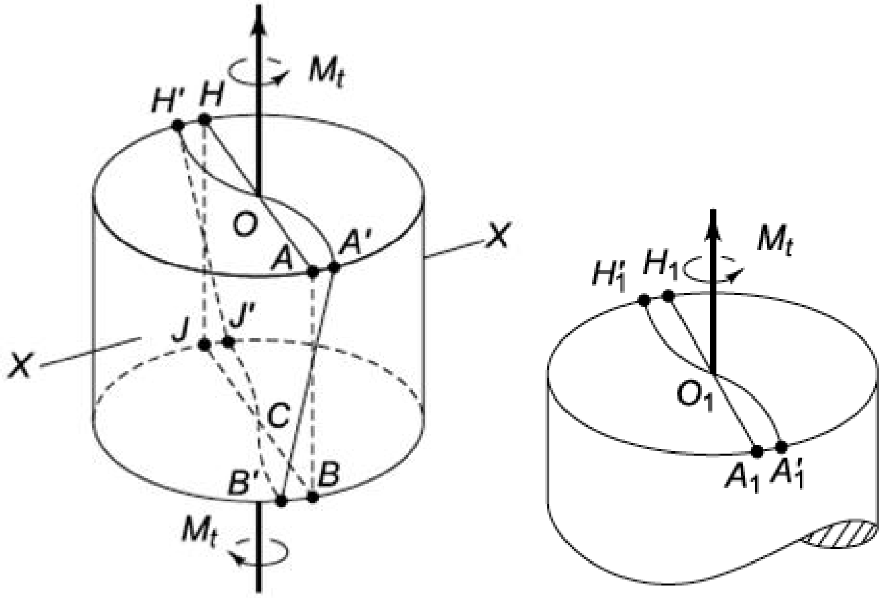


Fig. 6.5 The assumed shape of B′CJ′ does not match $A_1'O_1H_1'$

<Case.2>

We rotate the element in Fig. 6.6a about the axis XX which is perpendicular to the element $HOABCJ$. After a rotation of 180° the element is upside down, as shown in Fig. 6.6 (b). Now we compare Fig. 6.6 (a) with 6.6 (b). The elements are of identical shape and material and are subjected to identical loadings. Therefore, they should have identical deformations. The curvatures of diameters in the two elements, however, are of opposite sense.

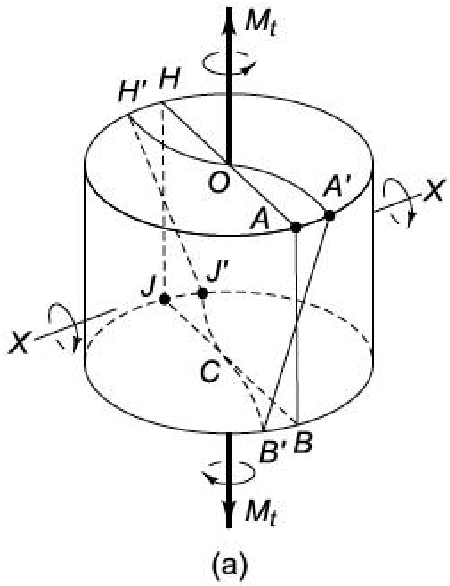


Fig. 6.6 *Rotating (a) about X-X through 180° yields (b), which has undergone different deformation even though the twisting moment and geometry are the same*

<Conclusion>

Thus the assumption that diametral lines deform into curved lines is ruled out by symmetry, and we are forced to the conclusion that the deformation pattern must be as indicated in Fig. 6.7.

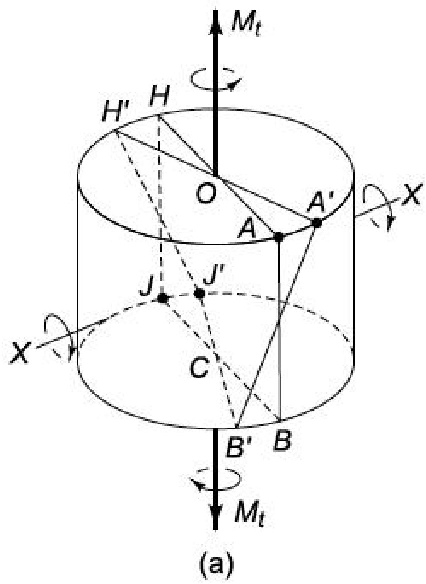


Fig. 6.7 *If the diameter HA remains straight during deformation, then rotation of (a) about X-X produces (b) which is identical in terms of deformation*

3 ▷ Symmetry of deformation has not ruled out a symmetrical expansion or contraction of the circular cross section or a lengthening or shortening of the cylinder. It does not seem plausible, however, that such dilatational deformations would be an important part of the deformation due to a twisting moment.

cf. We assume that $\epsilon_r = \epsilon_\theta = \epsilon_z = 0$ in this section.

4 ▷ On the basis of this assumption we shall arrive at a consistent theory which meets all the requirements of the theory of elasticity, providing the amount of twist is small.

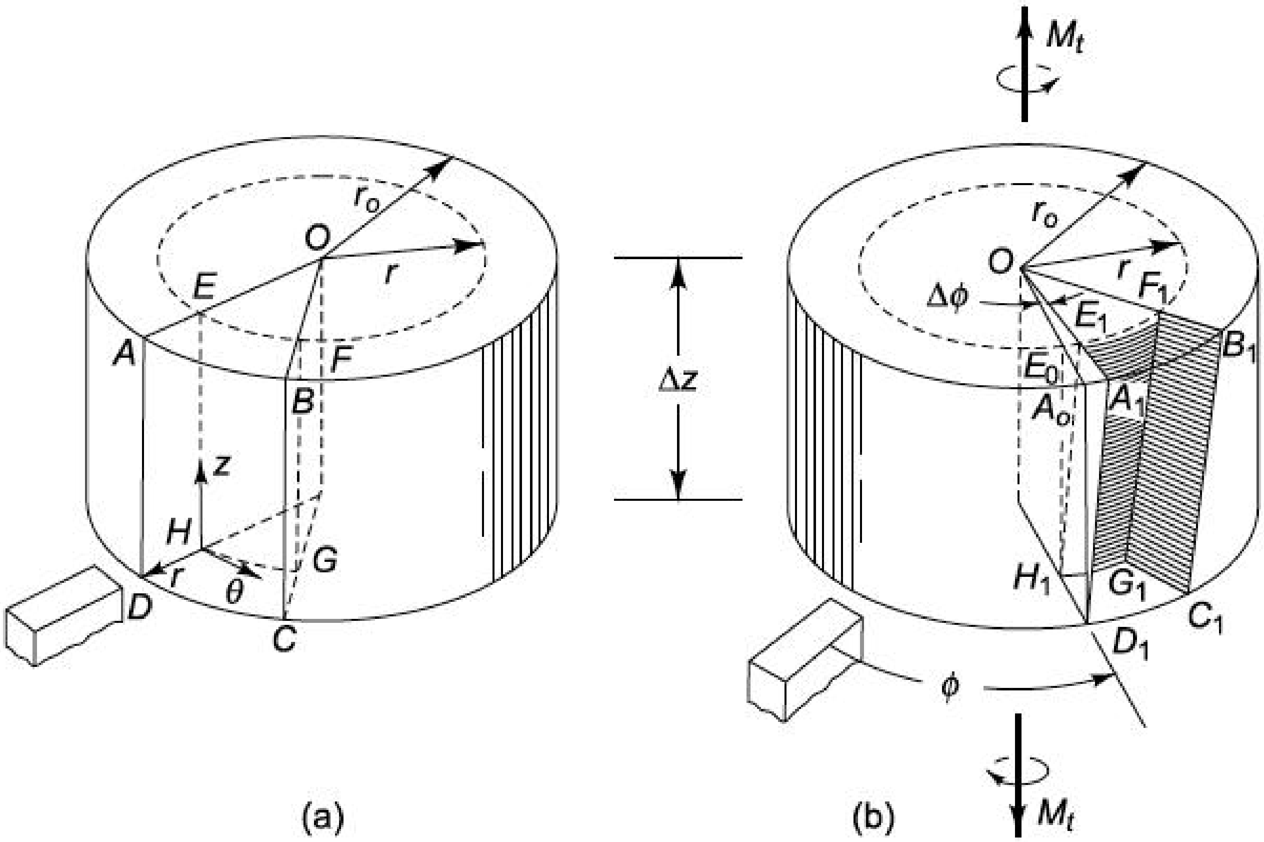


Fig. 6.8 *Analysis of deformation of a slice of circular shaft subjected to torsion*

$$\gamma_{\theta z} = lim_{\Delta z \to 0} \frac{E_0 E_1}{H_1 E_0} = lim_{\Delta z \to 0} \frac{r\Delta\phi}{\Delta z} = r\frac{d\phi}{dz} \quad (6.1)$$

→ It is important to emphasize that this states that the shear strain varies in direct proportion to the radius, from no shear at the center to a greatest shear at the outside, where $r = r_0$ (i.e., the element $A_1B_1C_1D_1$ in Fig. 6.8 has this greatest shear strain)

5 ▷ Twist per unit length (or Rate of twist)

We call $d\phi/dz$ the twist per unit length and it is a constant along a uniform section of shaft subjected to twisting moments at the ends.

