# 445.204

# Introduction to Mechanics of Materials

# (재료역학개론)

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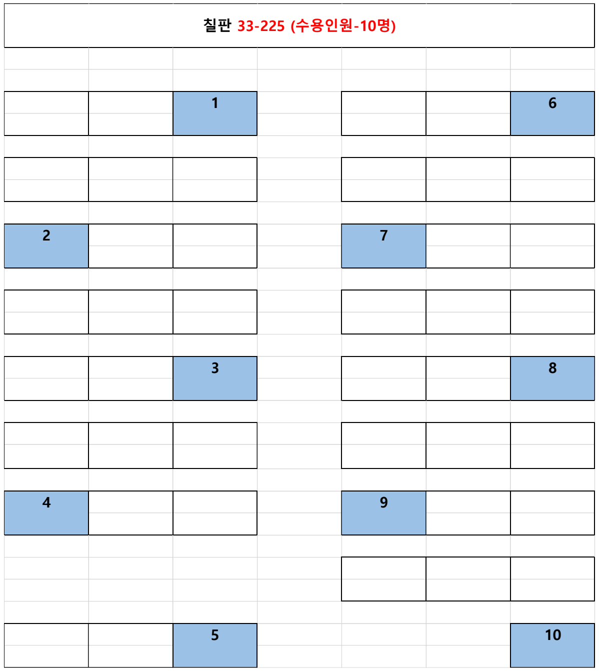
# Notice – Final exam.

- June 15 (Monday)
- 3:30pm to 6:15pm
- Eng. Building #33, Rooms: 225,226,228,229,230,231




**The schedule is subject to change (though little chance!). If there is any change, the schedule will be updated on ETL.**

# Homework #4

- Page 241-243: #6.1, 6.5, 6.11, 6.15, 6.19
- Page 255-259: #6.30, 6.32, 6.37,6.40, 6.56 6.59

- Page 250: Example 6.8, 6.10, 6.11

Due by <u>May 27 (Wed.) Midnight</u>!

Through ETL

# Chapter 7

# Stresses in beams

# Outline

**PART A PURE BENDING**

- Beam deformation in Pure Bending
- Beam Theory (with assumptions)
- Normal Strains in Beams
- Normal Stresses in Beams
- Stress Concentrations in Bending

**PART B SHEAR AND BENDING**

- Shear Stresses in Beams
- Shear Stress Distribution in Rectangular Beams
- Shear Stresses in Beams of Circular Cross Section

# Outline

- Shear Stress Distribution in Flanged Beams
- Comparison of Shear and Bending Stresses
- Design of Prismatic Beams
- (Option) Design of Beams of Constant Strength

PART C (Option) SPECIAL TOPICS

- **Composite Beams**
- Reinforced Concrete Beams
- Unsymmetric Bending
- Shear Center
- **Inelastic Bending**
- Curved Beams

# Introduction

- In ch. 6, shear force and bending moment for different types of beams are calculated.

- In this chapter, two important stresses in beams are discussed. 1) Bending stresses by moment load, and 2) shear stresses casued by shear force

# Part A

## **Pure Bending**

# Introduction- Beam bending

Three solid mechanics principles:

- Equilibrium (3 forces, 3 moments)
- Material behavior (Hooke's law, perfect plasticity etc.)
- Geometry of deformation or compatibility

# Beam deformation in Pure Bending

- **Pure bending** is caused when a beam is „Long“ or „slender“.

- Its length 5 or more times the largest cross sectional dimension. In this case, the shear stress (also called transverse shear) compared to bending stress will not be significant and hence can be neglected.

- In practice, the span/depth ratio is approximately 10 or more for metal beams of compact section, 15 or more for beams with relatively thin webs, and 24 or more for rectangular timber beams.

- Also, the slope of the deflection curve of the beam is almost always less than $5^0$ or 0.087 rad, and hence $(0.087)^2 = 0.00761$, which is a small number.

