Ch. 7
Stresses due to bending

7.1 Introduction

→ Our aim in this chapter is to determine the distributions of stresses which have the shear force V and the bending moment M_b as their resultant.

▶ <u>Beam</u>

When a slender member is subjected to transverse loading, we say it acts as a beam.

▶ <u>Pure bending</u>

When there is no shear force, and a constant bending moment is transmitted, we say it is a state of pure bending $\left(\because -V = \frac{dM}{dx}\right)$

cf. Our method of approach will be similar to that followed in the investigation of torsion in Chap. 6, and to a certain extent our results will be similar.

cf. In this chapter we shall also obtain an exact solution within the theory of elasticity of the special case of a beam subjected to pure bending. For more general cases we shall obtain approximate distributions of stresses on the basis of equilibrium considerations.

7.2 Geometry of deformation of a symmetrical beam subjected to pure bending

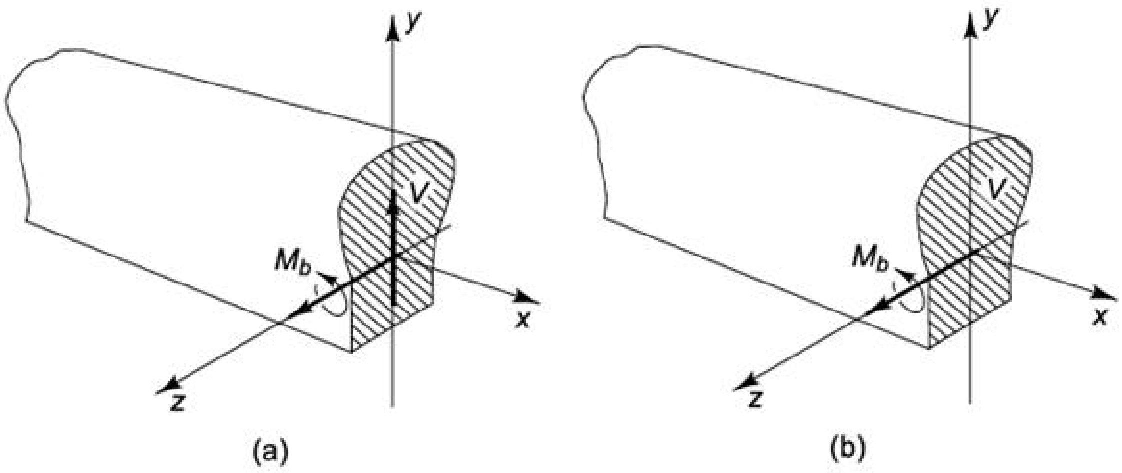


Fig. 7.2 *Symmetrical beam loaded in its plane of symmetry. (a) In general, both shear force and bending moment are transmitted; (b) in pure bending there is no shear force, and a constant bending moment is transmitted*

▶ Assumptions (See Fig. 7.2)

i) We consider an originally straight beam which is uniform along its length, whose cross sections is symmetrical.

ii) Its material properties are constant along the length of the beam.

iii) It is subjected to pure bending.

∴ The deformation pattern can be fixed by symmetry arguments alone.

cf. The result derived from these assumptions is valid to any types of beams whose materials are linear or nonlinear, elastic or plastic.

▶ Curvature

→ The curvature of a plane curve is defined as the rate of the slope angle change of the curve with respect to distance along the curve.

∴ For $\Delta s \to 0$ (see fig. 7.3)

$$\kappa = \frac{d\phi}{ds} = lim_{\Delta s \to 0} \frac{\Delta \phi}{\Delta s} = lim_{\Delta s \to 0} \frac{1}{\overline{O'B}} = \frac{1}{\rho} \qquad (7.1)$$

where $\rho = OB$ is *the radius of curvature* at point B.

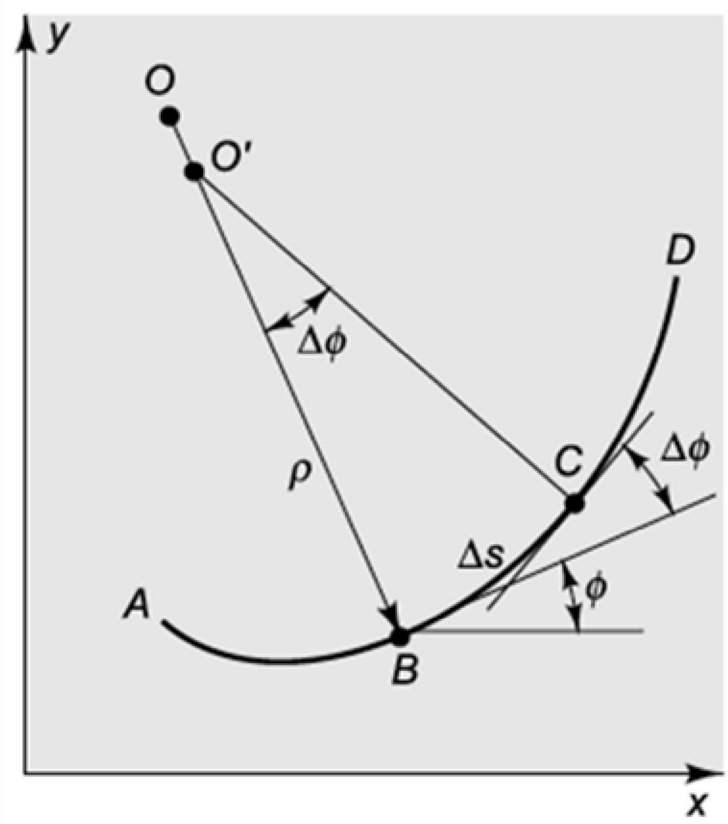


Fig. 7.3 *The line AD has curvature dϕ/ds = 1/ρ at point B, where ρ = OB is the radius if curvature at point B*

▶ Deformation behavior under pure bending

i) The surface A_1D_1, B_1E_1, C_1F_1 must be plane surfaces perpendicular to the plane of symmetry.

∴ In pure bending in a plane of symmetry plane cross sections remain plane.

ii) The fact that each element deforms identically means that the initially parallel plane sections now must have a common intersection, as illustrated by point O in Fig. 7.4b, and that the beam bends into the arc of a circle centered on this intersection.

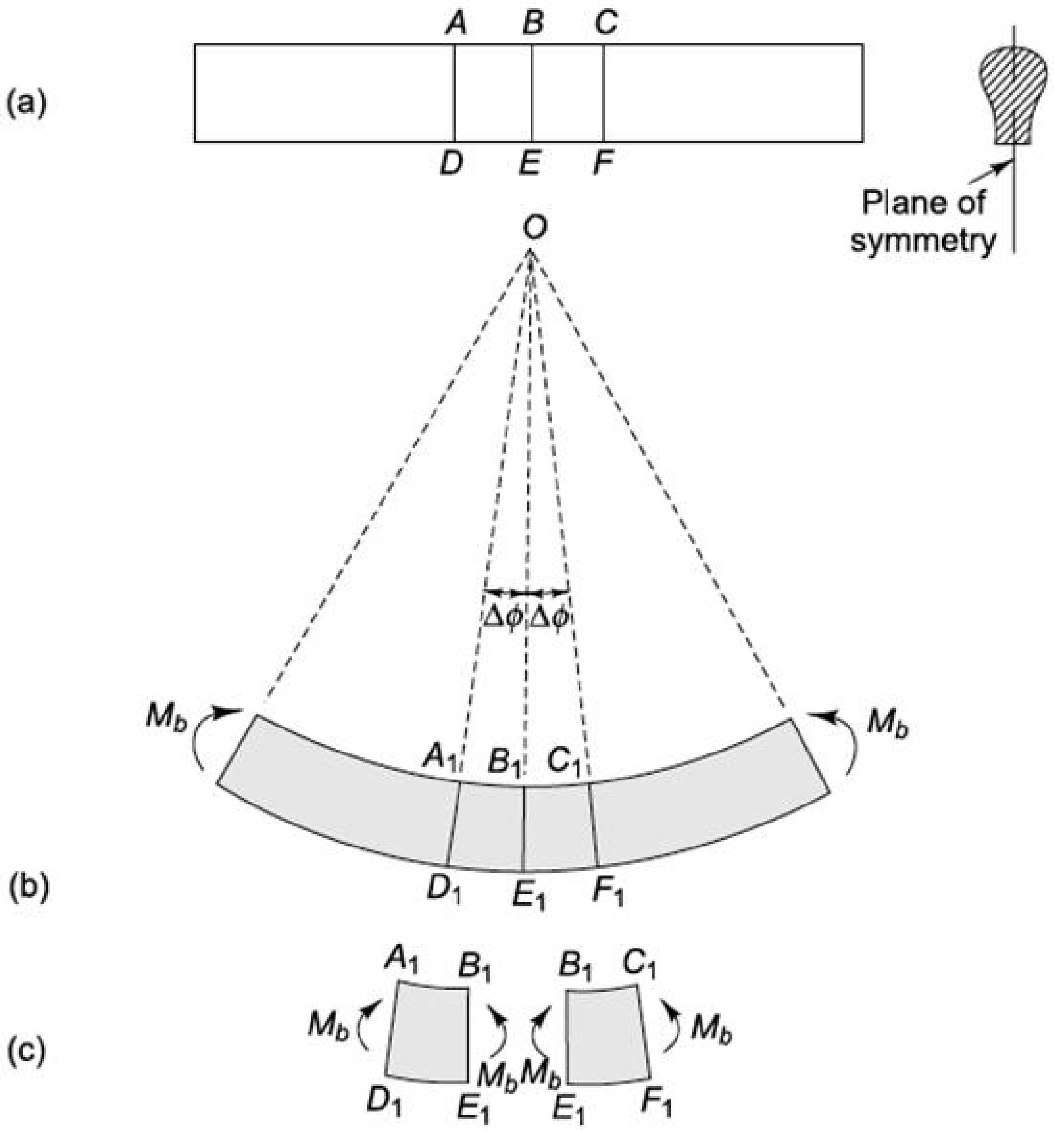


Fig. 7.4 *Overall deformation of a symmetrical beam subjected to pure bending in its plane of symmetry*

▶ Neutral Axis

Neutral axis is one line in the plane of symmetry which has not changed in length.

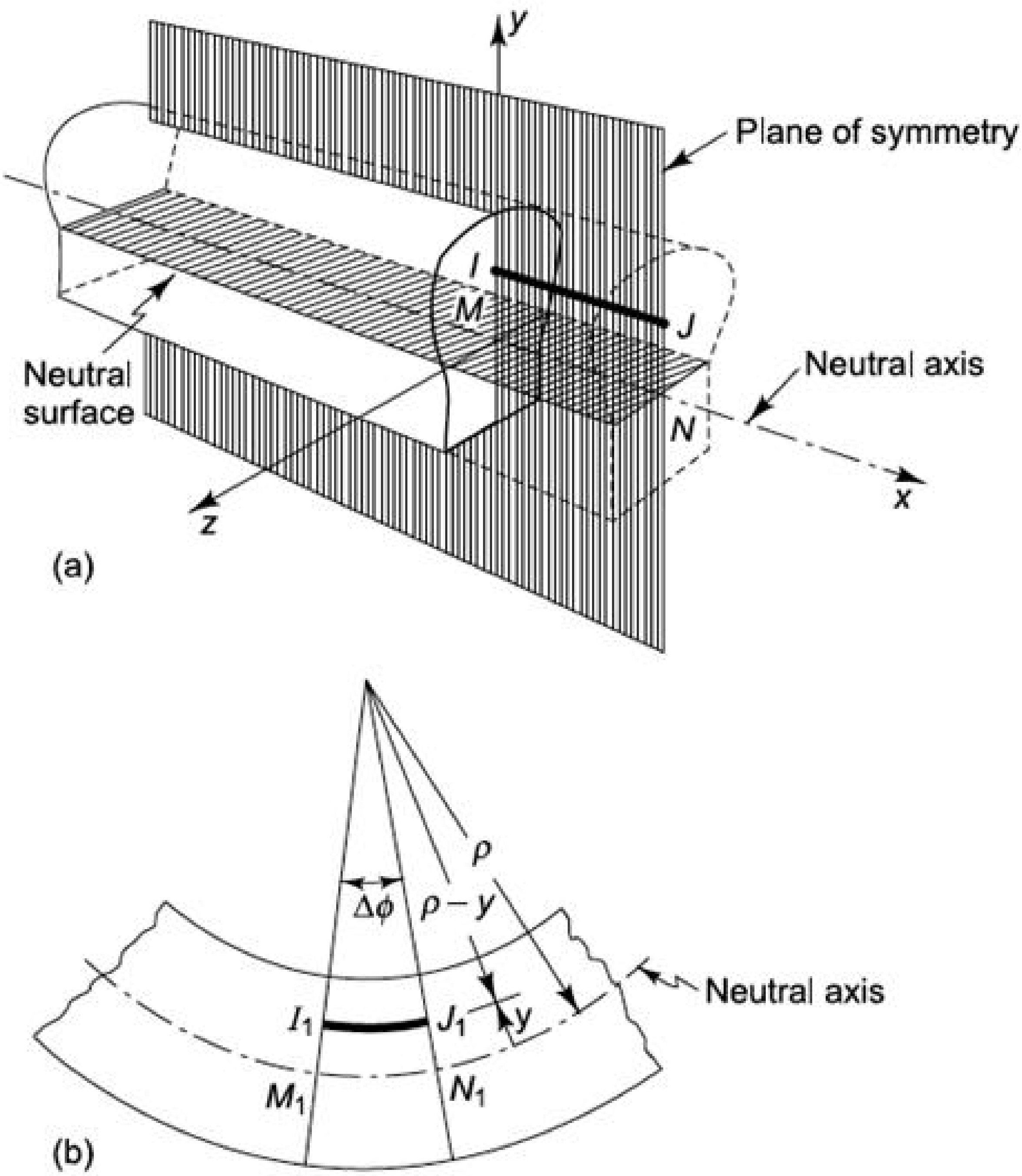


Fig. 7.5 *(a) Undeformed beam; (b) trace of deformed beam in the plane of symmetry*

▶ Distribution of strain (See Fig. 7.5)

$$\epsilon_x = \frac{I_1J_1 - IJ}{IJ} = \frac{I_1J_1 - M_1N_1}{M_1N_1} \ (\because IJ = M_1N_1) \tag{7.2}$$

$$\text{where } M_1N_1 = \rho\Delta\phi, \qquad I_1J_1 = (\rho - y)\Delta\phi \tag{7.3}$$

$$\therefore \quad \epsilon_x = -\frac{y}{\rho} = -\frac{d\phi}{ds}y = -\kappa y \tag{7.4}$$

▷ Remarks on Eq. (7.4)

i) Longitudinal strain of the beam ϵ_x is proportional to curvature κ (= bending deformation rate) and varies linearly with the distance from the neutral surface y.

ii) The derivation of (7.4) applies strictly only to the plane of symmetry, but we shall assume that (7.4) describes the longitudinal strain at all points in the cross section of the beam.

iii) This equation is irrelevant to the stress-strain relation of material.