6 ▷ Strains in the shaft

→ Thus, from symmetry and the plausible assumption that the extensional strains are zero, we have arrived at the following distribution of strains.

$$\epsilon_r = \epsilon_\theta = \epsilon_z = \gamma_{r\theta} = \gamma_{rz} = 0$$

$$\gamma_{\theta z} = r\frac{d\phi}{dz} \tag{6.2}$$

cf. These strains were derived from the geometrically compatible deformation of Fig 6.8 by simple geometry. We next turn to a consideration of the force-deformation relations of the shaft material.

6.3 Stresses obtained from Stress-Strain Relation

Using Hooke's law in cylindrical coordinates, we find that the stress components related to the strain components given by (6.2) are

$$\sigma_r = \sigma_\theta = \sigma_z = \tau_{r\theta} = \tau_{rz} = 0$$

$$\tau_{\theta z} = G\gamma_{\theta z} = Gr\frac{d\phi}{dz} \tag{6.3}$$

∴ The only component acting is the tangential shear stress component $\tau_{\theta z}$, whose magnitude varies linearly with radius as given by (6.3).

cf. These stress components are shown as follow (Fig. 6.9).

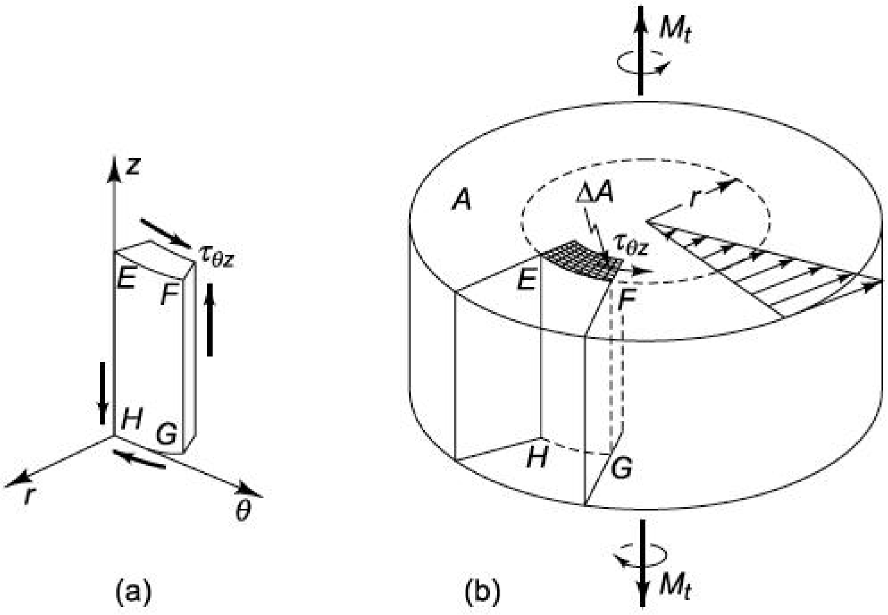


Fig. 6.9 *(a) Stress components acting on a small element; (b) distribution of shearing stress on cross section*

Inside each element, such as that shown in Fig.6.9a is in equilibrium because the shear stress $\tau_{\theta z}$ does not change in the θ direction (because of symmetry) nor in the z direction. (Because of the uniformity of the deformation and stress pattern along the length of the shaft.)

6.4 Equilibrium Requirement

▶ <u>From Fig. 6.9</u>

i) Both the shear strain $\gamma_{\theta z}$ and the shear stress $\tau_{\theta z}$ are proportional to the rate of twist $d\phi/dz$

ii) The stress distribution given by Eq. (6.3) and shown in Fig. 6.9b leaves the external cylindrical surface of the shaft free of stress, as it should.

iii) Both $\gamma_{\theta z}$ *and* $\tau_{\theta z}$ don't change in the θ nor in the *z* direction.

iv) The shearing stress is therefore the same on each *z* and θ face of the element in Fig.6.9a, and thus the element is in equilibrium.

▶ Equilibrium

$$\int_A r(\tau_{\theta z} dA) = M_t \tag{6.4}$$

6.5 Stress and Deformation in a Twisted Elastic Circular Shaft.

▶ $\tau_{\theta z} - M_t - \phi$ Relations

$$M_t = \int_A r(\tau_{\theta z}\, dA) = \int_A r\left[Gr\frac{d\phi}{dz}\, dA\right]$$

$$= G\frac{d\phi}{dz}\int_A r^2\, dA = I_z\frac{d\phi}{dz}G \tag{6.5}$$

where $I_z = \pi r_o^4 / 2 = \pi d^4/32$ (6.6)

From Eq. (6.5), we obtain the rate of twist $d\phi/dz$ in terms of the applied twisting moment

$$\frac{d\phi}{dz} = \frac{M_t}{GI_z} \tag{6.7}$$

$$\phi = \int_0^L \frac{M_t}{GI_z} dz = \frac{M_t L}{GI_z}\ [rad] \tag{6.8}$$

When we substitute $d\phi/dz$ from (6.7) into (6.3), we obtain the stress in terms of the applied twisting moment.

$$\tau_{\theta z} = Gr\frac{d\phi}{dz} = Gr\frac{M_t}{GI_z} = \frac{M_t r}{I_z} \qquad (6.9)$$

▶ Confer

i) The conditions for Eqs. (6.2) and (6.3) satisfy

① The fundamental equations of elasticity

② The requirements of equilibrium for every small element

③ Geometric compatibility

④ Hooke's law

⑤ No stress on the outside cylindrical surface

ii) Edge effect

If the shaft is reasonably long, our estimate (6.8) of the total twist is probably not very much affected by the manner of loading at the ends. We cannot, however, use (6.9) to predict the local stresses at the ends.

iii) With respect to central axis,

$$I_z = \pi r_o^4 / 2 = \pi d^4/32 \qquad (6.6)$$

▶ Torsional Stiffness

$$k = \frac{M_t}{\phi} = \frac{GI_z}{L}$$

cf. It gives the twisting moment per radian of twist.

cf. This ratio is analogous to a spring constant which gives tensile force per unit length of stretch.

▶ Example 6.2 A couple of 70N·m is applied to a 25-mm-diameter 2024-0 aluminum-alloy shaft, as shown in Fig. 6.11 (a). The ends *A* and *C* of the shaft are built-in and prevented from rotating, and we wish to know the angle through which the center cross section *O* of the shaft rotates.