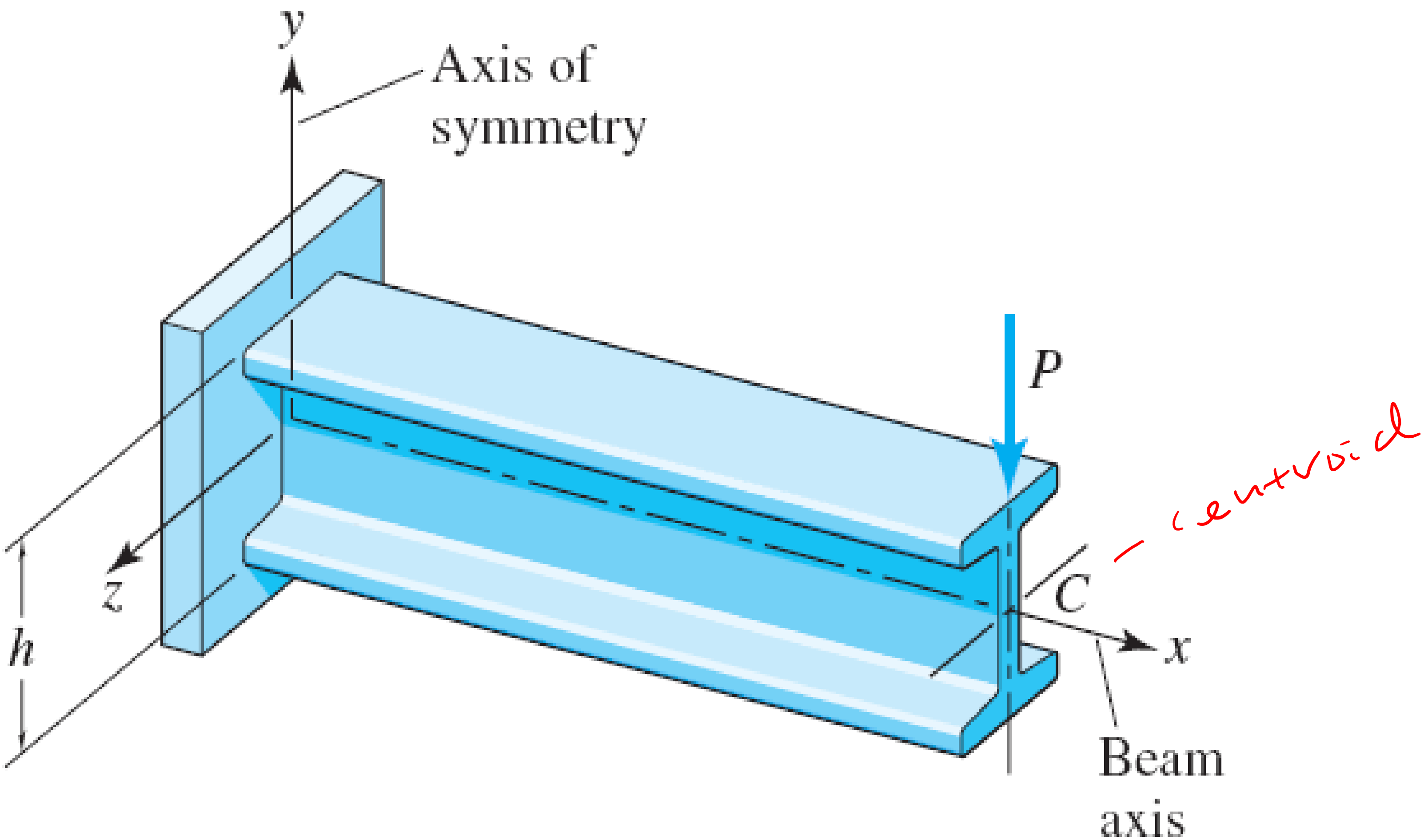
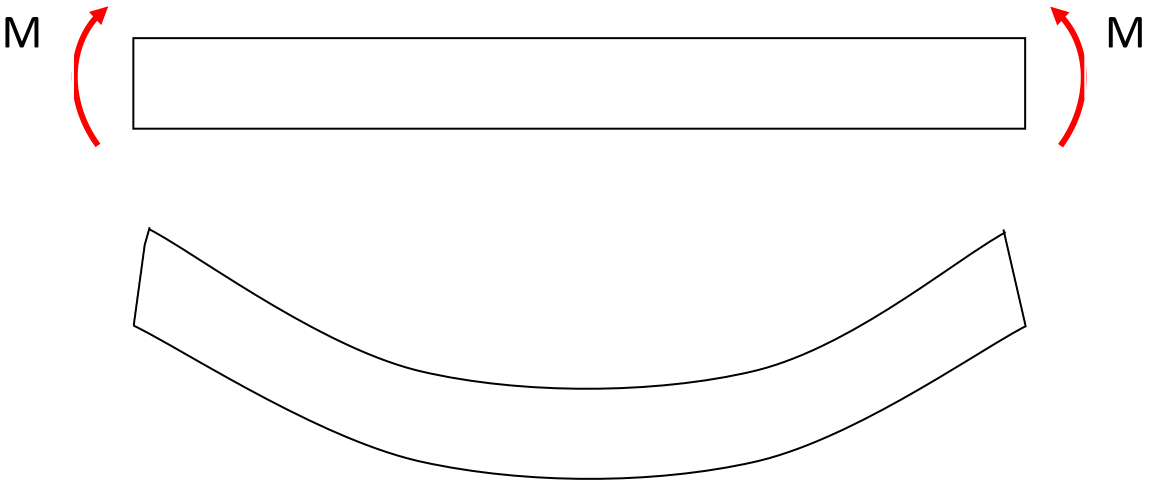