▷ Other strain components of strain

$$\gamma_{xy} = \gamma_{xz} = 0 \tag{7.5}$$

cf. We can make no quantitative statements about the strains ϵ_x, ϵ_y and γ_{yz} beyond the remark that they must be symmetrical with respect to the *xy* plane.

7.3 Stresses obtained from stress-strain relations

→ In this section we shall restrict ourselves to beams made of linear isotropic elastic material, i.e., to materials which follow Hooke's law.

▶ Strain components

$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] = -\frac{y}{\rho}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = 0 \qquad (7.6)$$

$$\gamma_{xz} = \frac{\tau_{xz}}{G} = 0$$

→ In pure bending, $\tau_{xy} = \tau_{xz} = 0$

7.4 Equilibrium requirements

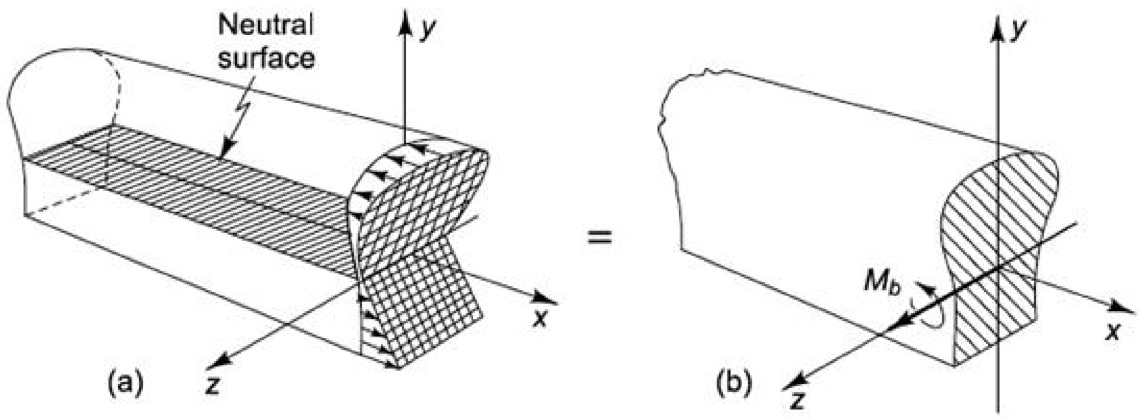


Fig. 7.6 *The resultant of the stress distribution in pure bending must be the bending moment M_b*

▶ Considering equilibrium (Fig. 7.6)

$$\sum F_x = \int_A \sigma_x \, dA = 0$$

$$\sum M_y = \int_A z\sigma_x \, dA = 0 \qquad (7.7)$$

$$\sum M_z = -\int_A y\sigma_x \, dA = M_b$$

cf. We make the fundamental assumption that the deformation of the cross section is sufficiently small so that we can use the undeformed coordinates to locate points in the deformed cross section.

7.5 Stress and deformation in symmetrical elastic beams subjected to pure bending

▶ Aim of this section

→ We shall find the solution satisfying strain requirements, Eqs. (7.6) and (7.7).

▶ Assumption

→ Considering that there is no normal or shear stress on the external surface of Δx and that the beam is slender, we can assume as follow.

$$\sigma_y = \sigma_z = \tau_{yz} = 0 \qquad (7.8)$$

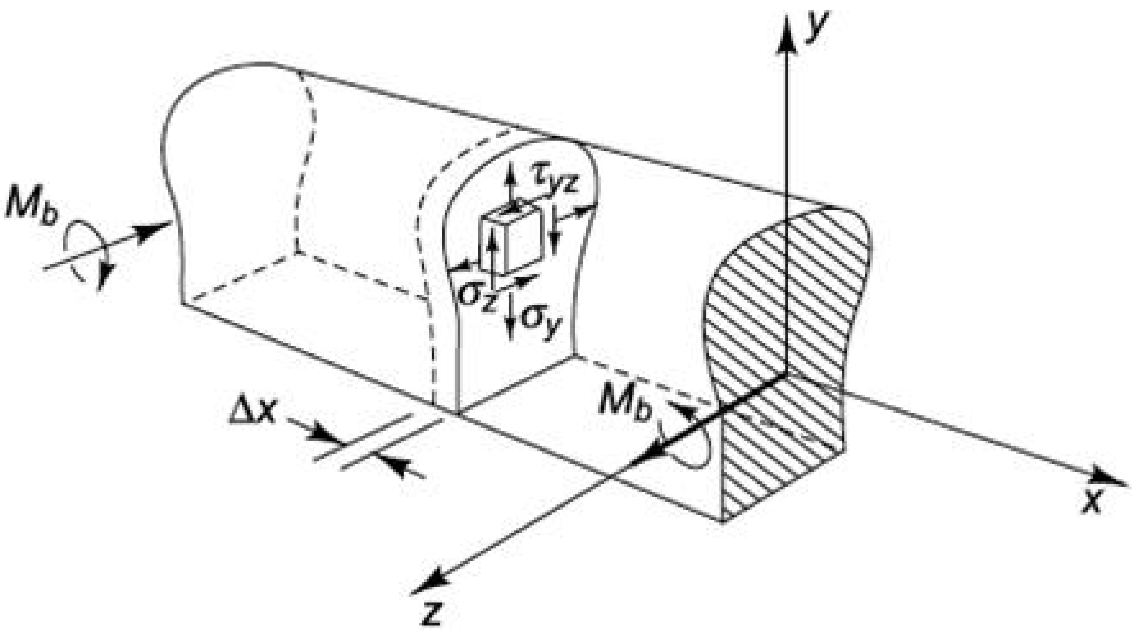


Fig. 7.8 *The transverse stresses σ_y, σ_z, and τ_{yz} are assumed to be zero*

▶ Analysis of stresses

Given above assumption, in the beam under pure bending, which follows Hooke's low, the only stress component is

$$\sigma_x = -E\frac{y}{\rho} = -\kappa E y = -E\frac{d\phi}{ds}y \tag{7.9}$$

▷ Equilibrium

i) $\sum F_x = \int_A \sigma_x \, dA = -\frac{E}{\rho}\int_A y \, dA = 0$ (7.10)

→ ∴ Since $\int_A \sigma_x \, dA = 0$, the neutral surface must pass through the centroid of the cross-sectional area.

cf. In case of a composite or nonlinear beam, it's possible to apply $\sum F_x = 0$ but the neutral surface doesn't pass through the centroid.

ii) $\sum M_y = \int_A z\sigma_x \, dA = -\frac{E}{\rho}\int_A yz \, dA = 0$ (7.11)

→ As the cross section is symmetrical with respect to *xy* plane, $\int_A yz \, dA = 0$

iii) $\sum M_z = -\int_A y\sigma_x \, dA = \frac{E}{\rho}\int_A y^2 \, dA = M_b$ (7.12)

where $I_z = \int_A y^2 \, dA$ (7.13)

→ ∴ Eq. (7.12) is

$$\kappa = \frac{1}{\rho} = \frac{d\phi}{ds} = \frac{M_b}{EI_{zz}} \tag{7.14}$$

cf. See the similarity with $\theta = \frac{d\phi}{dz} = \frac{M_t}{GI_z}$ (6.7)

$$\therefore \frac{M_b}{EI_z} = \kappa = -\frac{\epsilon_x}{y} \quad \rightarrow \quad \epsilon_x = -\frac{M_b y}{EI_z} \tag{7.15}$$

$$\therefore \sigma_x = E\epsilon_x = -\frac{M_b y}{I_z} \tag{7.16}$$

cf. See the similarity with $\tau_{\theta z} = \frac{M_t r}{I_z}$ (6.9)

▶ Analysis of strains

$$\epsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)] = \frac{1}{E}\left[0 - \nu\left(0 - \frac{M_b y}{I_z}\right)\right]$$
$$= \frac{\nu M_b y}{E I_z} = -\nu\epsilon_x$$
$$\epsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] = \frac{1}{E}\left[0 - \nu\left(-\frac{M_b y}{I_z} + 0\right)\right] \qquad (7.17)$$
$$= \frac{\nu M_b y}{E I_z} = -\nu\epsilon_x$$
$$\gamma_{yz} = \frac{\tau_{yz}}{G} = 0$$

▷ Remarks on lateral strain

i) Since the axial normal strain is compressive at the top of the beam and tensile at the bottom, the top of the cross section expands while the bottom of the cross section contracts.

ii) The trace of the neutral surface on the cross section has become an arc with curvature $-\nu(1/\rho)$.

∴ The deformed neutral surface is a surface of double curvature ($1/\rho$ and $-\nu/\rho$). A further result of the anticlastic curvature is that the neutral axis is the only line in the deformed neutral surface whose curvature is in a plane parallel to the original plane of symmetry of the beam

▷ This transverse curvature of the beam is called anticlastic curvature.

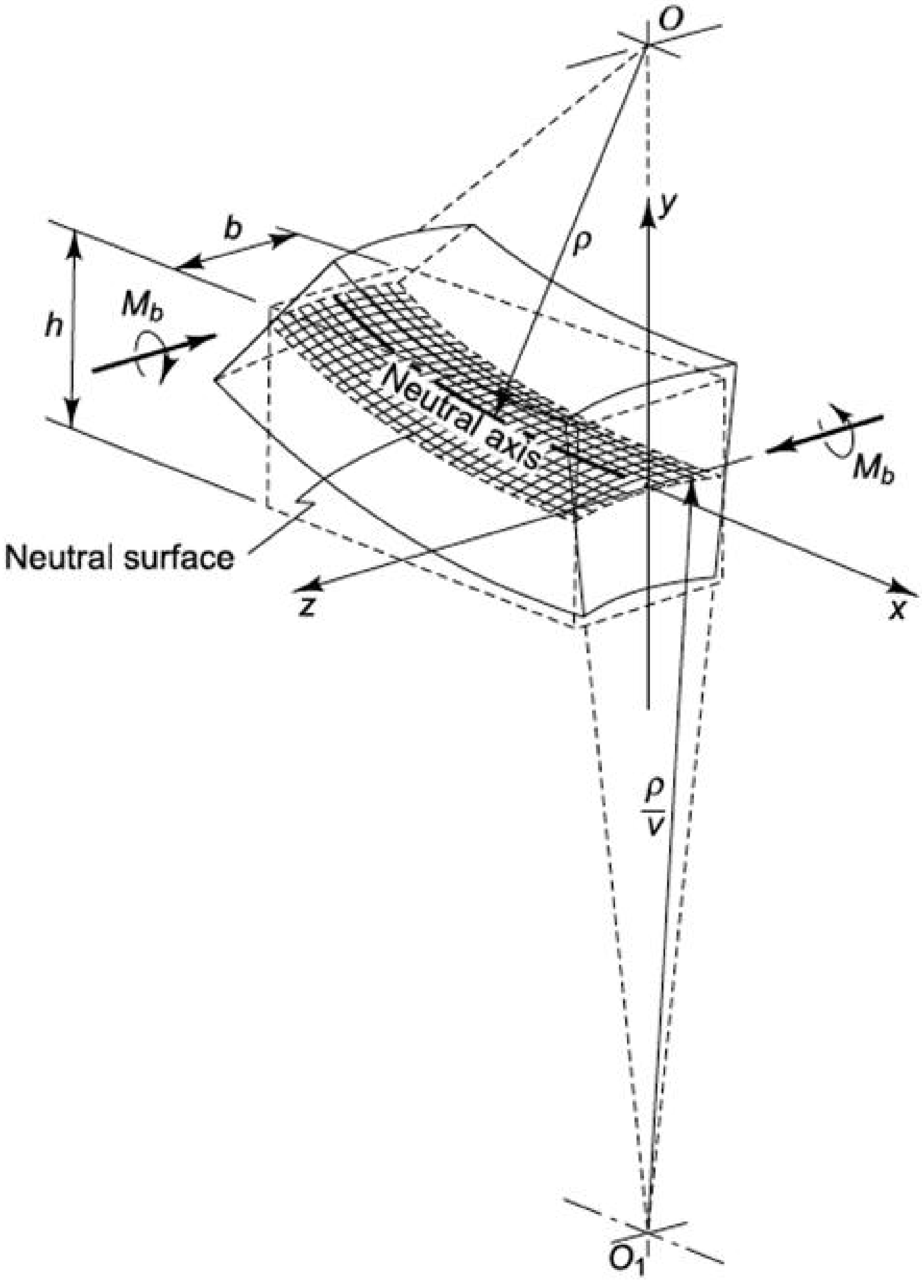


Fig. 7.9 *Deformed shape of an originally rectangular beam subjected to pure bending in a plane of symmetry*

▷ Validity of the assumption

i) The strains (7.5), (7.15), and (7.17) are geometrically compatible; the stresses (7.6), (7.8), and (7.16) satisfy the differential equations of equilibrium; and at every point the stresses and strains satisfy Hooke's law.

ii) Our solution is still very accurate in the central portion of the beam in accord with St. Venant's principle and only becomes appreciably in error near the ends. (The length of these transition regions at the ends is of the order of the depth of the beam cross section.)