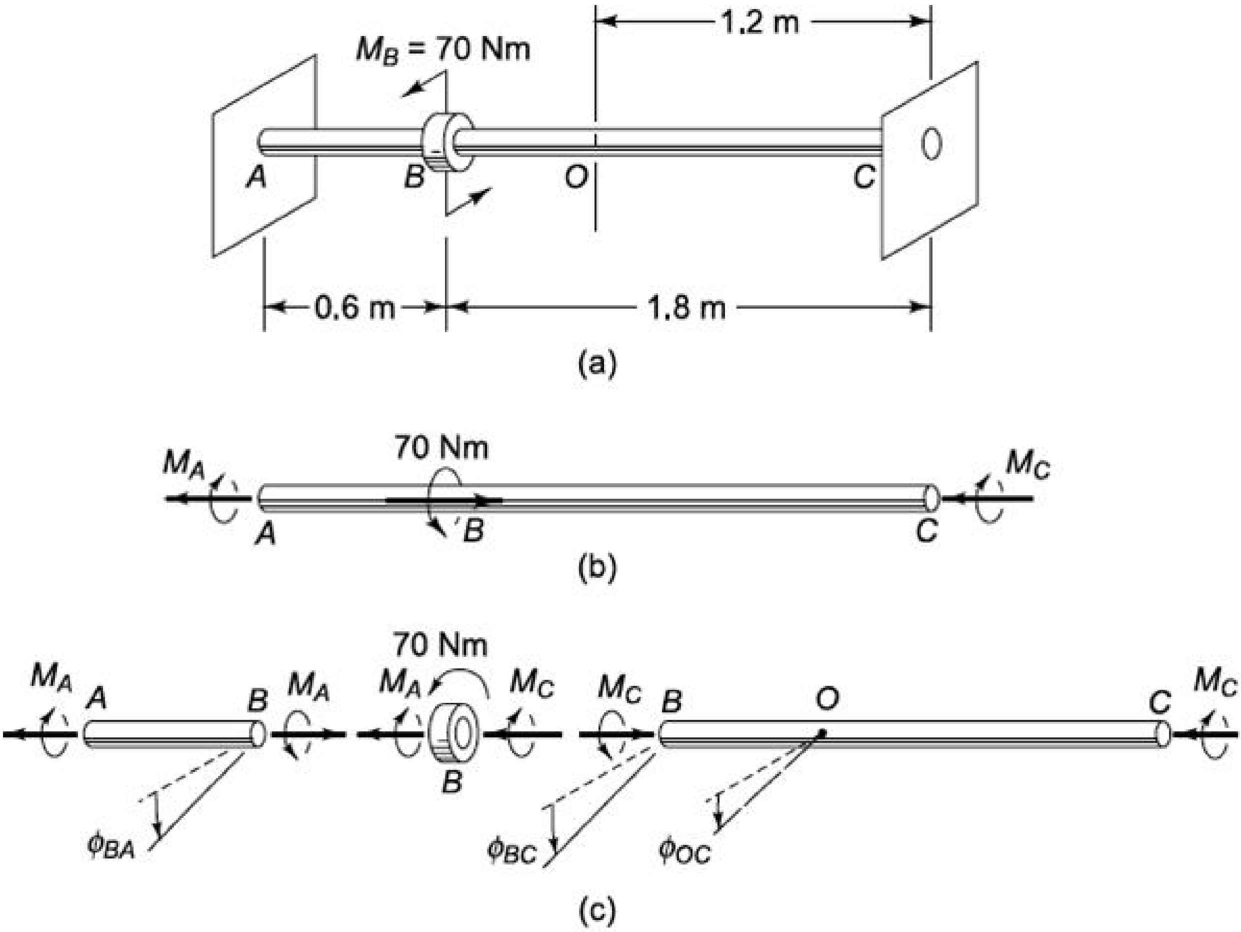


Fig. 6.11 *Example 6.2*

1 ▷ Equilibrium

From fig. (b)

$$M_A + M_C - 70 = 0 \quad \text{(a)}$$

2 ▷ Geometry

$$\phi_{BC} = \phi_{BA} \tag{b}$$

3 ▷ Load-Deformation

$$\phi_{BA} = \frac{M_A L_{AB}}{GI_z}, \qquad \phi_{BC} = \frac{M_C L_{BC}}{GI_z}, \qquad \phi_{OC} = \frac{M_C L_{OC}}{GI_z} \tag{c}$$

→ From Eq. (b)

$$\frac{M_A L_{AB}}{GI_z} = \frac{M_C L_{BC}}{GI_z}$$

$$\therefore \quad M_C = \frac{L_{AB}}{L_{BC}} M_A$$

→ From Eq.(a)

$$M_A + \frac{L_{AB}}{L_{BC}} M_A - 70 = 0$$

$$M_A = \frac{70}{1 + L_{AB}/L_{BC}} = 52.5\ N \cdot m$$

$$\therefore \quad M_C = 70 - M_A = 17.5\ N \cdot m$$

$$\therefore \ \phi_{OC} = \frac{M_C L_{OC}}{GI_z} = \frac{17.5(1.2)}{[26(10)^9][\pi(0.025)^4/32]} = 0.021\ rad = 1.20°$$

▶ Summary

$$\gamma_{\theta z} = r\frac{d\phi}{dz} \tag{6.2}$$

$$\tau_{\theta z} = \gamma_{\theta z} G = rG\frac{d\phi}{dz} \tag{6.3}$$

$$M_t = \int_0^r \tau_{\theta z} r(2\pi r\, dr) = G\frac{d\phi}{dz}\int r^2\, dA = I_z G\frac{d\phi}{dz} \tag{6.5}$$

$$\frac{d\phi}{dz} = \frac{M_t}{GI_z}, \qquad \phi = \frac{d\phi}{dz}L = \frac{M_t L}{GI_z} \qquad (6.7)\ (6.8)$$

$$\tau_{\theta z} = Gr\left(\frac{M_t}{GI_z}\right) = \frac{M_t r}{I_z} \tag{6.9}$$

6.6 Torsion of Elastic Hollow Circular Shafts

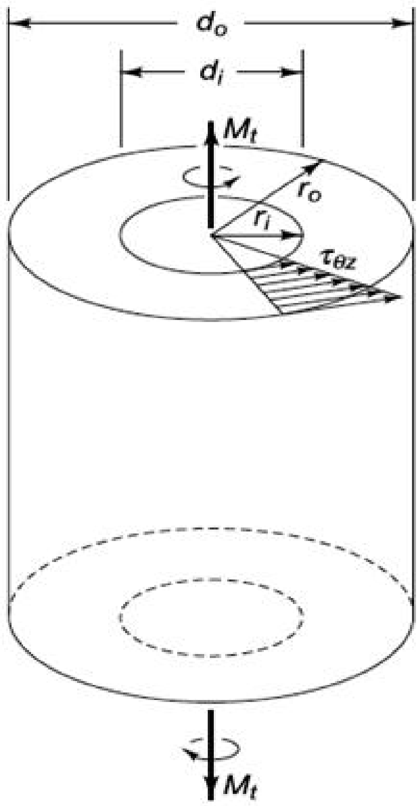


Fig. 6.12 *Stress distribution in elastic hollow circular shaft*

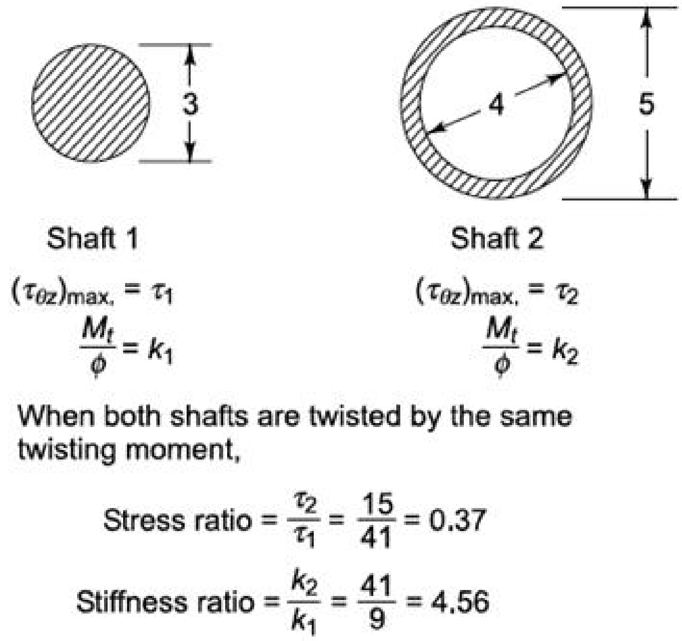


Fig. 6.13 *Illustration of advantages of hollow shaft over solid shaft of same cross-sectional area*

→ The only difference is that the integral in (6.4) now extends over an annulus instead of a complete circle.

$$I_z = \frac{\pi r_0^4}{2}\left(1 - \frac{r_L^4}{r_0^4}\right) = \frac{\pi d_0^4}{32}\left(1 - \frac{d_i^4}{d_0^4}\right) \qquad (6.11)$$

cf. $\int_A r(\tau_{\theta z} dA) = M_t \qquad (6.4)$

▶ Analysis

i) Making a concentric hole in a shaft does not reduce the torsional stiffness in proportion to the amount of material removed.