FIGURE 7.1 A cantilever beam loaded in its plane of symmetry.

plane of bending = xy plane

# Pure bending: example




A moment load M causes no shear force.

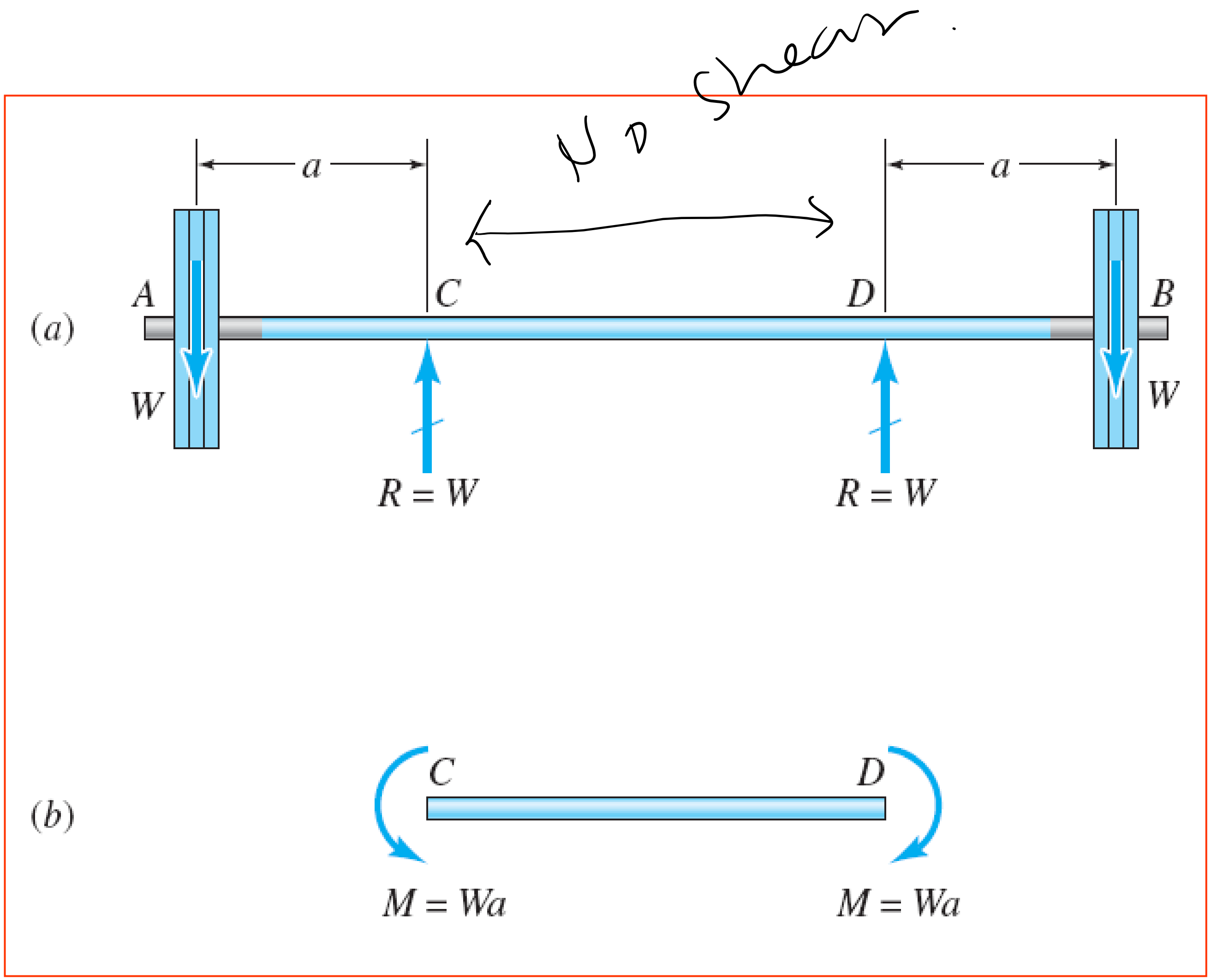
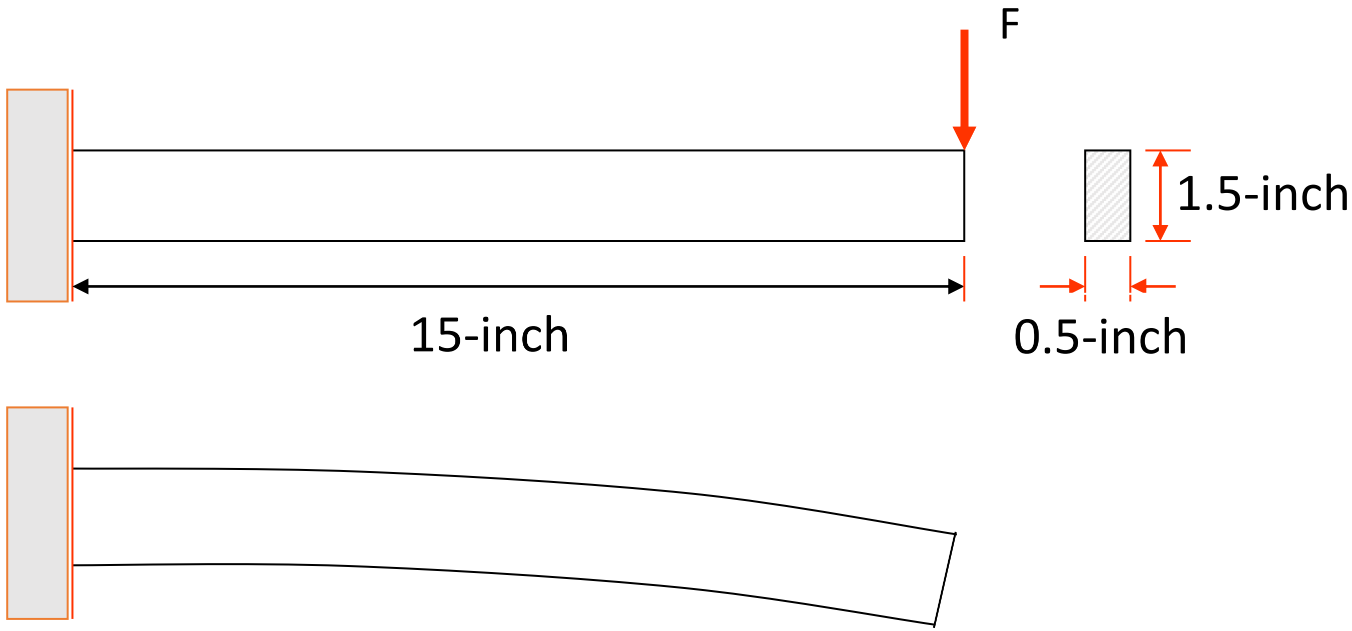