▷ The analysis of the pure bending of curved beams is reasonably accurate for the non-uniform bending of curved beams.

▶ Section modulus, S

$$S = \frac{M_{max}}{\sigma_{allow}} \qquad \text{(a)}$$

or $S_1 = \dfrac{I_z}{y_{max}}$, $S_2 = \dfrac{I_z}{y_{min}}$

cf. $\sigma_{max} = -\dfrac{M_b}{S_2}$, $\sigma_{min} = -\dfrac{M_b}{S_1}$

→ It's convenience to define a required section modulus when we select the beam.

i) The cross section of the beam must be used when the S is larger than the value that obtained from Eq. (a).

ii) It is desirable to select the cross section that has satisfactory section modulus and the smallest cross sectional area.

iii) On the rectangular cross section, the greater height h is, the larger S is.

iv) The square cross section beam is more efficient than the circular cross section beam with respect to the same area.

v) To design the beam economically, material should be placed in location that is away from the neutral axis as possible. (But, in an excessive case, there is a danger of buckling.)

▶Example 7.1

A steel beam 25 mm wide and 75 mm deep is pinned to supports at points A and B, as shown in Fig. 7.11a, where the support B is on rollers and free to move horizontally. When the ends of the beam are loaded with 5kN loads, find the maximum bending stress at the mid-span of the beam and also the angle $\Delta\phi_0$ subtended by the cross sections at A and B in the deformed beam.

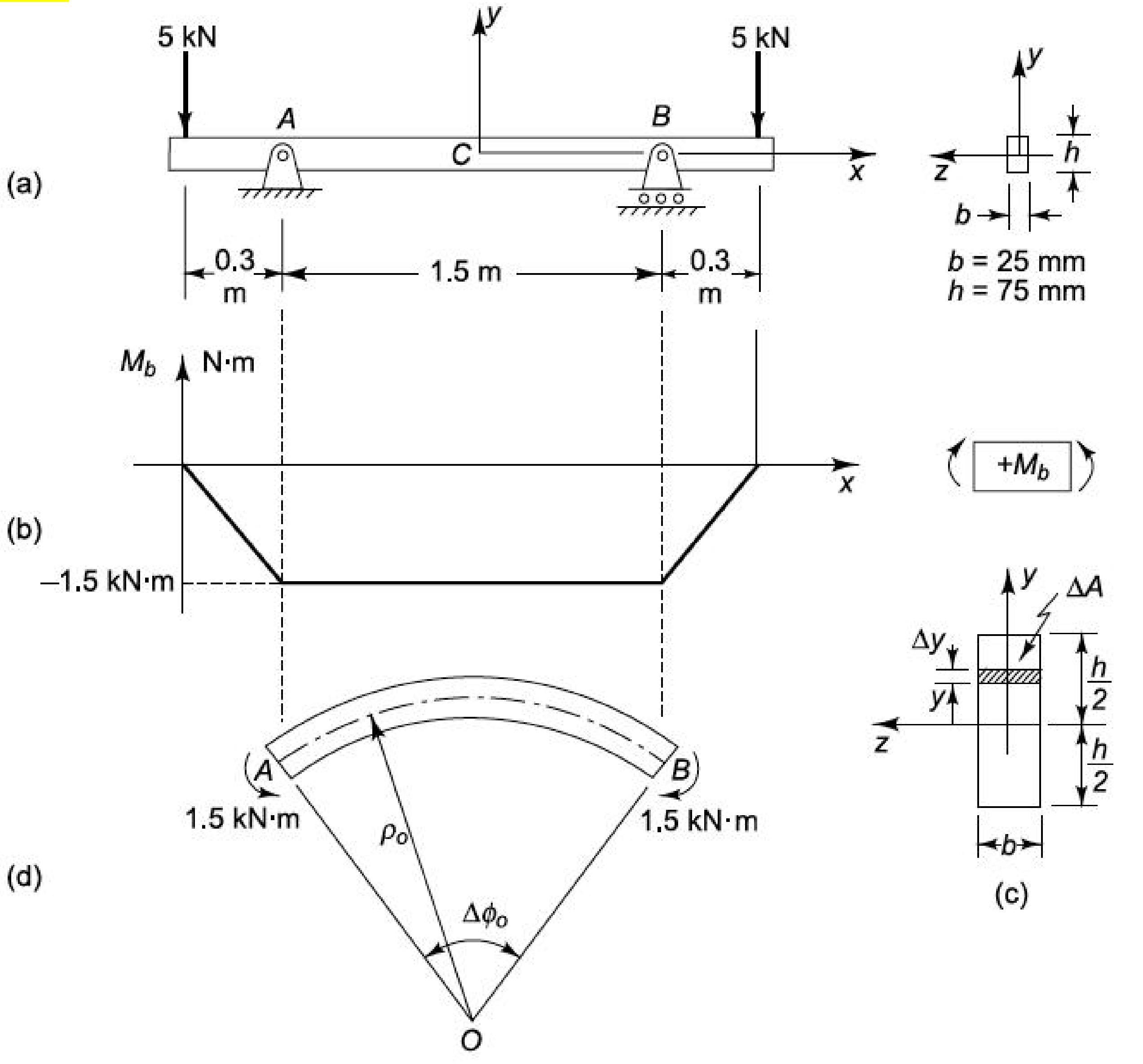


Fig. 7.11 *Example 7.1*

Sol)

From Fig. (c)

$$I_z = \int_{-h/2}^{h/2} y^2 b\, dy = \frac{bh^3}{12} = 8.789 \times 10^5 \text{ mm}^4$$

$$\therefore \quad \sigma_{max} = -\frac{M_b(h/2)}{I} = -\frac{(-1500)(37.5\times10^{-3})}{8.789(10)^{-7}} = 64.0 \text{ MN/m}^2$$

$$\kappa = \frac{d\phi}{ds} = \frac{M_b}{EI_z}$$

$$\therefore \quad \phi_B - \phi_A = \int_{-L/2}^{L/2} \frac{d\phi}{ds}\, ds = \frac{M_b L}{EI_z} = \frac{-1500(1.5)}{E(8.789\times10^{-7})}$$

Here, we let $E = 205$ GPa

$$\Delta\phi_0 = \frac{-1500(1.5)}{(205\times10^9)(8.789\times10^{-7})} = -0.0125 \text{ rad} = 0.7155°$$

Now, $\rho_0 = \frac{1}{\kappa} = \frac{EI_z}{M_b} = -120.12$ m

▶ Example 7.2

Find the maximum tensile and compressive bending stresses in the symmetrical T beam of Fig. 7.12 (a) under the action of a constant bending moment M_b.

Sol)

$$\bar{y} = \frac{\sum_i \bar{y}_i A_i}{\sum_i A_i} = \frac{(3/2)h(2bh)+(h/4)(3bh)}{2bh+3bh} = \frac{3}{4}h \qquad \text{(a)}$$

$$(I_{zz})_1 = \frac{b(2h)^3}{12} + 2bh\left(\frac{3}{4}h\right)^2 = \frac{43}{24}bh^3 \qquad \text{(b)}$$

$$(I_{zz})_2 = \frac{6b(h/2)^3}{12} + 3bh\left(\frac{h}{2}\right)^2 = \frac{13}{16}bh^3 \qquad \text{(c)}$$

Then, for the entire cross section

$$I_{zz} = (I_{zz})_1 + (I_{zz})_2 = 125/48\, bh^3 \qquad \text{(d)}$$

$$\therefore \begin{cases} \sigma_{max} = -\dfrac{M_b(-3/4\, h)}{(125/48)bh^3} = \dfrac{36}{125}\dfrac{M_b}{bh^2} \\ \sigma_{min} = -\dfrac{M_b(7/4\, h)}{(125/48)bh^3} = -\dfrac{84}{125}\dfrac{M_b}{bh^2} \end{cases}$$

cf. $|\sigma_{min}| \approx |2.3 \cdot \sigma_{max}|$

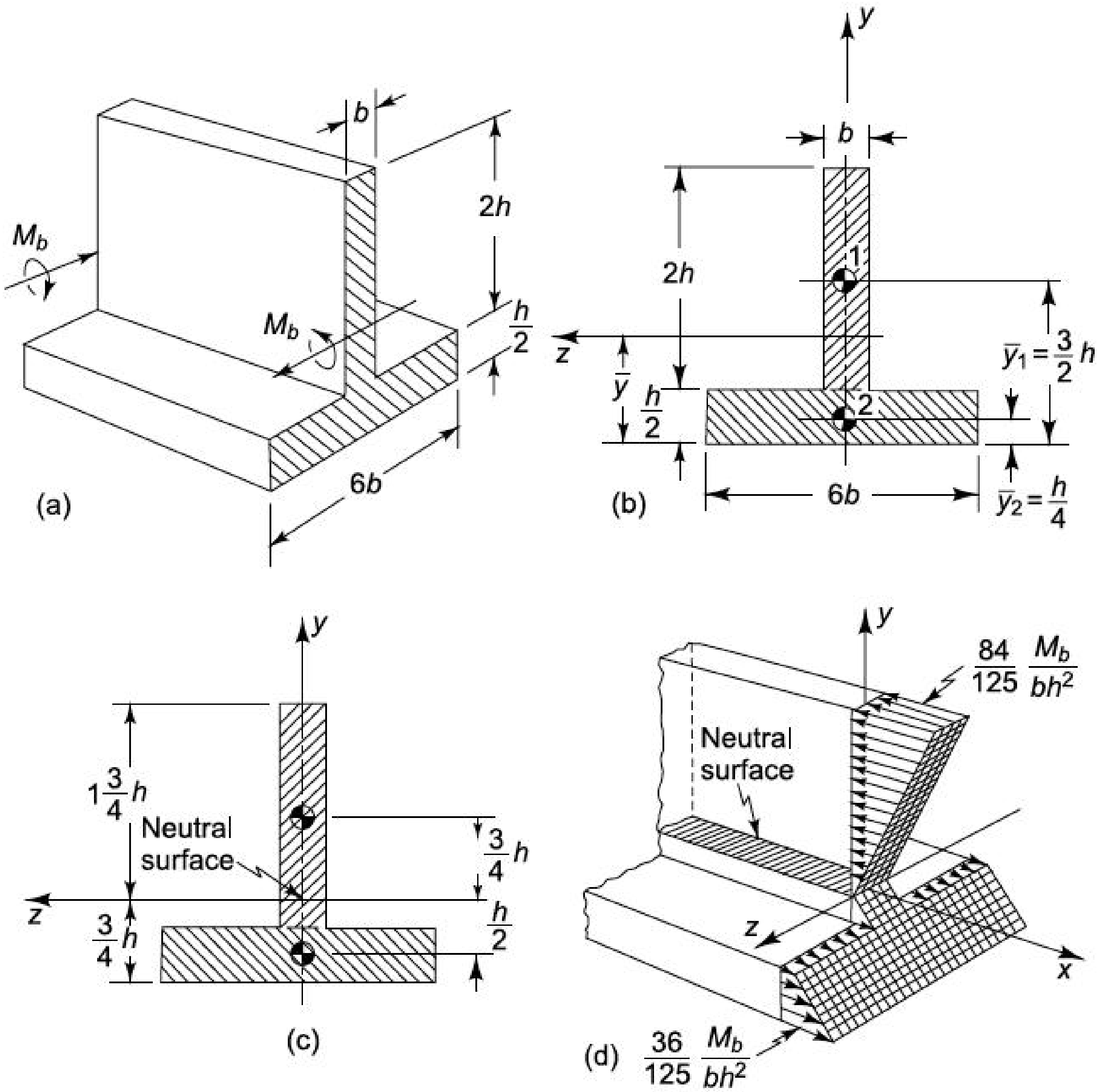


Fig. 7.12 *Example 7.2*

7.6 Stress in symmetrical elastic beam transmitting both shear force and bending moment

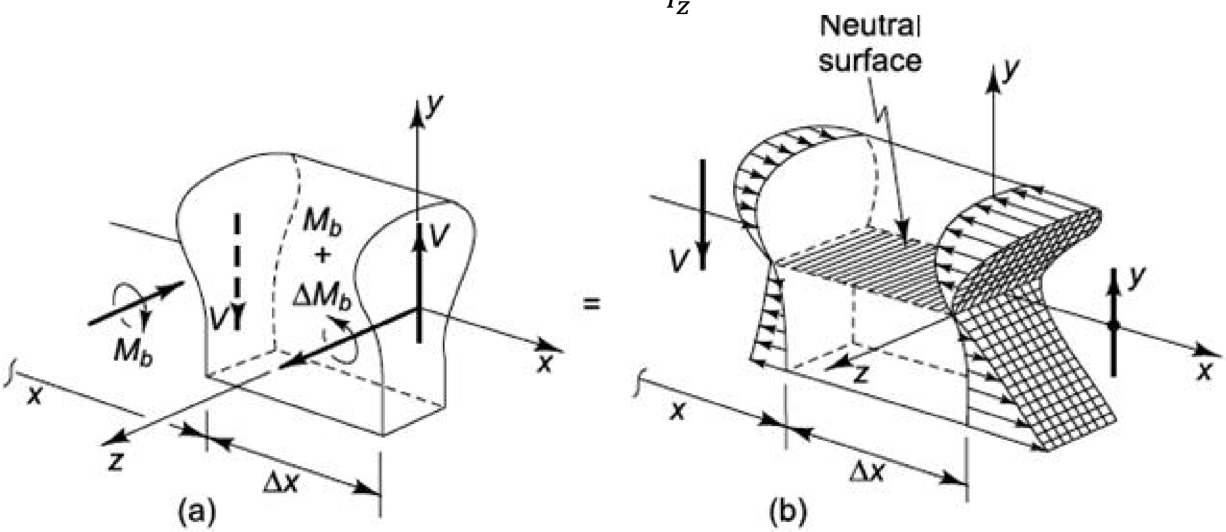
→ It is more difficult to obtain an exact solution to this problem since the presence of the shear force means that the bending moment varies along the beam and hence many of the symmetry arguments of Sec 7.2 are no longer applicable. Therefore, in this section we shall describe what is frequently referred to as the engineering theory of the stresses in beam.