→ An element of material near the center of the shaft has a low stress and a small moment arm and thus contributes less to the twisting moment than an element near the outside of the shaft.

ii) The torsional stiffness for a given length of given material depends only on the polar moment of inertia I_Z.

iii) It is apparent that a given amount of material is used most efficiently in torsion when it is formed into a hollow shaft.

cf. There is a limit on the increase in effectiveness that can be obtained by increasing the diameter and decreasing the wall thickness. (If the wall is made too thin, the cylinder wall will buckle due to compressive stresses which act in the wall on surfaces inclined at $45°$ to the axis of the cylinder.)

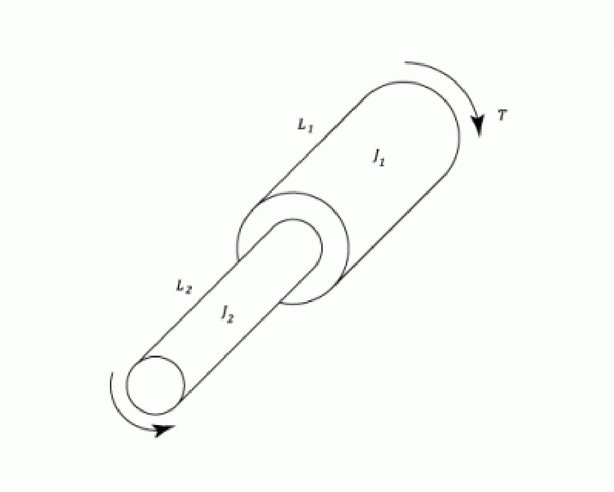
iv) Compare the hollow shaft and solid shaft in Fig 6.13 which have the same cross-sectional area but markedly different maximum stresses and deformation.

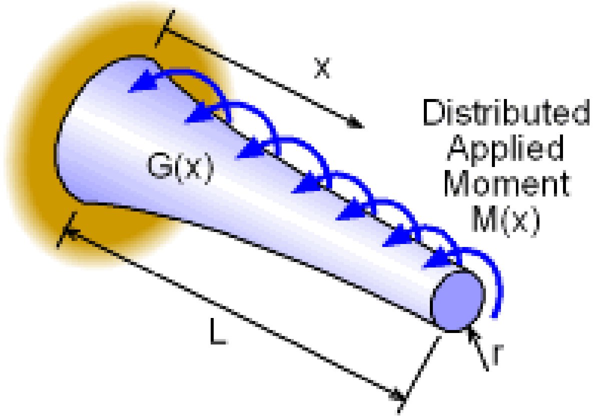
$$\frac{\tau_2}{\tau_1} = \frac{M_t r_2/I_2}{M_t r_1/I_1} = \frac{I r_2}{I_2 r_2} = \frac{(3^4)(5)}{(5^4-4^4)3^4} = \frac{135}{369} = \frac{15}{41} = 0.37$$

$$\frac{k_2}{k_1} = \frac{GI_2/L_2}{GI_1/L_1} = \frac{I_2}{I_1} = \frac{(5^4-4^4)}{3^4} = \frac{41}{9} = 4.56$$

cf. The shear-stress ratio is same with yield M_t ratio and stiffness ratio means ratio of torsion angle.

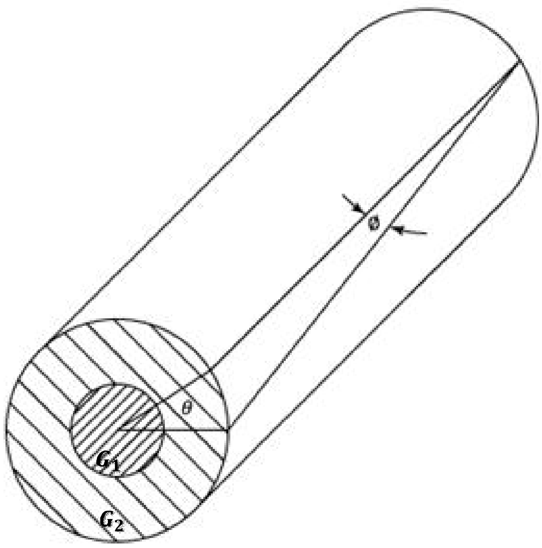
▶ Non-uniform torsion examples





$$\phi = \sum_{i=1}^{n} \frac{T_i L_i}{G_i I_i} \qquad \phi = \int_0^L d\phi = \int_0^L \frac{T_x dx}{G I_x}$$

▶ Composite shaft



$$\begin{cases} M_t = M_{t1} + M_{t2} \\ \phi_1 = \phi_2 = \frac{M_{t1} L}{G_1 I_1} = \frac{M_{t2} L}{G_2 I_2} \end{cases}$$

$$\begin{cases} M_{t1} = M_t \left(\frac{G_1 I_1}{G_1 I_1 + G_2 I_2} \right) \\ M_{t2} = M_t \left(\frac{G_2 I_2}{G_1 I_1 + G_2 I_2} \right) \end{cases}$$

$$\therefore \quad \phi_1 = \phi_2 = \frac{M_t L}{G_1 I_1 + G_2 I_2}$$

$$\frac{\tau_2}{\tau_1} = \frac{G_2 \gamma_2}{G_1 \gamma_1} = \frac{G_2}{G_1}$$

cf. Above ratio can be smaller than 1.

cf. Shear strains in two parts which are attached have same value, but each material has different coefficient and therefore stress is different.