FIGURE 7.2 Bar with symmetrical loading: (a) free-body diagram; (b) center portion in pure bending.

# An almost pure bending




L/depth = 15/1.5 = 10; therefore this is considered as a „long“ beam and so the shear force effect is neglected.

# Geometry of deformation

- Deflection (elastic) curve
- Longitudinal axis of the beam
- Plane sections of the beam
- Tension versus compression in the longitudinal fibers of the beam
- Neutral axis (neutral surface)
- Radius of curvature ($\rho$) of the beam

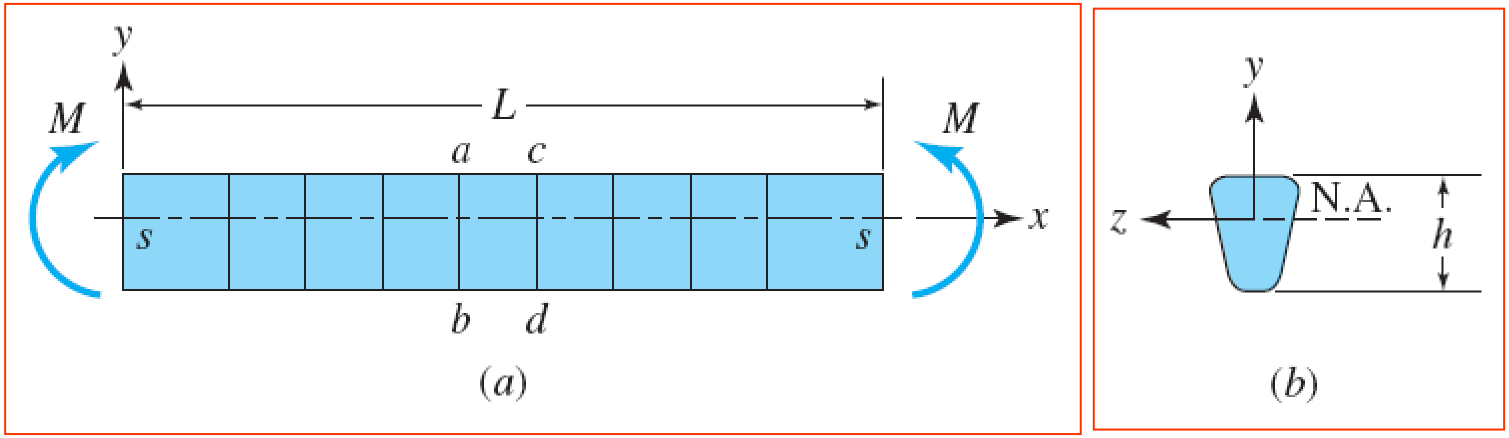



FIGURE 7.3 Beam in pure bending: (a) before deformation; (b) cross section;