▶ Engineering theory of beams

▷ Assumption

→ The bending-stress distribution (7.16) is valid even when the bending moment varies along the beam, i.e., when a shear force is present.

$$\sigma_x = E\epsilon_x = -\frac{M_b y}{I_z} \qquad (7.16)$$



▷ Analysis

i) Fig. (a)

→ We take the case where there is no external transverse load acting on the element so that the transverse shear force V is independent of x.

→ We assume the shear force is constant through the beam to simplify the analysis.

ii) Fig. (b)

→ Due to the increase ΔM_b, in the bending moment over the length Δx, the bending stresses acting on the positive x face of the beam element will be somewhat larger than those on the negative x face.

→ We assume that the bending stresses are given by (7.16).

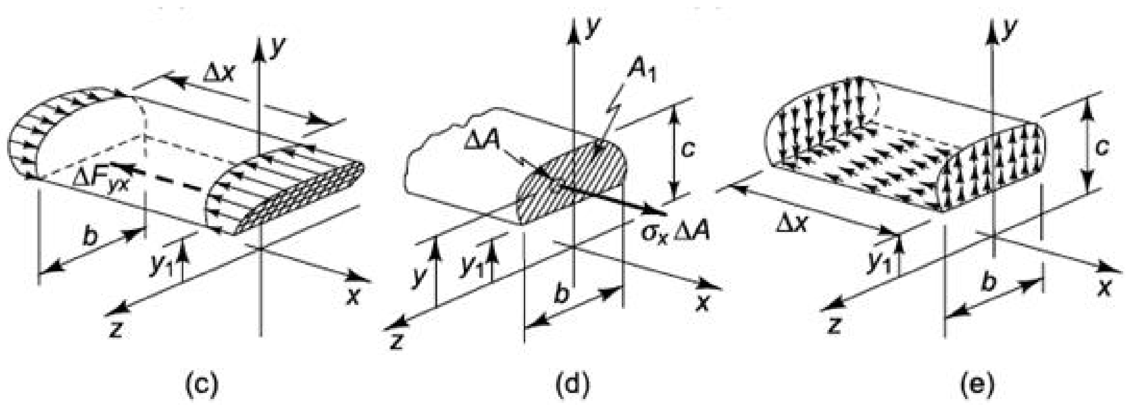


Fig. 7.13 *Calculation of shear stress τ_{xy} in a symmetrical beam from equilibrium of a segment of the beam*

iii) Fig. (c), Fig. (d)

→ We next consider the equilibrium of the segment of the beam shown in Fig. 7.13 (c), which we obtain by isolating that part of the beam element of Fig. 7.13 (b) above the plane defined by $y = y_1$. Due to the unbalance of bending stresses on the ends of this segment, there must be a force ΔF_{yx} acting on the negative y face to maintain force balance in the x direction.

$$\sum F_x = \left[\int_{A_1} \sigma_x \, dA\right]_{x+\Delta x} - \Delta F_{yx} - \left[\int_{A_1} \sigma_x \, dA\right]_x = 0 \qquad (7.18)$$

$$\rightarrow \Delta F_{yx} = -\int_{A_1} \frac{(M_b+\Delta M_b)y}{I_{zz}} \, dA + \int_{A_1} \frac{M_b y}{I_{zz}} \, dA$$

$$= -\frac{\Delta M_b}{I_{zz}} \int_{A1} y \, dA \qquad (7.19)$$

$$\therefore \frac{dF_{yx}}{dx} = \lim_{\Delta x \to 0} \frac{\Delta F_{yx}}{\Delta x} = -\frac{dM_b}{dx}\frac{1}{I_{zz}}\int_{A1} y\, dA \qquad (7.20)$$

where

$$\frac{dF_{yx}}{dx} = q_{yx} \qquad (7$$

$$\frac{dM_b}{dx} = -V \qquad ($$

$$\int_{A_1} y\, dA = Q \qquad (7$$

$$\therefore \quad q_{xy} = \frac{VQ}{I_{zz}} \qquad (7.23)$$

→ The quantity q_{yx}, which is the total longitudinal shear force transmitted across the plane defined by $y = y_1$ per unit length along the beam, is called the shear flow. The shear flow q_{yx} obviously is the resultant of a shear stress τ_{yx} distributed across the width b of the beam. If we make the assumption that the shear stress is uniform across the beam, we can estimate the shear stress τ_{yx} at $y = y_1$ to be

$$\tau_{xy} = \frac{q_{yx}}{b} = \frac{VQ}{bI_{zz}} = \tau_{yx} \qquad (7.25)$$

cf.

i) The foregoing theory can be proved to be internally consistent in that it can be shown that for a beam of arbitrary cross section the resultant of the stress distribution (7.25) over the cross section is in fact the shear force V.

ii) The shear stress distribution at the bottom and the top is zero.

▶ Shear stress distribution in rectangular beam

→ The equilibrium equations (4.13) apply.

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} = 0 \end{cases} \qquad (4.13)$$

→ If we deal with a case where the shear force does not vary with x, the shear stress also will be independent of x, and the second of (4.13) is automatically satisfied since the normal stress σ_y has been assumed to be zero.

∴ 1st equation becomes,

$$-\frac{\partial \tau_{xy}}{\partial y} = \frac{\partial \sigma_x}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{M_b y}{I_{zz}}\right) = \frac{V}{I_{zz}} y \tag{7.26}$$

$$\therefore \quad -\int_{y_1}^{h/2} \frac{\partial \tau_{xy}}{\partial y}\, dy = \frac{V}{I_{zz}} \int_{y_1}^{h/2} y\, dy = \frac{V}{I_{zz}} \left[\frac{y^2}{2}\right]_{y_1}^{h/2}$$

$$\therefore \ -\left(\tau_{xy}\right)_{h/2} + \left(\tau_{xy}\right)_{y_1} = \frac{V}{2I_{zz}} \left[\left(\frac{h}{2}\right)^2 - y_1^2\right] \tag{7.27}$$

→ The shear stress is a maximum at the neutral surface and falls off parabolically, as illustrated in Fig. 7.15.

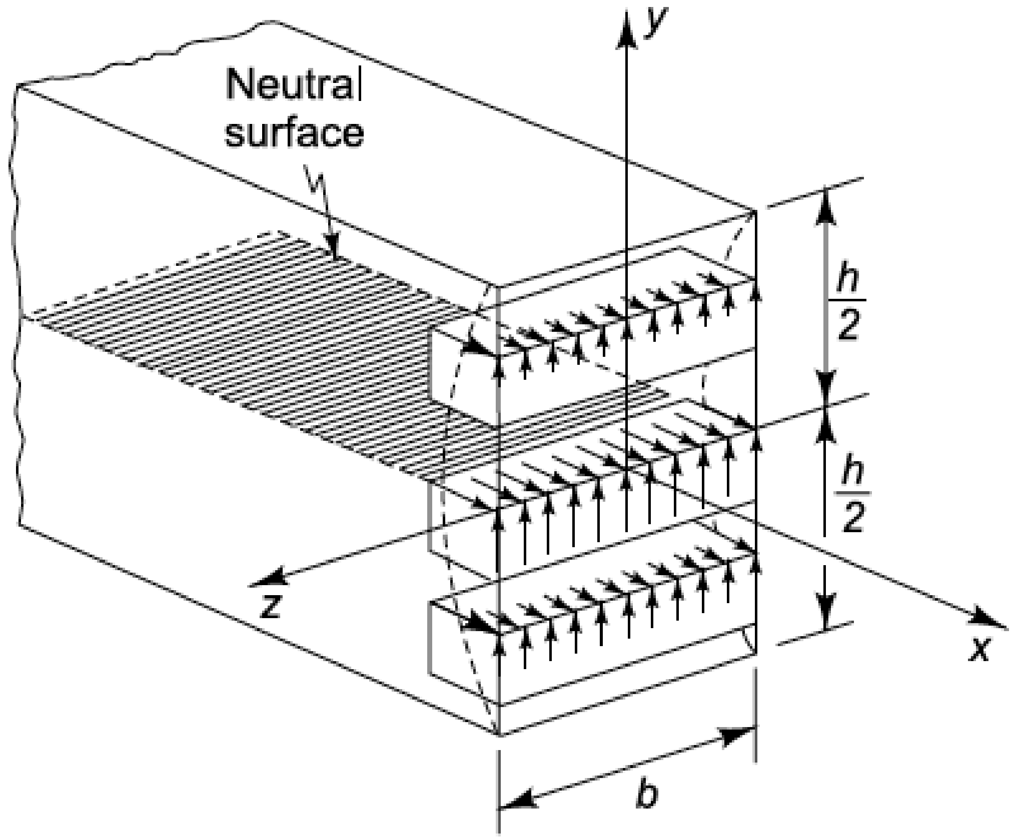


Fig. 7.15 *Illustration of parabolic distribution of shear stress τ_{xy} in a rectangular beam*

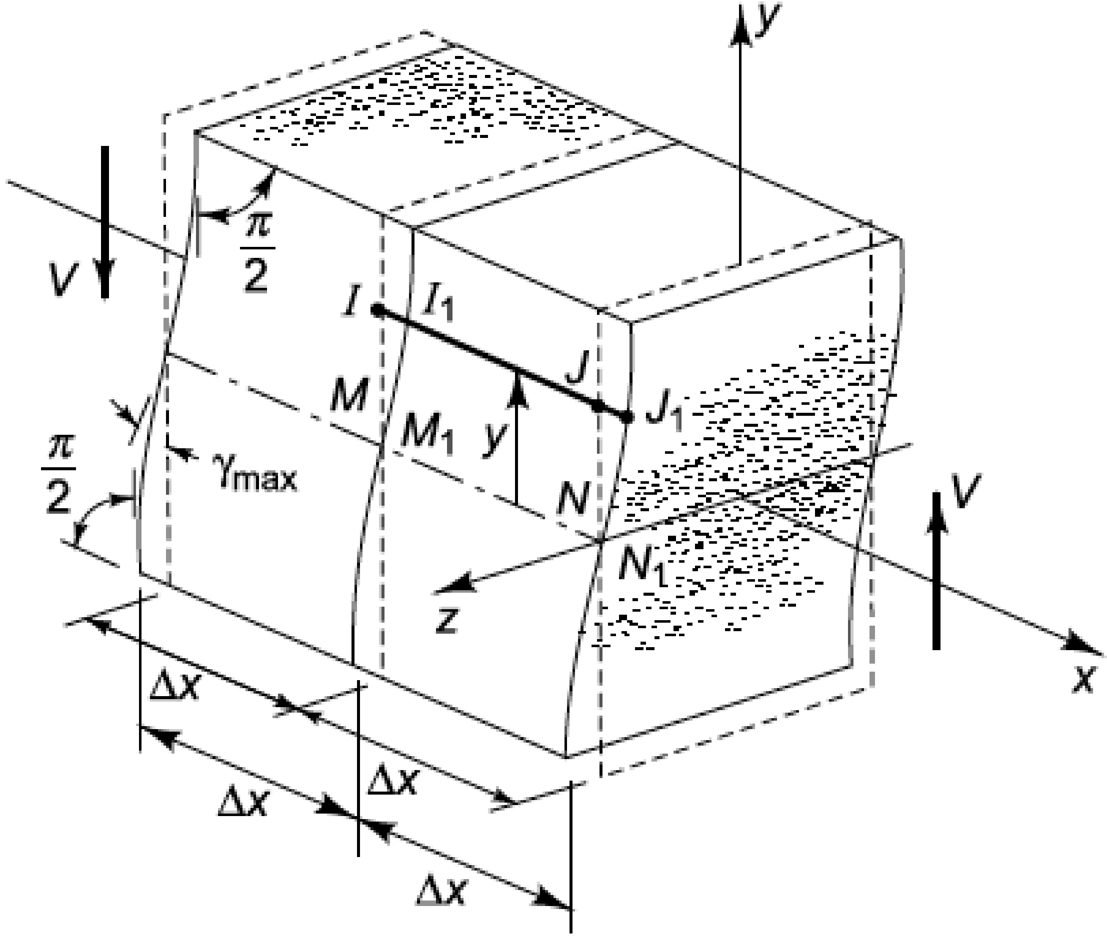


Fig. 7.16 *Distortion of rectangular beam due to shear force which is constant along the length of the beam*

▷ The relation between shear stress and shear strain in a rectangular beam

i) By substituting the stress distribution (7.27) into Hooke's law (5.2), we find that the shear strain γ_{xy}, also varies parabolically across the section.

ii) If the shear force is constant along the length of the beam, any longitudinal line IJ does not change its length as it deforms into the position I_1J_1. From this we would suppose that the presence of a constant shear force would have little effect on the bending-stress distribution (7.16).

cf. The exact solution from the theory of elasticity shows that (7.14) and (7.16) are still correct when there is a constant shear force. This means that the expression (7.23) for the shear flow is also exact for the case of constant shear force.

cf. Both (7.14) and (7.16) are in error when the shear force varies along the beam, but the magnitude of error is small for long, slender beams and, consequently, (7.23) represents a good estimate even in the presence of a varying shear force.