6.7 Stress Analysis in Torsion; Combined Stress

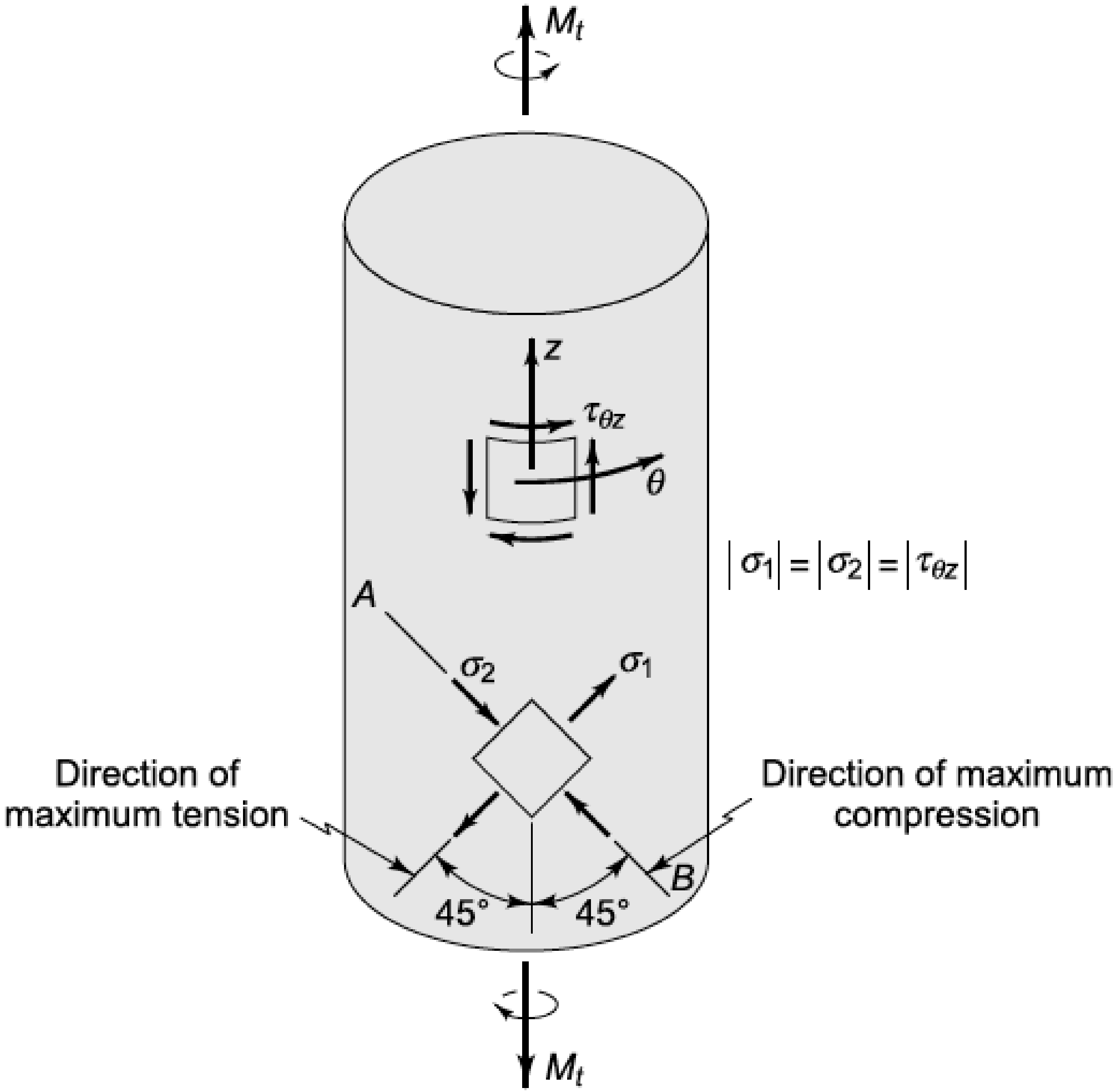


Fig. 6.14 *The principal stresses in torsion are equal tension and compression acting on faces inclined at 45° to the axis of the shaft*

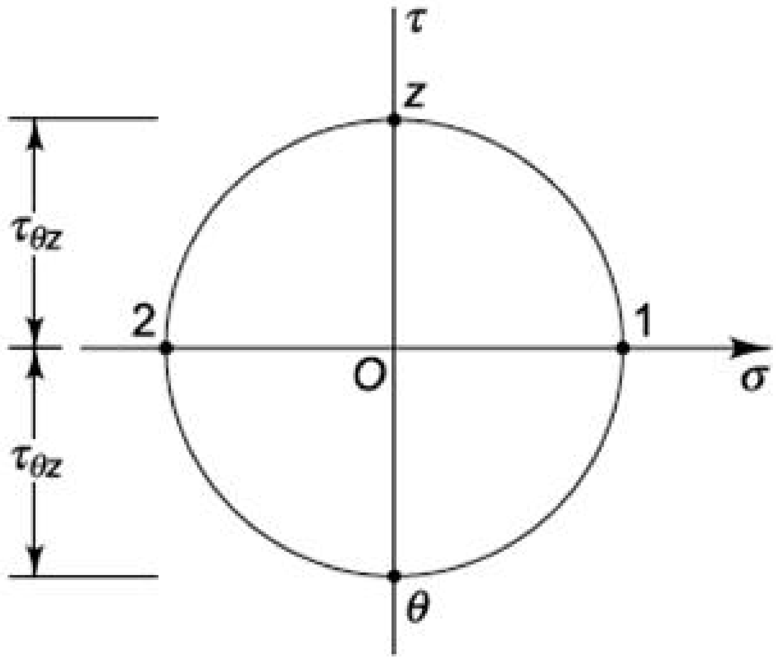


Fig. 6.15 Mohr's circle for stress for element of shaft in torsion

→ When shaft is twisted, it is on the pure shear stress state. And a convenient way to determine these stress components is to use Mohr's circle for stress.

cf. We may use the two-dimensional Mohr's circle because there is no stress in the r-direction.

▶ Magnitudes of principal stresses (from Mohr circle)

$|\sigma_1| = |\sigma_2| = |\tau_{\theta z}|$

$\theta_P = 45°$

cf. If a piece of chalk (which is a brittle material with a low tensile strength and much larger strength in compression and shear) is twisted, the chalk will fracture along a spiral line normal to the direction of maximum tension (e.g., along the line AB in Fig. 6.14)

▶ Combined-stress

→ The stresses and strains contributed by one form of loading are not altered by the presence of another kind of loading.

→ The justification for superposition lies in the linearity of Eqs. (5.6), (5.7), and (5.8) underlying the theory of elasticity.

▶ Example 6.3

In Fig. 6.16 (a) an uniform, homogeneous, circular shaft is shown subjected simultaneously to an axial tensile force P and a twisting moment M_t. In Fig. 6.16 (b) the individual stress distributions are sketched for the separate loads.