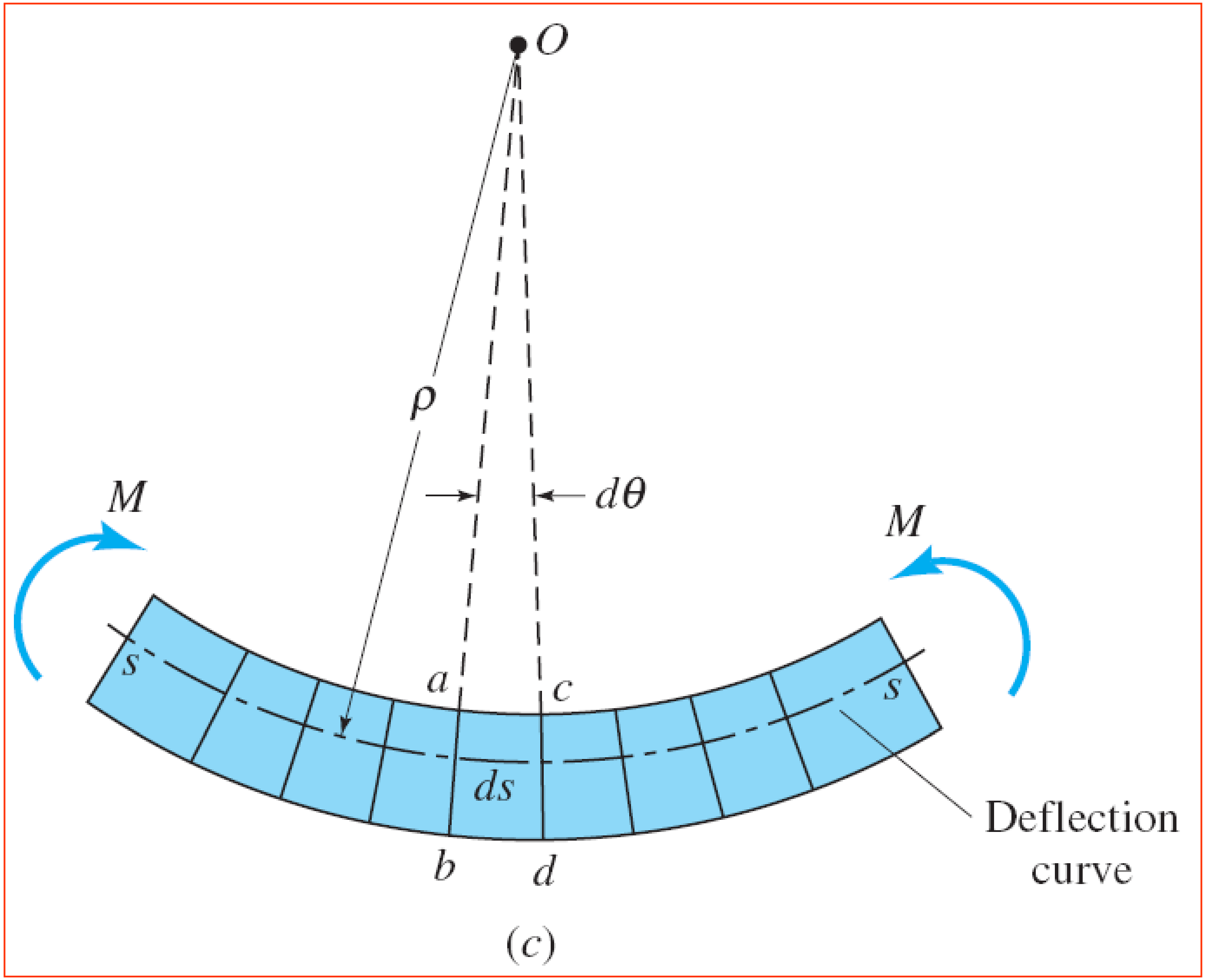



FIGURE 7.3 Beam in pure bending: (c) after bending

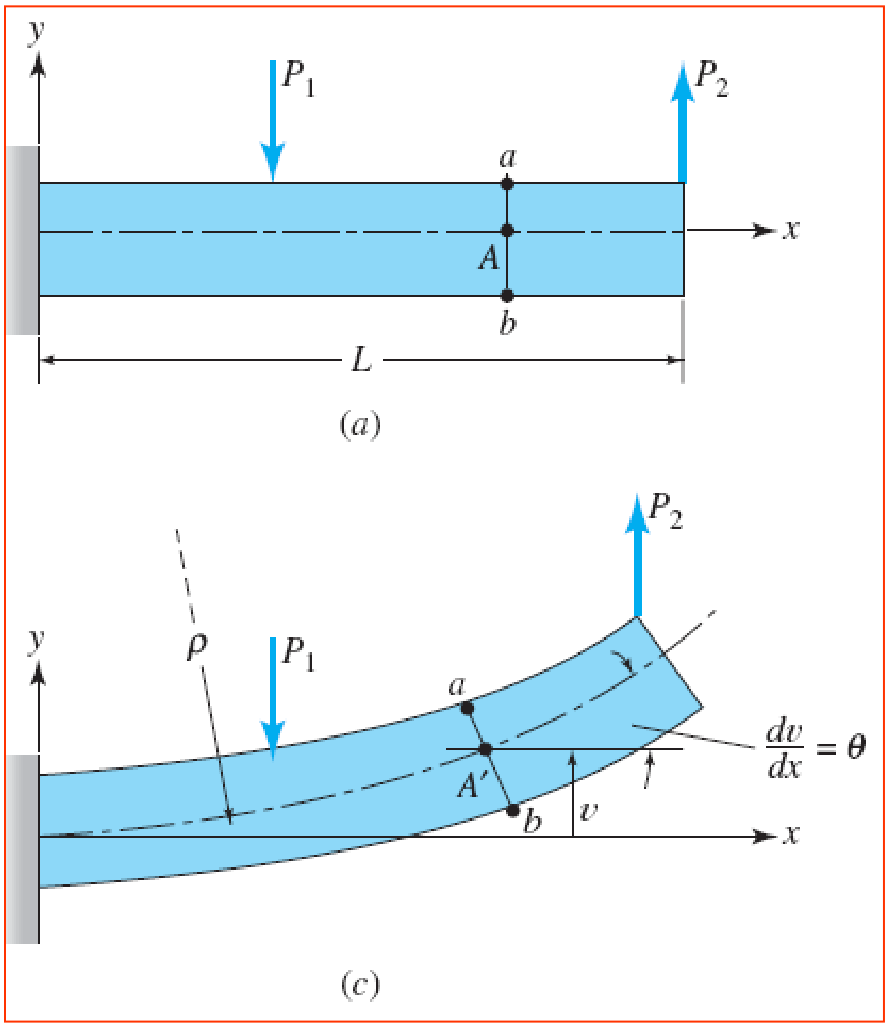
# Radius of curvature formula

$$\boxed{\kappa = \frac{1}{\rho} = \frac{d\theta}{dx}}$$

# Assumptions of beam theory

1. The **deflection of the beam axis is small** compared with the span of **the beam. The angle** of rotation of the deflection curve is also very small, and approximately equal to the slope, $\theta = dv/dx$. If the beam is slightly curved initially, the curvature is in the plane of the bending, and the radius of curvature is large in relation to its depth ($\rho \geq 10h$).
2. **Plane sections initially normal to the beam axis remain plane and normal to that axis after bending** (for example, ab). This means that the **shearing strains $\gamma_{xy}$ are negligible**. The deflection of the beam is thus associated principally with the longitudinal normal or bending strains $\varepsilon_x$.
3. The **effects of transverse normal strains $\varepsilon_y$ and the remaining strains ($\varepsilon_z$, $\gamma_{xz}$, $\gamma_{yz}$) on the distribution of the $\varepsilon_x$ may also be ignored**.
4. The distribution of the **normal or bending stresses $\sigma_x$** is not affected by the deformation due to shear stresses $\tau_{xy}$. The **stresses normal to the neutral surface, $\sigma_y$, are small compared with $\sigma_x$** and may also be omitted. This supposition becomes unreliable in the vicinity of large concentrated transverse loads.