▶ Comments on rectangular beam

i) From $\tau_{xy} = \frac{V}{2I}[\left(\frac{h}{2}\right)^2 - y_1^2]$, (7.27)

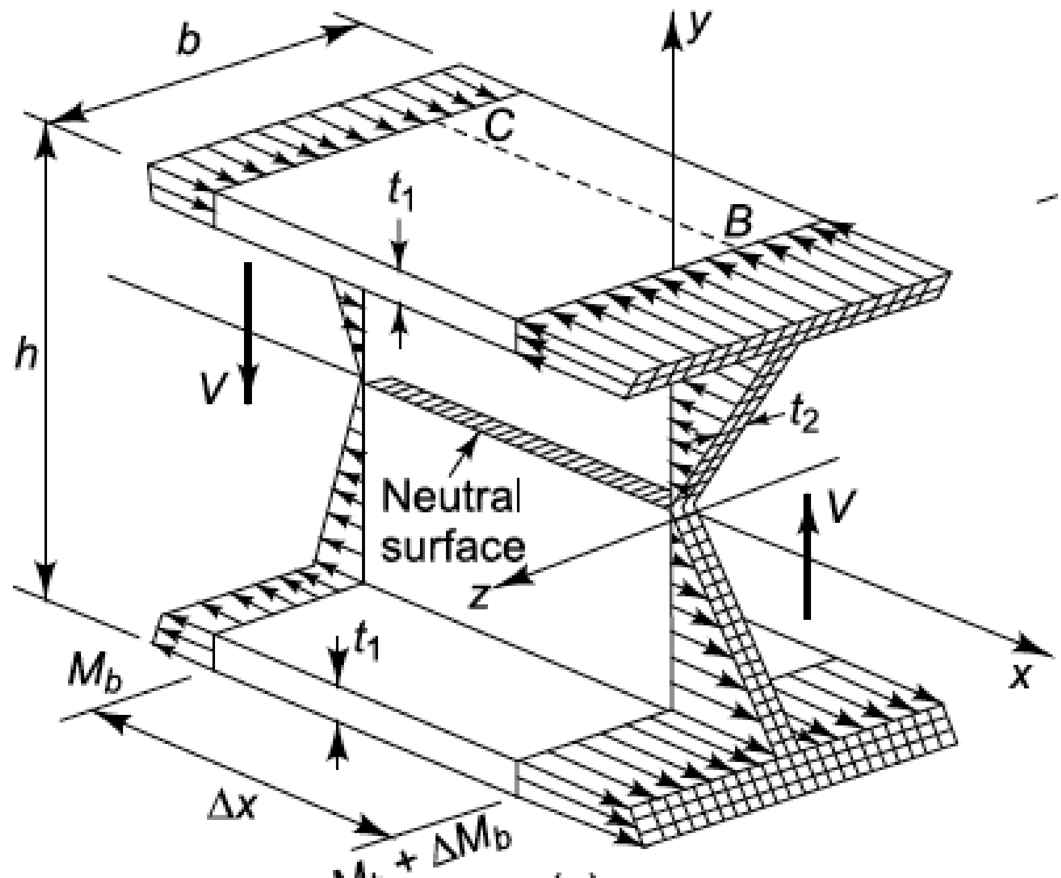
$\tau_{max} = \frac{Vh^2}{8I} = \frac{3V}{2A} = 1.5\tau_{avg}$

$\rightarrow \therefore \tau_{max}$ is 50% greater than $\tau_{avg}(= V/A)$

ii) Eq. (7.27) is useful only for linear elastic beams.

iii) This equation is more accurate when b is smaller than h. If b is same with h, true $\tau_{\max}$ is 13% greater than $\tau_{\max}$ that is derived from Eq. (7.27).

▶ Shear-stress distribution in I-beam



(a)

▷ Assumptions

i) The shear stress is uniform across the thickness t_1, t_2.

ii) We neglect the effect of small fillet at the connection of flange and web.

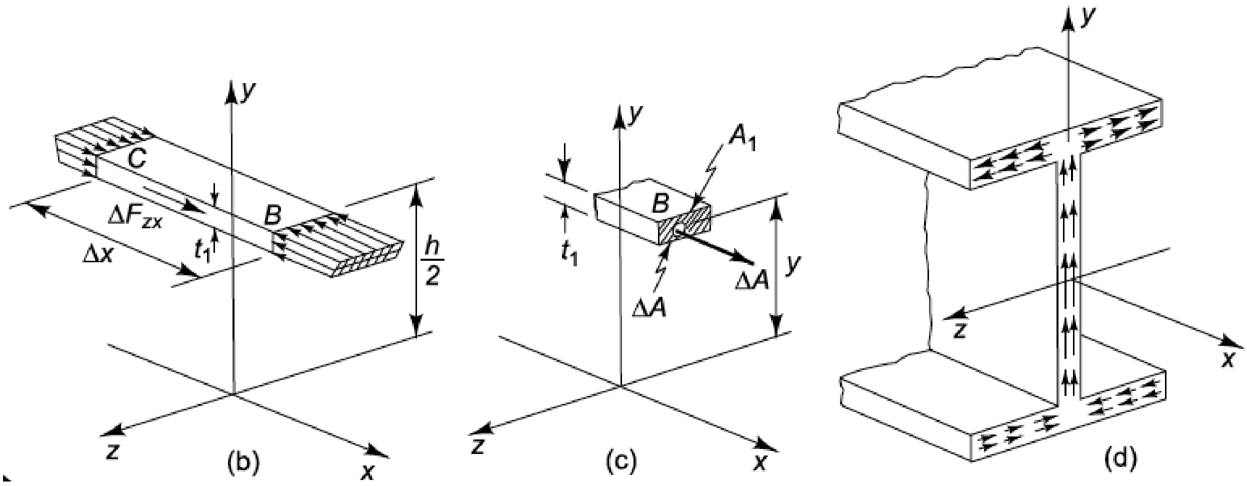


Fig. 7.17 *Calculation of shear stress in an I beam*

▷ From Fig. (b)

$$q_{zx} = -\frac{VQ}{I_{zz}} \tag{7.28}$$

$$\tau_{xz} = \tau_{zx} = \frac{q_{zx}}{t_1} = -\frac{VQ}{t_1 I_{zz}} \tag{7.29}$$

▷ Shear-stress distribution

→ In Fig. 7.17 (d) we show the shear-stress distribution over the cross section of the beam; in each flange the stress τ_{xz} varies linearly from a maximum at the junction with the web to zero at the edge, while in the web the stress τ_{xy} has a parabolic distribution.

cf. The stress distribution at the junction of the web and flange is quite complicated; standard rolled I beams are provided with generous fillets at these points to reduce the stress concentration.

cf. On a typical wide-flange beam, mean shear-stress is within the $\pm 10\%$ of the true maximum shear-stress.

▶ Note

▷ Proof of the Eq. (7.28)

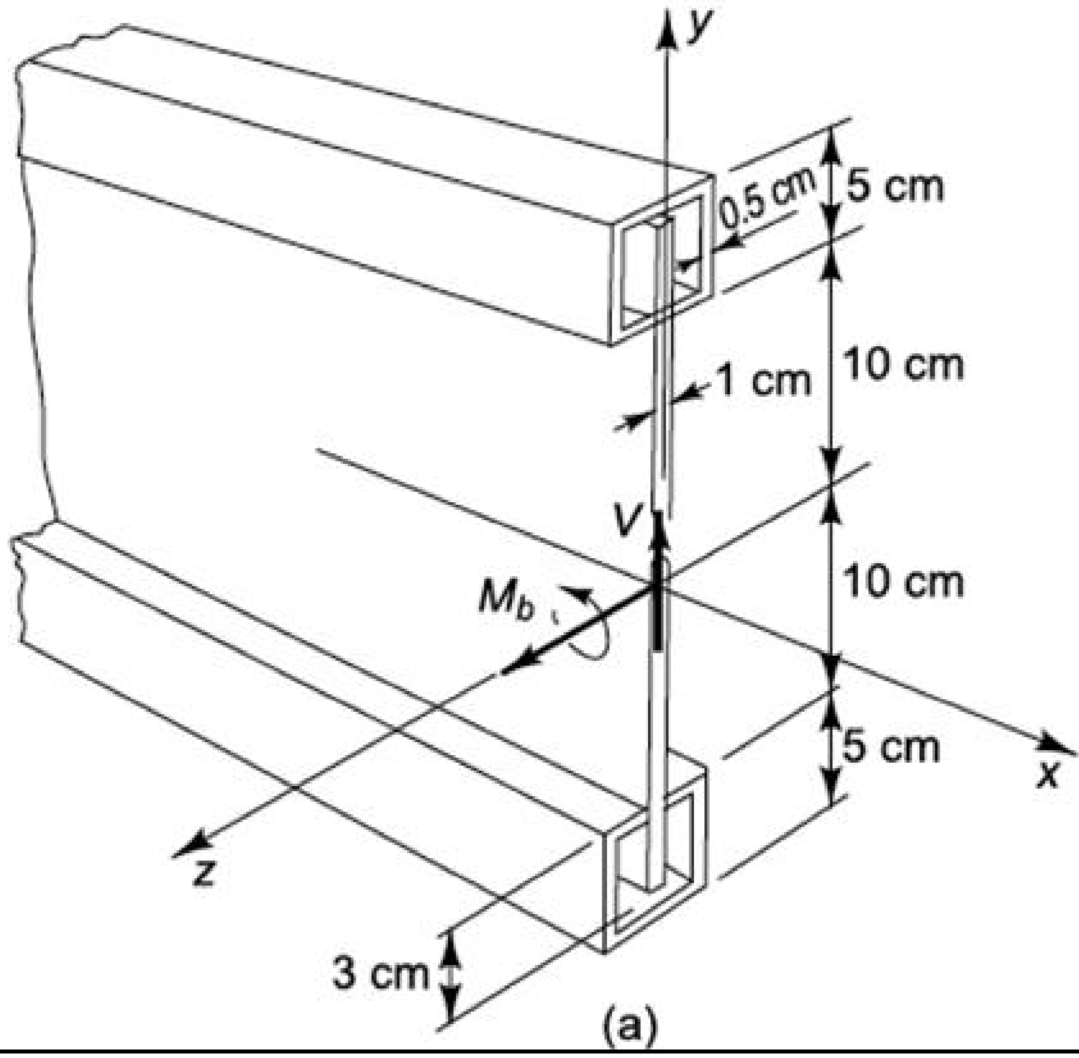
$$\sum F_x = \left[\int_{A_1} \sigma_x \, dA\right]_{x+\Delta x} + \Delta F_{zx} - \left[\int_{A_1} \sigma_x \, dA\right]_x = 0$$

$$\therefore \Delta F_{zx} = \int_{A_1} \frac{(M_b+\Delta M_b)y}{I_{zz}} \, dA - \int_{A_1} \frac{M_b y}{I_{zz}} \, dA = \int_{A_1} \frac{\Delta M_b y}{I_{zz}} \, dA = \frac{\Delta M_b Q}{I_{zz}}$$

$$\therefore q_{zx} = lim_{\Delta x \to 0} \frac{\Delta F_{zx}}{\Delta x} = \frac{dM_b}{dx}\frac{Q}{I_{zz}} = -\frac{VQ}{I_{zz}} \qquad (7.28)$$

▶ Example 7.3

In making the brass beam of Fig. 7.18 (a), the box sections are soldered to the 1 cm plate, as indicated in Fig. 7.18 (b). If the shear stress in the solder is not to exceed 1000 $\mathrm{N/cm^2}$, what is the maximum shear force which the beam can carry?



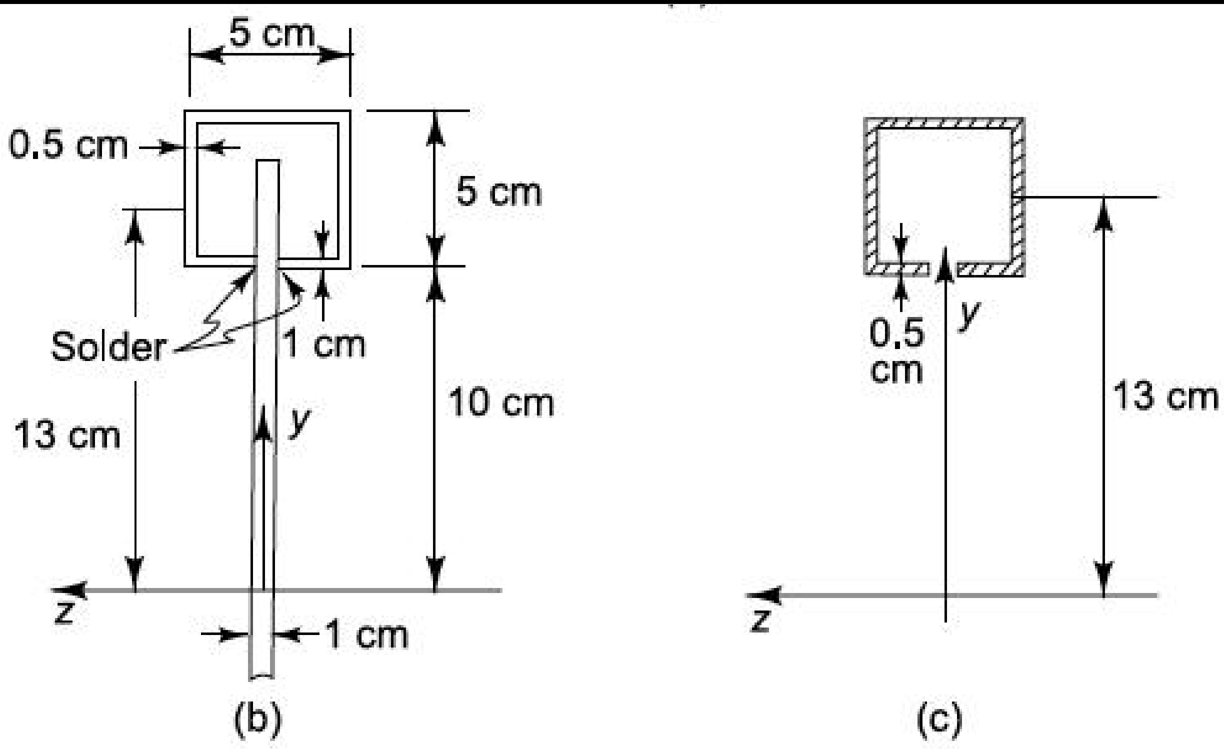


Fig. 7.18 *Example 7.3*

$$2q_{zx} = \frac{VQ}{I_{zz}} \tag{a}$$

where

$$q_{zx} = 1000(0.5) = 500 \text{ N/cm}$$

$$Q = 12.5[5^2 - 4^2] = 112.5 \text{ cm}^3$$

$$\therefore \ V = \frac{2q_{zx}I_{zz}}{Q}$$

$$= \frac{2(500)(4337)}{112.5} = 38551 \text{ N}$$

▶ Example 7.4

A rectangular beam is carried on simple supports and subjected to a central load, as illustrated in Fig 7.19. We wish to find the ratio of the maximum shear stress $(\tau_{xy})_{\max}$ to the maximum bending stress $(\sigma_x)_{\max}$.