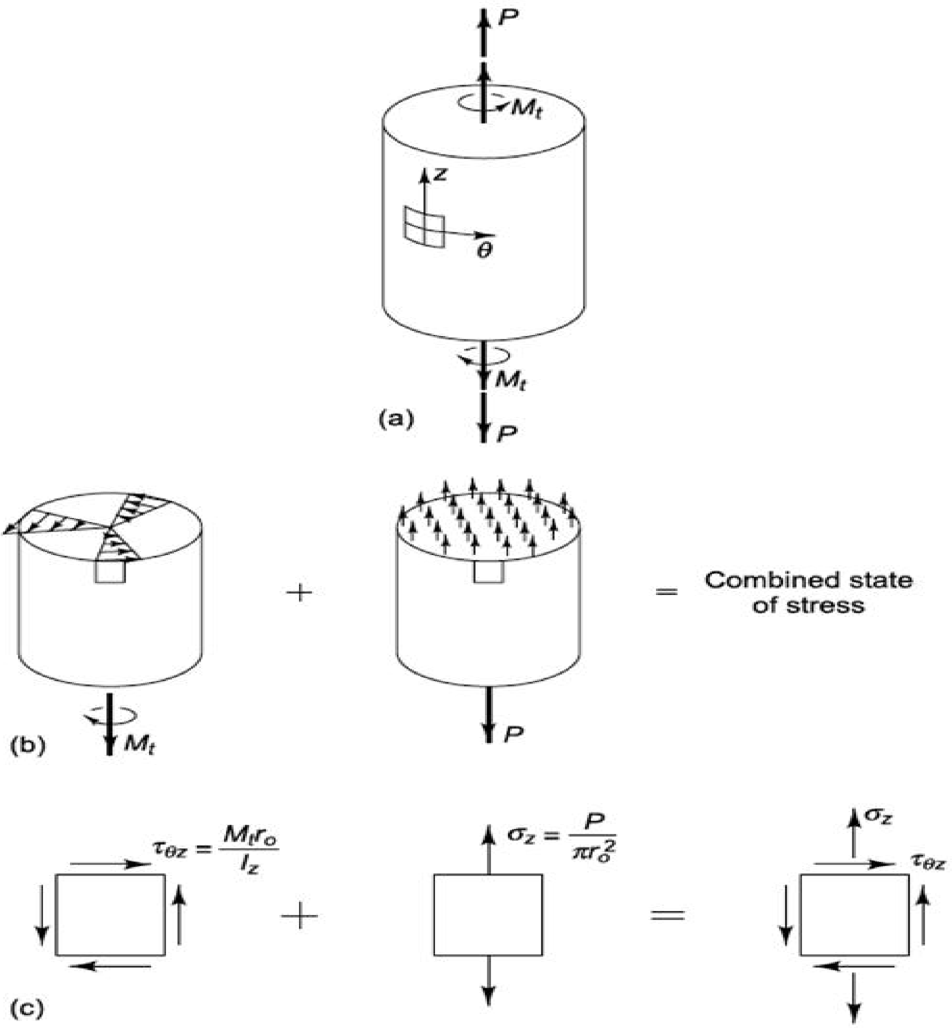


Fig. 6.16 *Example 6.3. Combined stresses due to torsion and tension*

From the Fig. 6.16 (a),

$$\tau_{\theta z} = \frac{M_t r_0}{J_z} \quad \text{(a)}$$

$$\sigma_z = \frac{P}{\pi r_0^2} \quad \text{(b)}$$

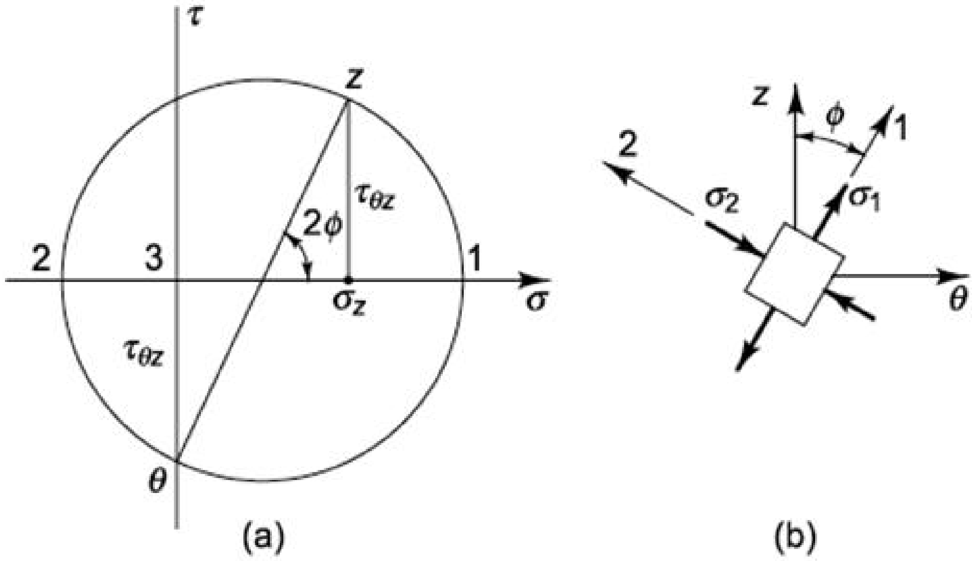


Fig. 6.17 *Example 6.3. Principal directions and principal stresses*

cf. The most convenient method of describing the combined-stress state is to use the principal stress components.

cf. Note that this element is in a state of plane stress, i.e., the third principal stress σ_3 is zero.

→ Positive shear stress τ_{xy} (see Fig. 4.11) is plotted downward at x and upward at y. Negative shear stress is plotted upward at x and downward at y.

▶ Note

▷ In pure shear state,

$$\epsilon_1 = \gamma_{\theta z}/2$$

6.8 Strain Energy Due to Torsion

→ In this section we apply that result specifically to the case of torsion of circular members and consider an example of Castigliano's theorem applied to torsional deformation.

cf. Obtaining the strain energy is important in many ways such as dynamic analysis and structure theory.

▶ For circular shaft [Isotropic-linear-elastic]

→ The only non-vanishing stress and strain components are $\tau_{\theta z}$ and $\gamma_{\theta z}$. The total strain energy (5.17) thus reduces to

$$U = \frac{1}{2}\int_V \tau_{\theta z}\gamma_{\theta z}\, dV \qquad (6.12)$$

$$= \frac{1}{2}\int_V \frac{1}{G}\left[\frac{M_t r}{I_z}\right]^2 dV = \frac{1}{2}\int_L \frac{M_t^2}{GI_z^2} dz \int_A r^2\, dA$$

$$= \int_L \frac{M_t^2}{2GI_z}\, dz = \int_L \frac{GI_z}{2}\left(\frac{d\emptyset}{dz}\right)^2 dz \qquad (6.13)$$

$$\rightarrow dU = \frac{1}{2}M_t\, d\phi = \frac{1}{2}M_t \frac{d\phi}{dz}\, dz = \frac{GI_z}{2}\left(\frac{d\phi}{dz}\right)^2 \qquad (6.14)$$

▷ For uniform torsion

$$\begin{cases} u = \dfrac{U}{V} = \dfrac{\tau_{\theta z}\gamma_{\theta z}}{2} = \dfrac{\tau_{\theta z}^2}{2G} = \dfrac{G\gamma_{\theta z}^2}{2} \\ U = \dfrac{M_t^2 L}{2GI_z} = \dfrac{GI_z\emptyset^2}{2L} = \dfrac{I_z\emptyset}{2} \end{cases}$$

cf. We illustrate the application of Castigliano's theorem ($\delta_i = \partial U/\partial P_i$) to a torsional system in the following example 6.4.

▶ Example 6.4

Consider a closely wound coil spring of radius R loaded by a force P (Fig. 6.18 (a)). The spring consists of n turns of wire with wire radius r. We wish to find the deflection of the spring and hence the spring constant.

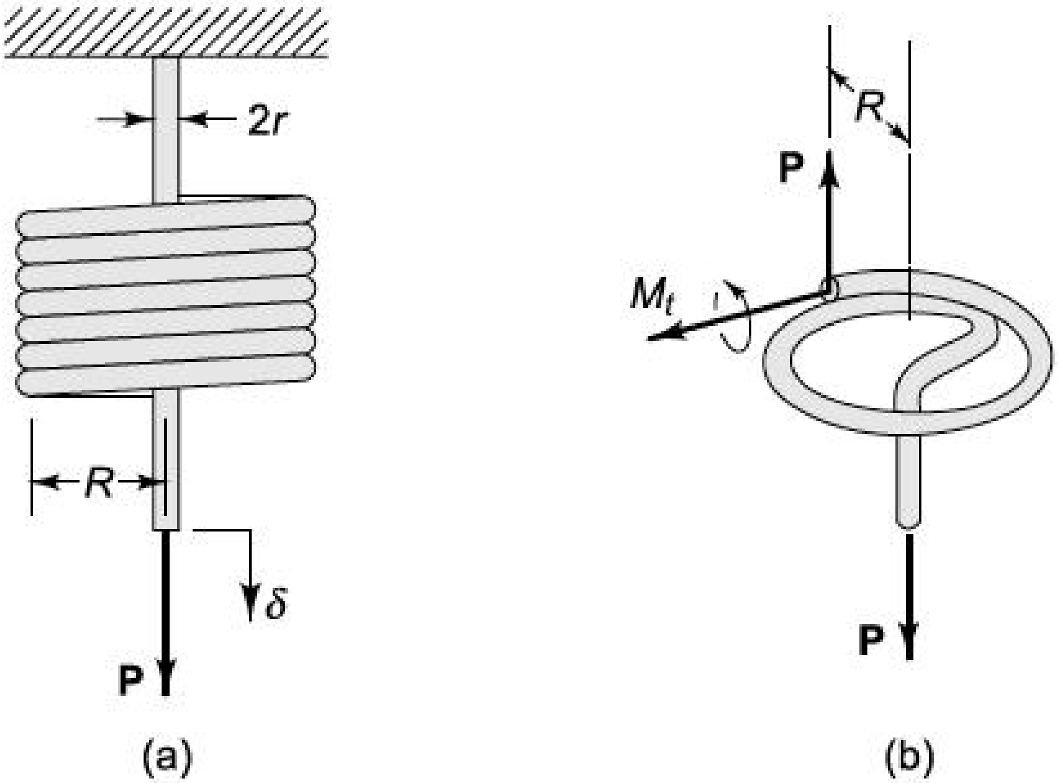


Fig. 6.18 Example 6.4

1 ▷ The strain energy associated with the twisting moment

$$U = \int_L \frac{P^2R^2}{2GI_z}\,dz = \int_0^{2\pi n} \frac{P^2R^2}{2GI_z} R\,d\theta = \frac{P^2R^3}{2GI_z} 2\pi n \qquad \text{(a)}$$

2 ▷ Strain energy due to the transverse shear force

→ There is additional strain energy in the spring due to the transverse shear force P. It can be shown, however, that the ratio of strain energy due to transverse shear to strain energy due to torsion is proportional to $\left(\frac{r}{R}\right)^2$ and hence is small for springs of usual design.

3 ▷ Application of Castigliano's theorem

$$\delta = \frac{\partial U}{\partial P} = \frac{PR^3}{GI_z} 2\pi n \qquad \text{(b)}$$

$$\rightarrow \quad \therefore \quad k = \frac{P}{\delta} = \frac{GI_z}{2\pi n R^3} \qquad \text{(c)}$$

Upon substituting for the moment of inertia I_z in (c), we find that

$$k = \frac{Gr^4}{4nR^3}$$

→ We see that the spring constant is inversely proportional to the number of coils n and directly proportional to the fourth power of the wire radius. For example, if we increase the wire radius by 19 percent, the spring constant is doubled.

6.9 The Onset of Yielding in Torsion

→ In order to apply either criterion to a particular material it is necessary to obtain (experimentally) the yield stress Y in uniaxial tension.

→ Then, to decide whether yielding will occur in a general state of stress, we compute the equivalent or effective stress $\bar{\sigma}$(or $\bar{\tau}$) according to the criterion employed and compare with Y.

▶ The principal stresses acting on an element of a shaft in torsion

$$\sigma_1 = \tau_{\theta z}, \quad \sigma_2 = -\tau_{\theta z}, \quad \sigma_3 = 0 \qquad (6.15)$$

1 ▷ Using the Mises criterion

$$\bar{\sigma} = Y = \sqrt{\frac{1}{2}[(2\tau_{\theta z})^2 + (-\tau_{\theta z})^2 + (-\tau_{\theta z})^2]} = \sqrt{3}\tau_{\theta z} \qquad (6.16)$$

thus an element of a shaft in torsion would be expected to begin yielding when

$$\therefore \quad \tau_{\theta z} = \frac{1}{\sqrt{3}} Y = 0.577\, Y \tag{6.17}$$

2 ▷ Using the maximum shear-stress criterion

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{Y}{2} \tag{6.18}$$

The equivalent shear stress is $\bar{\tau} = \tau_{\theta z}$

$$\therefore \quad \tau_{\theta z} = \frac{1}{2} Y = 0.500Y \tag{6.19}$$

→ As can be seen from (6.17) and (6.19), this discrepancy is about 15 percent. From the point of view of the designer trying to avoid yielding, it is more conservative to design on the basis of (6 19).

cf. Since the shear stress $\tau_{\theta z}$ is proportional to the radius r in an elastic shaft, it is clear that according to either criterion the elements on the outer surface of the shaft will reach the yield condition first.

6.10 Plastic Deformations

→ It is important to remember that in passing from elastic to plastic behavior there is no alteration in the conditions of equilibrium or in the conditions of geometric compatibility. The only change is in the stress-strain relation.

cf. The only non-vanishing strain component was $\gamma_{\theta z}$ remain valid whether the material is elastic or plastic. What will be different is the relation between $\gamma_{\theta z}$ and $\tau_{\theta z}$.

▶ Two ways to obtain the relation between $\gamma_{\theta z}$ and $\tau_{\theta z}$ in plastic region

i) Direct experiment in which the material is subjected to uniform pure shear

ii) To make use of tension test data and to predict the relation between $\gamma_{\theta z}$ and $\tau_{\theta z}$ in torsion by using one of the plastic flow rules → it is less exact, but simpler.

cf. In this chapter we shall confine our analytical treatment to the elastic-perfectly plastic material. (∴ Strain hardening does not exist.)

→ In plastic region, $\tau_{\theta z} = \tau_{r} = constant$.

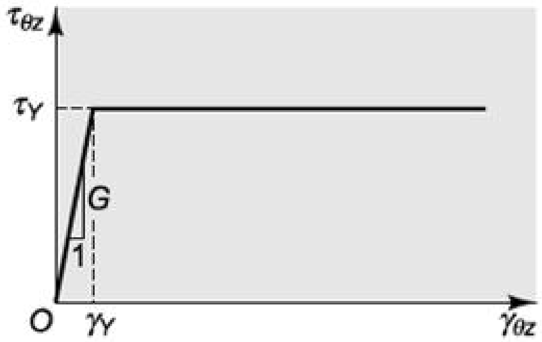


Fig. 6.19 *Shear-stress–shear-strain curve for elastic-perfectly plastic material*

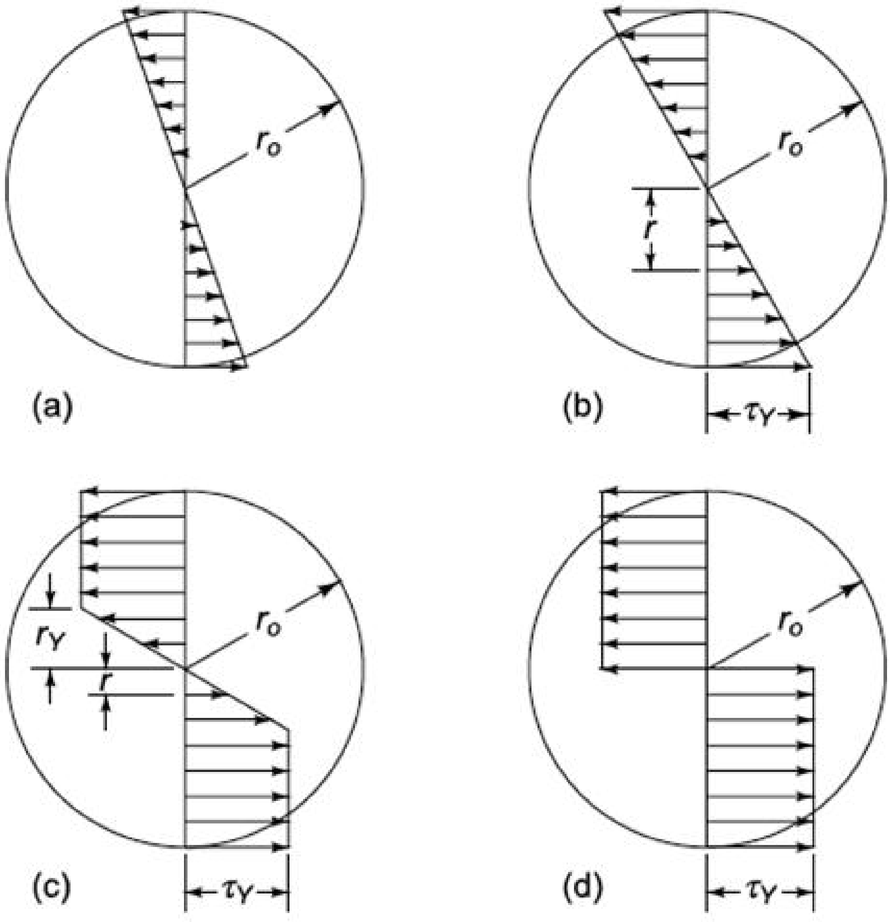


Fig. 6.20 *Shear-stress distribution in a twisted shaft of material having the stress-strain curve of Fig. 6.19. (a) Entirely elastic; (b) onset of yield; (c) partially plastic; (d) fully plastic*

▶ Analysis

1 ▷ To obtain quantitative representations of the sketches in Fig. 6.20, we proceed as follows. The elastic relations (6.8) and (6.9) apply until the yield-point situation in Fig. 6.20 (b) is reached.

$$\phi = \frac{d\phi}{dz}L = \frac{M_t L}{G I_z} \tag{6.8}$$

$$\tau_{\theta z} = Gr\left(\frac{M_t}{G I_z}\right) = \frac{M_t r}{I_z} \tag{6.9}$$

2 ▷ Let us call the twisting moment and twisting angle associated with this (b) stress distribution T_Y and ϕ_Y, respectively. Then from (6.8) and (6.9) we have

$$T_Y = \frac{\tau_Y I_z}{r_0} = \frac{\pi}{2}\tau_Y r_0^3 \qquad (6.20.\ a)$$

$$\therefore \phi_Y = T_Y \frac{L}{GI_z} = \left(\frac{\pi}{2}\tau_Y r_0^3\right)\frac{L}{G\pi r^4/2} = \frac{\tau_Y L}{G r_0} \quad (6.20.\ b)$$

cf. $\phi = \int_0^L \frac{M_t}{G\ I_z}\ dz = \frac{M_t\ L}{G\ I_z}$ (6.8)

$\tau_{\theta z} = \frac{M_t r}{I_z}$ (6.9)

3 ▷ Now as the shaft is twisted further the shear strain at the outer radius becomes larger than γ_Y. We still have the geometric relation (6.1) between shear strain and twist angle

$$\gamma_{\theta z} = r\frac{d\phi}{dz} = r\frac{\phi}{L} \quad (6.21)$$

4 ▷ At some intermediate radius r_Y the strain will be just equal to γ_Y. We can solve for r_Y when $\phi > \phi_Y$

$$r_Y = \frac{L\gamma_Y}{\phi} \quad (6.22)$$

5 ▷ Using the fact that $\tau_Y = G\gamma_Y$ and introducing the second of (6.20), we find

$$r_Y = \frac{L\gamma_Y}{\phi} = r_0\frac{\phi_Y}{\phi} \quad (6.23)$$

6 ▷ Next, we obtain a quantitative representation for the stress distribution $\tau_{\theta z}$ corresponding to the strain distribution $\gamma_{\theta z}$ of (6.21) by using the stress-strain relation of Fig. 6.19. In the inner elastic core $0 < r < r_Y$,

$$\tau_{\theta z} = G\gamma_{\theta z}$$

$$= G\frac{\phi}{L}r = G\frac{[L\tau_Y/G r_Y]r}{L} = \tau_Y\frac{r}{r_Y} \quad (6.24)$$