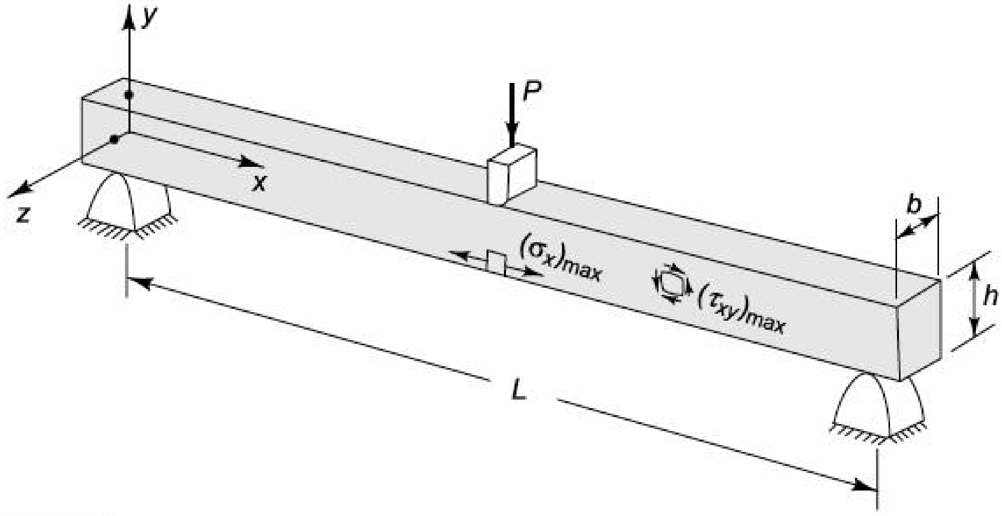


Fig. 7.19 *Example 7.4. Rectangular beam on simple supports and with a central load*

$$(M_b)_{max} = PL/4 \tag{a}$$

$$I_{zz} = bh^3/12 \tag{b}$$

Substituting (a) and (b) in (7.16)

$$(\sigma_x)_{max} = -\frac{(M_b)_{max}(-h/2)}{I_{zz}} = -\frac{(PL/4)(-h/2)}{bh^3/12} = \frac{3}{2}\frac{PL}{bh^2} \tag{c}$$

Substituting $y_1 = 0$ in (7.27),

$$(\tau_{xy})_{max} = \frac{P/2}{2(bh^3\backslash 12)}\left[\left(\frac{h}{2}\right)^2 - 0^2\right] = \frac{3}{2}\frac{P/2}{bh} = \frac{3}{4}\frac{P}{bh} \tag{d}$$

$$\therefore \frac{(\tau_{xy})_{max}}{(\sigma_x)_{max}} = \frac{1}{2}\frac{h}{L} \tag{e}$$

→ The bending and shear stresses are of comparable magnitude only when L and h are of the same magnitude. (the factor of 1/2 in (e) can be as large as 3 or 4 for I beams with thin webs.)

cf. If a different loading is put on the beam m Fig 7.19, the ratio of the maximum stresses will again be found to depend upon the ratio of the depth to the length of the beam, although, of course, the factor of proportionality will differ from that just found. If beams of other cross-sectional shape are investigated, similar results are obtained.

▶ Localized buckling in I beams

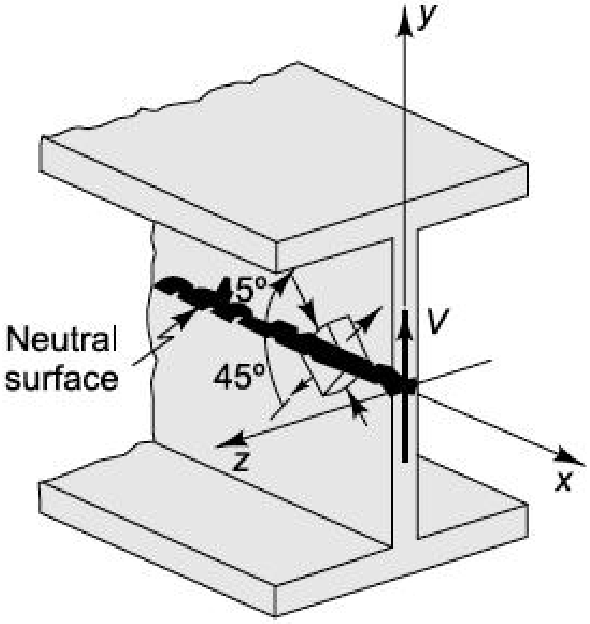


Fig. 7.20 *Illustration of compressive and tensile stresses acting on an element at the neutral surface in the web of an I beam transmitting a shear force (see Example 7.5)*

→ From the point of view of reducing bending stress, it is apparent from (7.16) that for a given cross-sectional area of beam it is best to distribute that area so that I_{zz} is as large as practical, i.e., to concentrate the area as far as possible from the centroid. But there are restrictions due to the side effects of buckling.

1 ▷ If the cross-sectional area of the I beam was kept constant while the depth was increased at the expense of a decrease in the flange thickness;
The beam might fail by a buckling of the compression flange at a stress level well below that at which the material would yield.

2 ▷ If an increase in beam depth was accomplished at the expense of a decrease in web thickness;
The compressive stresses resulting from the transmission of shear along the beam might cause buckling of the web.

7.8 Strain Energy Due to Bending

- We consider first the case of pure bending where the only nonvanishing stress component is the longitudinal stress. The total strain energy (5.17) thus reduces to

$$U = \frac{1}{2}\iiint \sigma_x \epsilon_x \, dxdydz = \iiint \frac{\sigma_x^2}{2E} \, dxdydz \tag{7.30}$$

$$= \iiint \frac{1}{2E}\left(\frac{M_b y}{I_{zz}}\right)^2 dxdydz = \int_L \frac{M_b^2}{2EI_{zz}^2} \, dx \iint_A y^2 \, dydz$$

$$= \int_L \frac{M_b^2}{2EI_{zz}} \, dx \tag{7.31}$$

- This formula may also be derived by considering each differential element of length dx to act as a bending spring.

$$dU = \frac{M_b d\phi}{2} = \frac{1}{2} M_b \frac{d\phi}{dx} \, dx = \frac{1}{2} M_b \left(\frac{M_b}{EI_{zz}}\right) dx$$

$$\therefore \quad U = \int_L \frac{M_b^2}{2EI_{zz}} \, dx \tag{7.31}$$

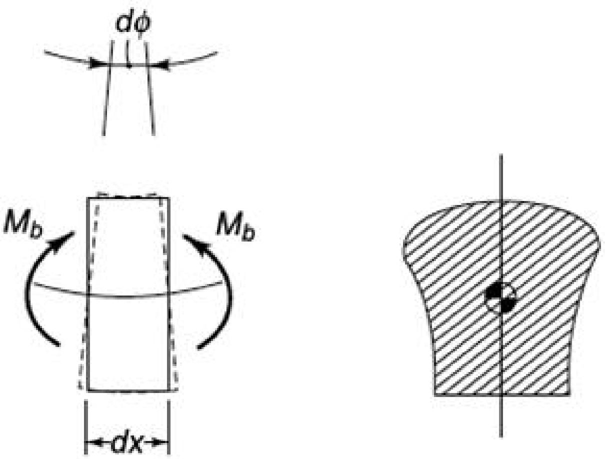


Fig. 7.24 *Differential element of beam bends through angle $d\phi$ under action of bending moment M_b*

- When a beam is subjected to transverse shear in addition to bending, there are, in general, transverse shear-stress components τ_{xy} and τ_{xz} in addition to the bending stressσ_x. The total strain energy (5.17) then becomes

$$U = \frac{1}{2}\iiint\left(\sigma_x\epsilon_x + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz}\right)dxdydz$$

$$= \iiint \frac{\sigma_x^2}{2E}\,dxdydz + \iiint \frac{\tau_{xy}^2+\tau_{xz}^2}{2G}\,dxdydz \tag{7.32}$$

cf. For slender members the latter contribution is almost always negligible in comparison with the former. This may be inferred from the discussion in Sec. 7.6 concerning the comparative magnitudes of the bending and shear stresses. If σ_x is an order of magnitude larger than τ_{xy} and τ_{xz}, then, since the integrals in (7.32) depend on the squares of the stresses, we see that the first integral is two orders of magnitude larger than the second. As a consequence, it is common to neglect the contribution to the strain energy due to the transverse shear stresses. The pure-bending formula (7.31) is then used to represent the total strain energy in a beam whether there is transverse shear or not.

▶ The contribution of $U_{bending}, U_{shear}$ in the rectangular beam

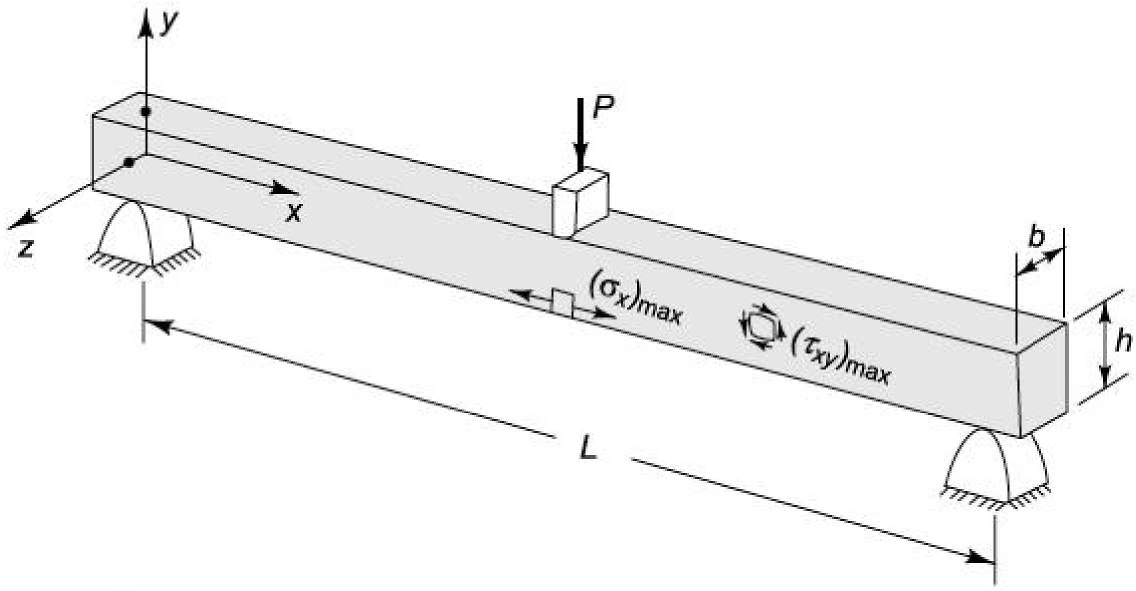


Fig. 7.19 Example 7.4. Rectangular beam on simple supports and with a central load

$$q_x = \frac{P}{2}\langle x\rangle_{-1} - P\left\langle x - \frac{L}{2}\right\rangle_{-1}$$

$$V(x) = -\frac{P}{2}\langle x\rangle^0 + P\left\langle x - \frac{L}{2}\right\rangle^0 = P\left\{-\frac{1}{2} + \left\langle x - \frac{L}{2}\right\rangle^0\right\}$$

$$M_b(x) = \frac{P}{2}x - P\left\langle x - \frac{L}{2}\right\rangle^1 = P\left\{\frac{x}{2} - \left\langle x - \frac{L}{2}\right\rangle^1\right\}$$

$\therefore$ For $0 < x < L, \quad -\frac{h}{2} < y < \frac{h}{2}, \quad -\frac{b}{2} < z < \frac{b}{2}$

$$\tau_{xy} = \frac{V(x)}{2I_{zz}}\left[\left(\frac{h}{2}\right)^2 - y^2\right] \tag{7.27}$$

$$\tau_{xz} = 0$$

$\therefore$ From Eq. (7.32) $(U = U_b + U_s)$,

$$U_b = \int_L \frac{M_b^2}{2EI_{zz}}\,dx = 2\int_0^{\frac{L}{2}} \frac{(Px/2)^2}{2EI_{zz}}\,dx = \frac{P^2L^3}{8Ebh^3} \tag{7.33}$$

$$U_s = \int_0^L \frac{V^2}{8GI_{zz}^2}\,dx \cdot \int_{-h/2}^{h/2}\left[\left(\frac{h}{2}\right)^2 - y^2\right]^2 dy \cdot \int_{-b/2}^{b/2} dz$$

$$= \frac{P^2Lbh^5}{960 \cdot GI_{zz}^2} = \frac{3}{20}\frac{P^2L}{Gbh} \tag{7.34}$$

$$\therefore\ U = U_b + U_s = \frac{P^2L^3}{8Ebh^3} + \frac{3}{20}\frac{P^2L}{Gbh} = \frac{P^2L^3}{8Ebh^3}\left[1 + \frac{6}{5}\frac{E}{G}\left(\frac{h}{L}\right)^2\right] \tag{7.35}$$

$\therefore$ The ratio of two contributions is

$$\frac{U_s}{U_b} = \frac{6}{5}\frac{E}{G}\left(\frac{h}{L}\right)^2 = \frac{12}{5}(1+\nu)\left(\frac{h}{L}\right)^2$$

cf.

i) For a beam with $L > 10h$ and with Poisson's ratio $\nu = 0.28$, the shear contribution is less than 3 percent of the bending contribution. (U_s/U_b does not depend on width b.)

ii) For beams with other loadings and other cross-sectional shapes, the ratio of U_s to U_b is always proportional to the square of the ratio of beam depth to beam length.

iii) The numerical factor of 6/5 in (7.35) can be as large as 12 for I beams.

7.9 Onset of Yielding in Bending

▶ For pure bending

$$\sigma_1 = \sigma_x \qquad \sigma_2 = \sigma_3 = 0 \tag{7.36}$$

→ ∴ In this case, the yielding condition is as follows;

$$\sigma_x = Y \tag{7.37}$$

▶ For combined load

$$\begin{cases} \text{Von Mises Criterion} \\ \text{Tresca Criterion} \end{cases}$$

cf. Even in relatively simple structures the most critically stressed point may not be obvious, and calculations may have to be made for more than one point.

▶ **Example 7.7**
A circular rod of radius r is bent into the U-shape to form the structure of Fig. 7.25 (a). The material in the rod has a yield stress Y in simple tension. We wish to determine the load P that will cause yielding to begin at some point in the structure.

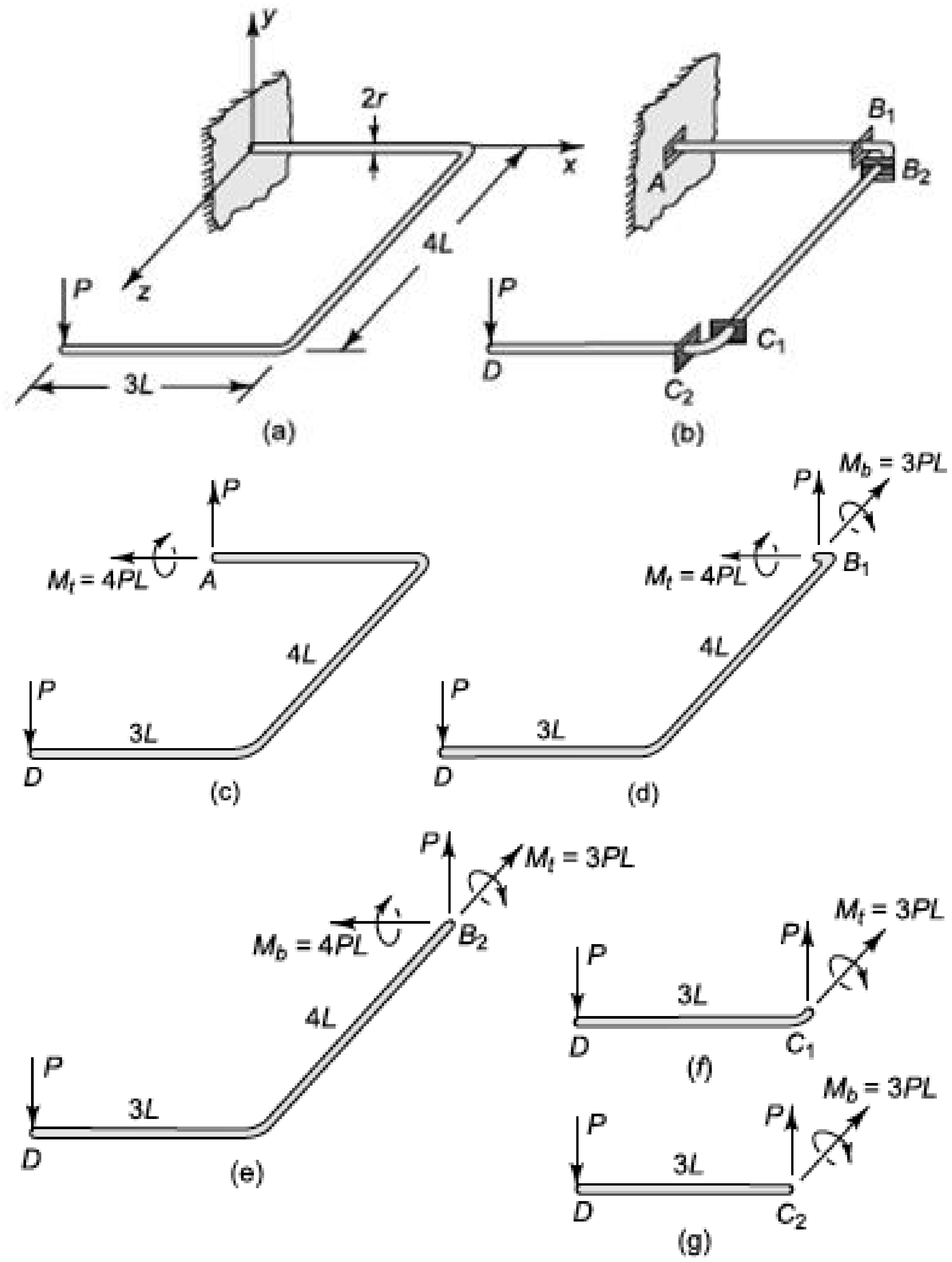


Fig. 7.25 *Example 7.7. Bending and twisting moments at five critical locations in a structure*

Sol) Referring the Fig. 7.25, we can guess that B_1 or B_2 are critical cross-sections.

1 ▷ For B_1 (see Fig. 7.26 (a))

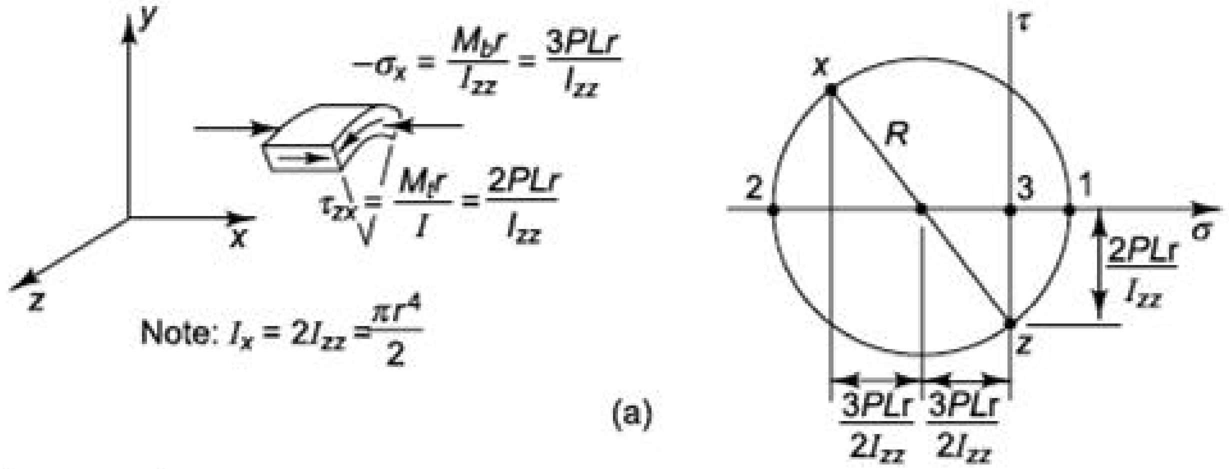


Fig. 7.26 *Example 7.7 (a) Maximum stress condition at location B_1; (b) maximum stress condition at location B_2*

$$R = \sqrt{\left(\frac{3}{2}\frac{PLr}{I_{zz}}\right)^2 + \left(\frac{2PLr}{I_{zz}}\right)^2} = \frac{5}{2}\frac{PLr}{I_{zz}} = (\tau_{xz})_{max} \tag{a}$$

Principal stresses are

$$\begin{cases} \sigma_1 = +\frac{PLr}{I_{zz}} \\ \sigma_2 = -4\frac{PLr}{I_{zz}} \\ \sigma_3 = 0 \end{cases} \tag{b}$$

i) By von Mises Criterion

$$\sqrt{\frac{1}{2}\left[\left(\frac{PLr}{I_{zz}} + 4\frac{PLr}{I_{zz}}\right)^2 + \left(-4\frac{PLr}{I_{zz}} - 0\right)^2 + \left(0 - \frac{PLr}{I_{zz}}\right)^2\right]} = Y \tag{c}$$

$\rightarrow \therefore$ The yiedling condition is

$$\therefore P = 0.218\frac{I_{zz}Y}{Lr} \tag{d}$$

ii) By Tresca Criterion

$$\tau_{max} = \frac{|\sigma_{max} - \sigma_{min}|}{2} = \frac{1}{2}\left(\frac{PLr}{I_{zz}} + 4\frac{PLr}{I_{zz}}\right) = \frac{Y}{2} \tag{e}$$

$$\therefore P = 0.200\frac{I_{zz}Y}{Lr} \tag{f}$$

→ $\therefore$ The difference between (d) and (f) is 9%.

2 ▷ For B_2 (see Fig. 7.26 (b))

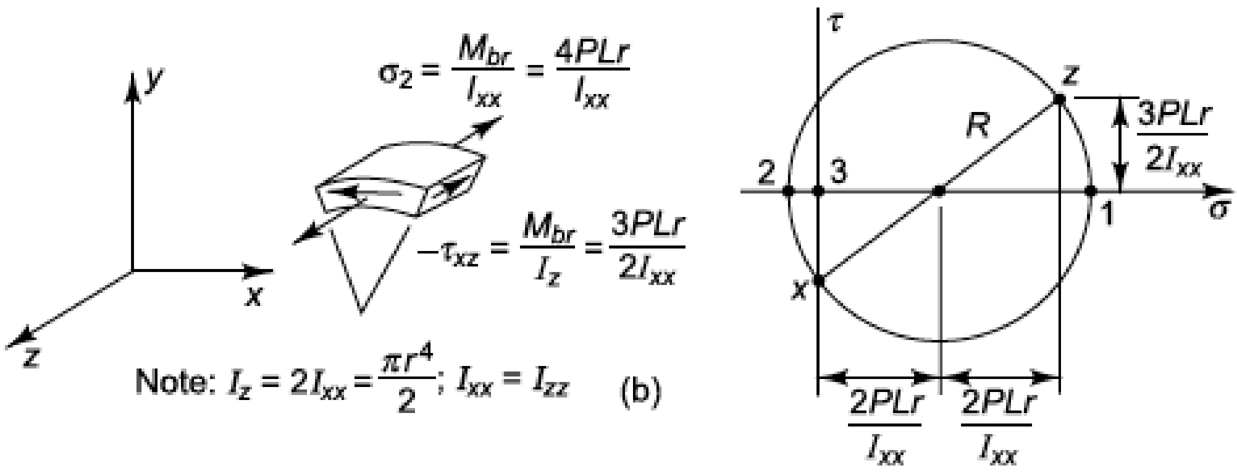


Fig. 7.26 *Example 7.7 (a) Maximum stress condition at location B_1; (b) maximum stress condition at location B_2*

Principal stresses are

$$\begin{cases} \sigma_1 = +\frac{9}{2}\frac{PLr}{I_{xx}} \\ \sigma_2 = -\frac{1}{2}\frac{PLr}{I_{xx}} \\ \quad \sigma_3 = 0 \end{cases} \qquad \text{(g)}$$

i) By von Mises Criterion

$$\sqrt{\frac{1}{2}[\left(\frac{9}{2}\frac{PLr}{I_{xx}} + \frac{1}{2}\frac{PLr}{I_{xx}}\right)^2 + \left(-\frac{1}{2}\frac{PLr}{I_{xx}} - 0\right)^2 + \left(0 - \frac{9}{2}\frac{PLr}{I_{xx}}\right)^2]} = Y \qquad \text{(h)}$$

→ ∴ The yiedling condition is

$$\therefore P = 0.210\frac{I_{xx}Y}{Lr} \qquad \text{(i)}$$

ii) By Tresca Criterion

$$\tau_{max} = \frac{|\sigma_{max} - \sigma_{min}|}{2} = \frac{1}{2}\left(\frac{9}{2}\frac{PLr}{I_{xx}} + \frac{1}{2}\frac{PLr}{I_{xx}}\right) = \frac{Y}{2} \qquad \text{(j)}$$

$$\therefore P = 0.200\frac{I_{xx}y}{Lr} \qquad \text{(k)}$$

→ ∴ The difference between (i) and (k) is 5%.

→ The maximum shear-stress criterion predicts yielding at locations B_1 and B_2 at the same load, indicating that the Mohr's circles in Fig. 7.26 (a) and (b) are of equal size. The Mises criterion identifies B_2 as the critical location and predicts yielding there at a load 5 percent greater than the load for yielding according to the maximum shear-stress criterion.

7.10 Plastic deformation

▶ Assumptions

i) We shall restrict our attention to symmetrical beams.

ii) We shall further restrict our inquiry to beams in which the material has the elastic-perfectly plastic stress-strain behavior.

iii) The Mises and the maximum shear-stress criteria predict yielding at the same bending-stress level since pure bending corresponds to a uniaxial state of stress.