7 ▷ In the outer plastic region $r_Y < r < r_o$,

$$\tau_{\theta z} = \tau_Y \quad (6.25)$$

8 ▷ The stress distribution defined by (6.24) and (6.25) is sketched in Fig. 6.20 (c). Finally, we use the equilibrium requirement that the stress distribution of Fig. 6.20 (c) should be equivalent to the applied twisting moment M_t.

$$M_t = \int_A r\tau_{\theta z} dA$$

$$= \int_0^{r_Y} r\left(\tau_Y\frac{r}{r_Y}\right) 2\pi r\ dr + \int_{r_Y}^{r_o} r\tau_Y\ 2\pi r\ dr$$

$$= \frac{2\pi}{3}\tau_Y r_0^3\left(1 - \frac{1}{4}\frac{r_Y^3}{r_0^3}\right) \tag{6.26}$$

9 ▷ This result can be put into a more useful final form by introducing the yield-point twisting moment from (6.20) and the twisting angle from (6.23)

$$M_t = \frac{4}{3}T_Y\left(1 - \frac{1}{4}\frac{\phi_r^3}{\phi^3}\right) \tag{6.27}$$

cf. This nonlinear relationship is valid when $\phi > \phi_Y$

10 ▷ The limit or fully plastic twisting moment T_L (Fig. 6.21)

when $\phi \to \infty$

$T_L \to \frac{4}{3}T_Y$

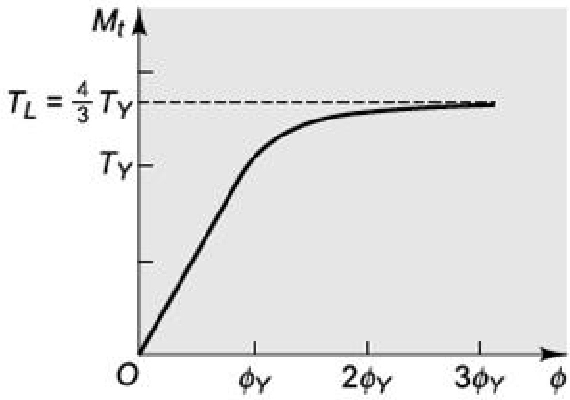


Fig. 6.21 *Twisting-moment–twisting-angle relationship for solid circular shaft made of material with stress-strain curve of Fig. 6.19*

6.11 Residual Stresses

▶ <u>From Fig. 6.22;</u>

If we assume that the material of the shaft unloads elastically after it has been strained plastically, then if at any stage the twisting moment were to be decreased, the twisting moment-twisting angle curve would trace out a straight line parallel to the original elastic relation of (6.8), as sketched in Fig. 6.22.

→ The justification for this lies in the fact that the geometric and equilibrium requirements for torsion remain unchanged while the stress-increment strain-increment relation is now elastic for the entire shaft.

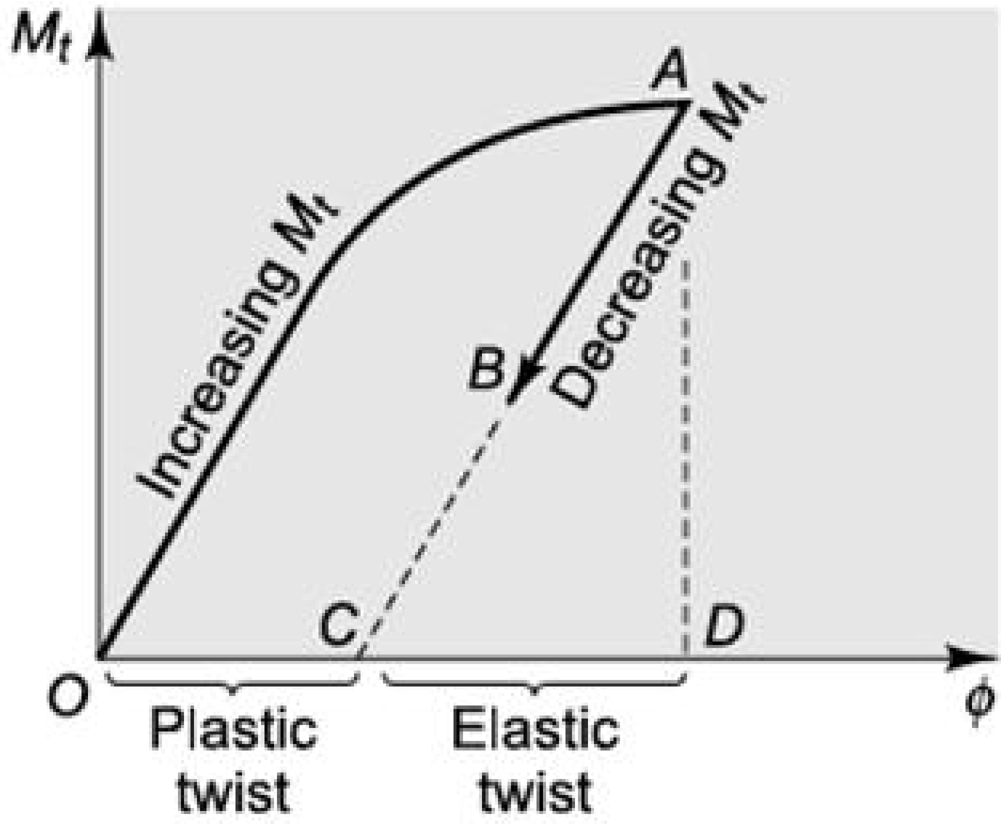


Fig. 6.22 *Unloading a plastically deformed shaft*

▶ <u>Residual stress</u>

Although there is no external load on the shaft in this condition, there is a distribution of self-balancing internal stresses in the shaft. These internal stresses which are "locked in" the material by the plastic deformation are called residual stresses.

→ The distribution of residual stresses can be found by using superposition.

▶ <u>Calculation of residual stress</u>

i) When parts (a) and (b) of Fig. 6.23 are superposed, we end up with no external twisting moment but with a distribution of residual stresses, as shown in Fig. 6.23 (c).

ii) The outer part of the shaft carries shearing stresses of the opposite sense to that imposed by the original application of the load, while the inner part carries stresses of the same sense as those originally imposed.

iii) Under some circumstances the reversed stresses obtained in this manner might be larger than the yield stress in the opposite direction. In this case simple linear superposition would not be applicable (see Prob. 6.41).

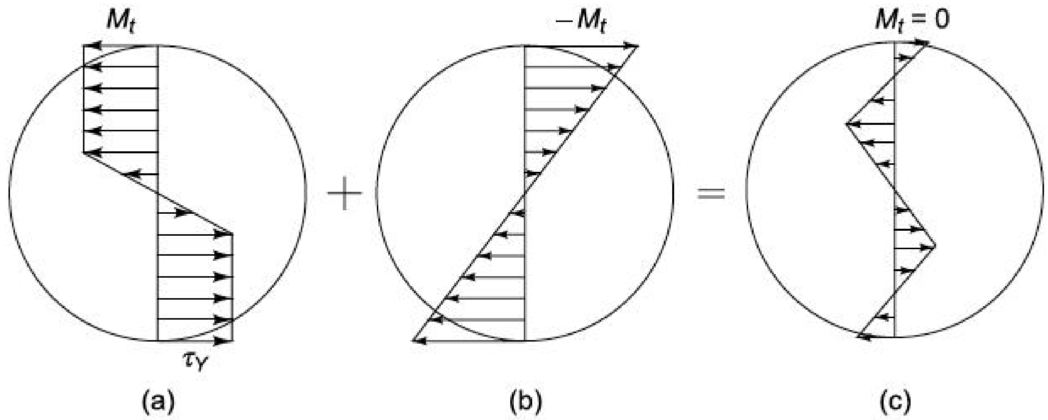


Fig. 6.23 *Residual shear-stress distribution in a shaft which has been twisted into the plastic region and unloaded*