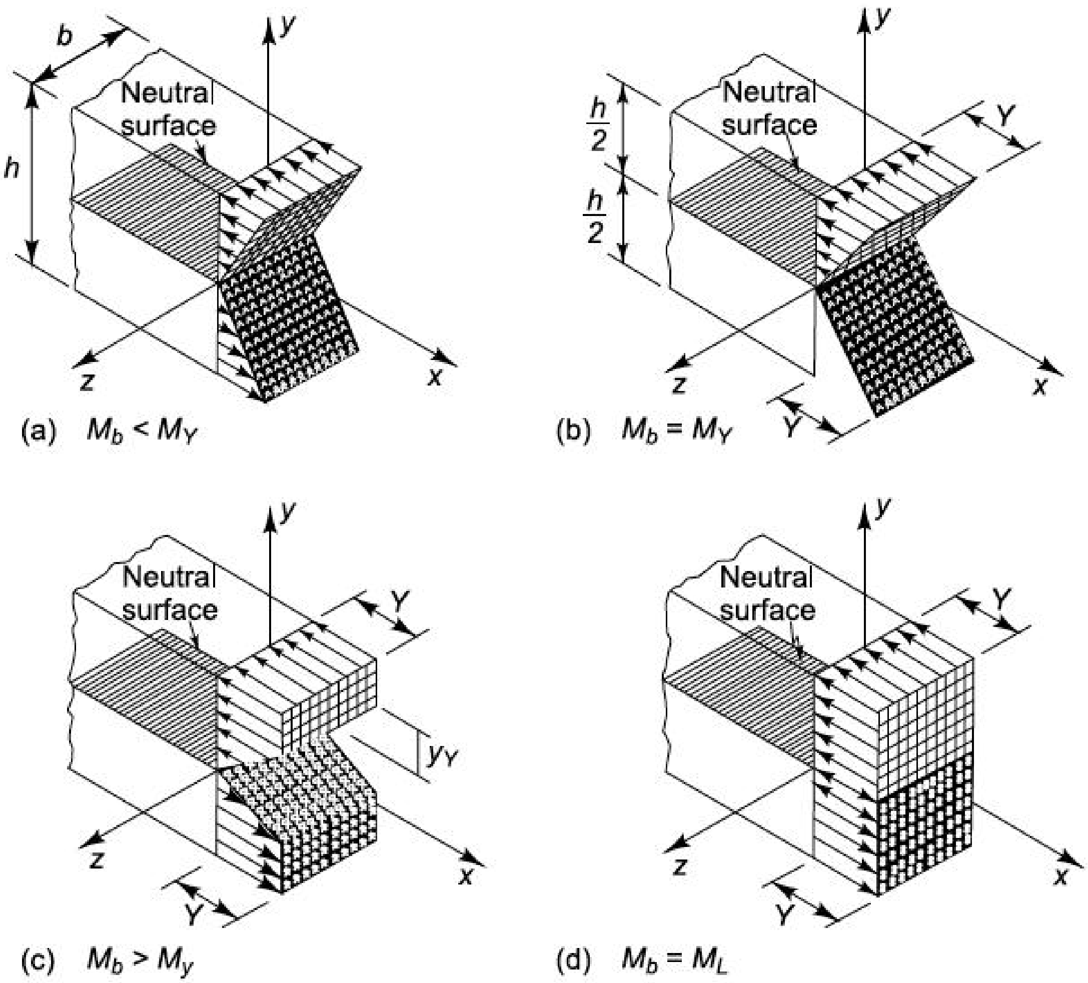


Fig. 7.28 *Bending-stress distribution in a rectangular beam of elastic-perfectly plastic material as the curvature is increased until the fully plastic moment M_L is reached at infinite curvature*

▶ From Fig. 7.28

The nature of the geometric deformation is independent of the stress-strain behavior of the material.

1▷ Elastic region ($0 < \sigma_{max} < Y$)

$$\epsilon_x = -\frac{y}{\rho} = -\frac{d\phi}{ds} y \tag{7.4}$$

2▷ Onset of yielding ($\sigma_{max} = Y$)

$$\frac{d\phi}{ds} = \frac{1}{\rho} = \frac{M_b}{EI_{zz}} \tag{7.14}$$

M_Y corresponds to the situation where $\sigma_x = -Y$ at $y = +h/2$.

$$M_Y = \frac{Y(bh^3/12)}{h/2} = \frac{bh^2}{6}Y \tag{7.38}$$

$$\left(\frac{1}{\rho}\right)_Y = \frac{\epsilon_Y}{h/2} \tag{7.39}$$

3 ▷ Between yielding and fully plastic ($\sigma_{max} = Y,\ \ M_Y < M_b < M_L$)

$$\begin{cases} \text{i) } For\ 0 < y < y_Y & ; \quad \sigma_x = -\frac{y}{y_Y}Y \\ \text{ii) } For\ y_Y < y < h/2 & ; \quad \sigma_x = -Y \end{cases} \tag{7.40}$$

→ Taking an element of area of size $\Delta A = b\Delta y$,

$$M_b = \int_A \sigma_x y\, dA$$

$$= 2\left(-\int_0^{y_Y} \sigma_x yb\, dy - \int_{y_Y}^{h/2} \sigma_x yb\, dy\right) \tag{7.41}$$

$$= \frac{bh^2}{4}Y\left[1 - \frac{1}{3}\left(\frac{y_Y}{h/2}\right)^2\right] \tag{7.42}$$

Since, $\frac{1}{\rho} = \frac{\epsilon_Y}{y_Y}$ (7.43)

From Eq. (7.39),

$$\frac{y_Y}{h/2} = \frac{(1/\rho)_Y}{1/\rho} \tag{7.44}$$

∴ Eq. (7.42) is;

$$M_b = \frac{bh^2}{4}\left(\frac{6}{bh^2}M_Y\right)\left\{1 - \frac{1}{3}\left[\frac{(1/\rho)_Y}{1/\rho}\right]^2\right\}$$

$$= \frac{3}{2}M_Y\left\{1 - \frac{1}{3}\left[\frac{(1/\rho)_Y}{1/\rho}\right]^2\right\} \tag{7.45}$$

when $\frac{1}{\rho} > \left(\frac{1}{\rho}\right)_Y$

4 ▷ Fully plastic region ($\sigma_x = Y, M_b = M_L$)

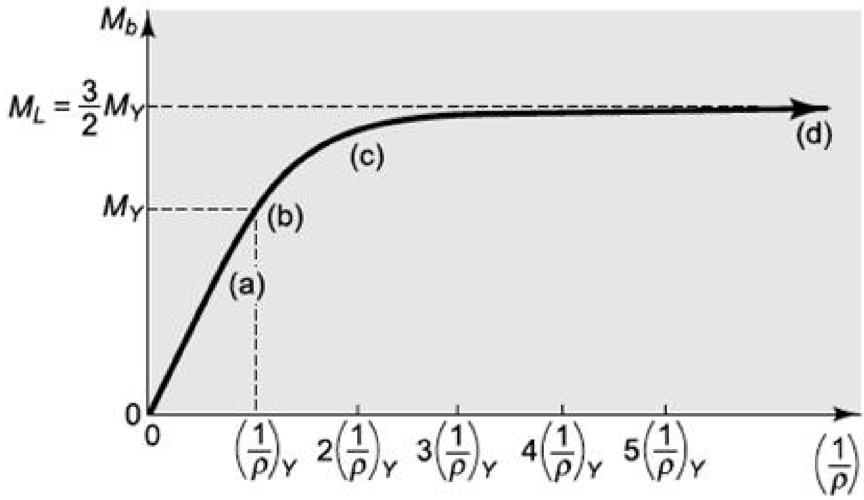


Fig. 7.29 *Moment-curvature relation for the rectangular beam of Fig. 7.28. The positions (a), (b), (c), and (d) correspond to the stress distributions shown in Fig. 7.28*

i) As the curvature increases, the moment approaches the asymptotic value $3/2M_Y$ which we call the fully plastic moment, or limit moment, and for which we use the symbol M_L.

ii) The ratio $K \equiv \frac{M_L}{M_Y}$ is a function of the geometry of the cross section.

Ex) Solid rectangular: $K = 1.5$

Solid circle: $K = 1.7$

Thin-walled circular tube: $K = 1.3$

Typical I beam: $K = 1.1{\sim}1.2$

iii) In the engineering theory the effect of shear force on the value of the bending moment corresponding to fully plastic behavior is negligible in beams of reasonable length.

▶ **Example 7.8**
An originally straight rectangular bar is bent around a circular mandrel of radius $R_0 - h/2$, as shown in Fig. 7.31 (a). As the bar is released from the mandrel, its radius of curvature increases to R_1, as indicated in Fig. 7.31 (b). This change of curvature is called elastic spring-back; it becomes a factor of great importance when metals must be formed to close

dimensional tolerances. Our interest here is in the amount of this spring-back and in the residual stresses which remain after the bar is released.

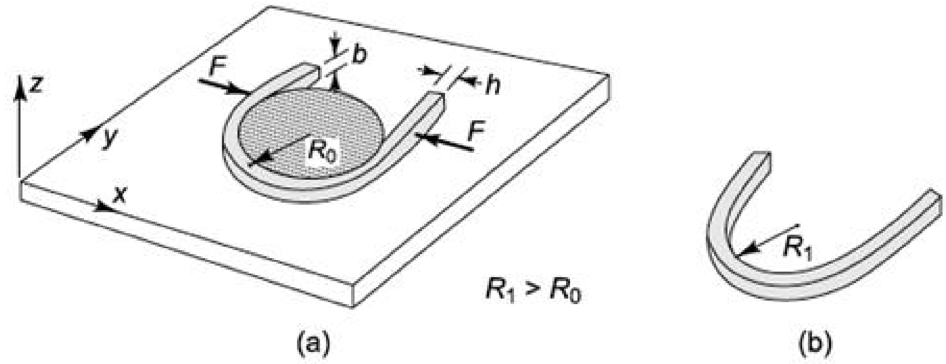


Fig. 7.31 *Example 7.8. Illustration of elastic springback which occurs when an originally straight rectangular bar is released after undergoing large plastic bending deformation*

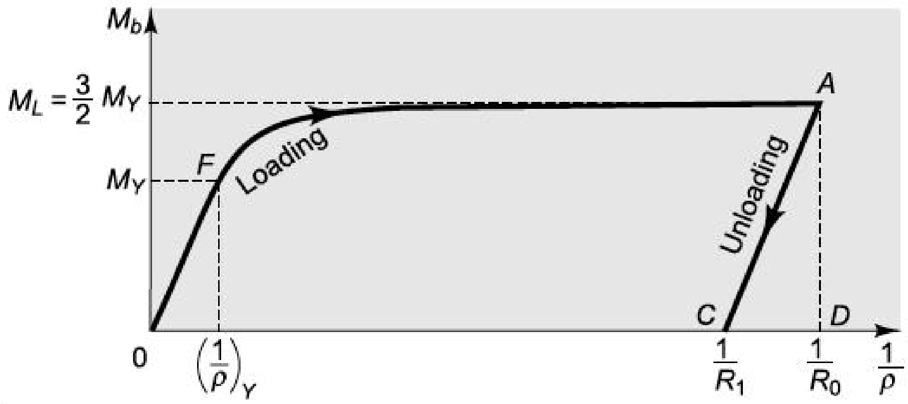


Fig. 7.32 *Example 7.8. Moment-curvature relation for the complete cycle of loading and unloading the rectangular bar in Fig. 7.31*

Sol) As you can see in Fig. 7.32, the decrease in curvature due to the elastic unloading is

$$\frac{1}{R_0} - \frac{1}{R_1} = \frac{3}{2}\left(\frac{1}{\rho}\right)_Y \qquad \text{(a)}$$

where,

$$\left(\frac{1}{\rho}\right)_Y = \frac{\epsilon_Y}{h/2} = \frac{Y}{E}\frac{2}{h} \qquad \text{(b)}$$

$$\therefore \frac{1}{R_0} - \frac{1}{R_1} = \frac{Y}{E}\frac{3}{h} \qquad \text{(c)}$$

▷ From Fig. 7.33

If we now added a further negative bending moment, we could decrease the curvature still further beyond the value $1/R_1$. At first, such action would be elastic, but when this additional bending moment exceeded the value $M_b = -\frac{1}{2}M_Y$, there would be reversed yielding at the inner and outer radii of the bar.

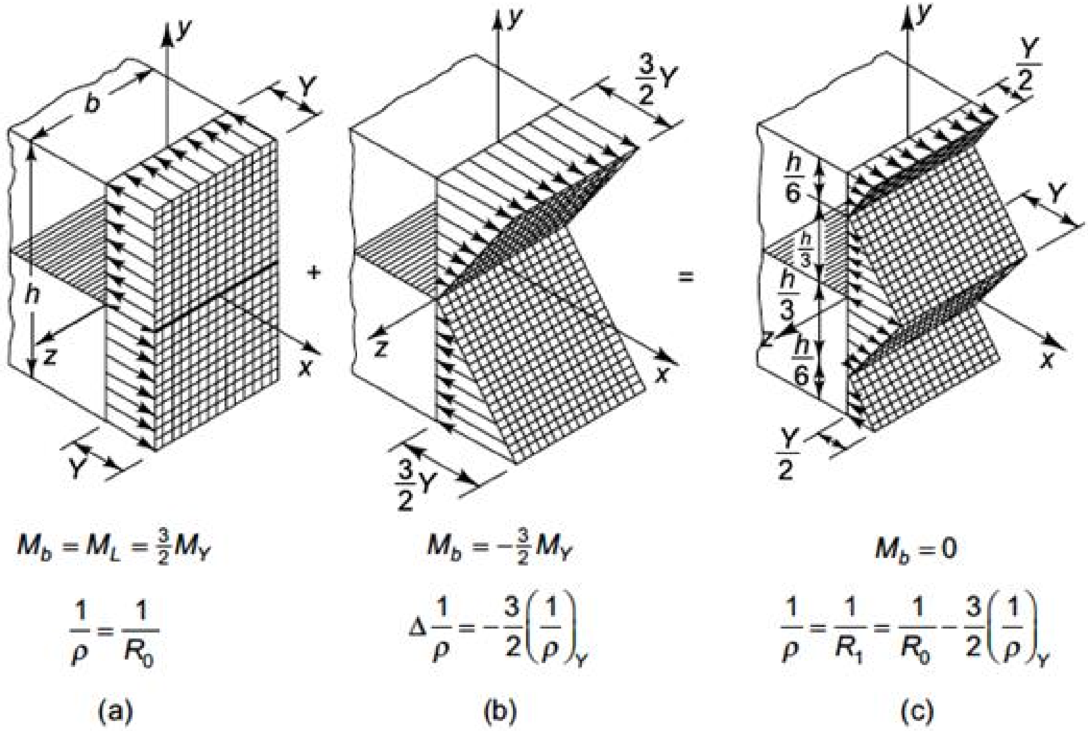


Fig. 7.33 *Example 7.8. Illustrating calculation of the residual-stress distribution in the bar of Fig. 7.31(b).*