# Example (cantilever beam)



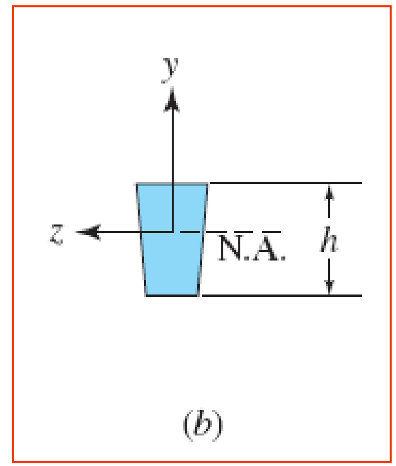




**FIGURE 7.4** Beam subjected to transverse loading: (*a*) before deformation; (*b*) cross section; (*c*) after deformation.

# Normal strains in beams

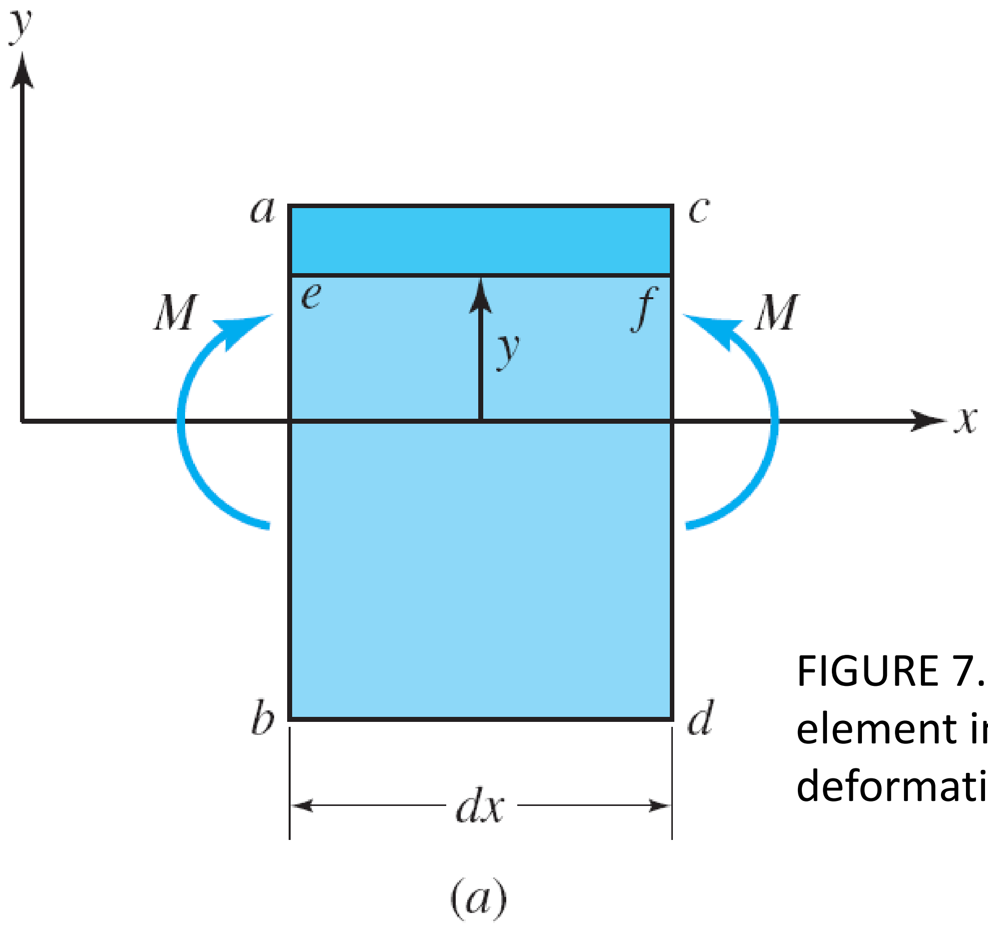



FIGURE 7.5 The geometry of an element in pure bending: (a) before deformation.

$$dx' - dx = (\rho - y)\, d\theta - \rho\, d\theta = -y\, d\theta$$

$$\boxed{\varepsilon_x = -\frac{y}{\rho} = -\kappa y}$$

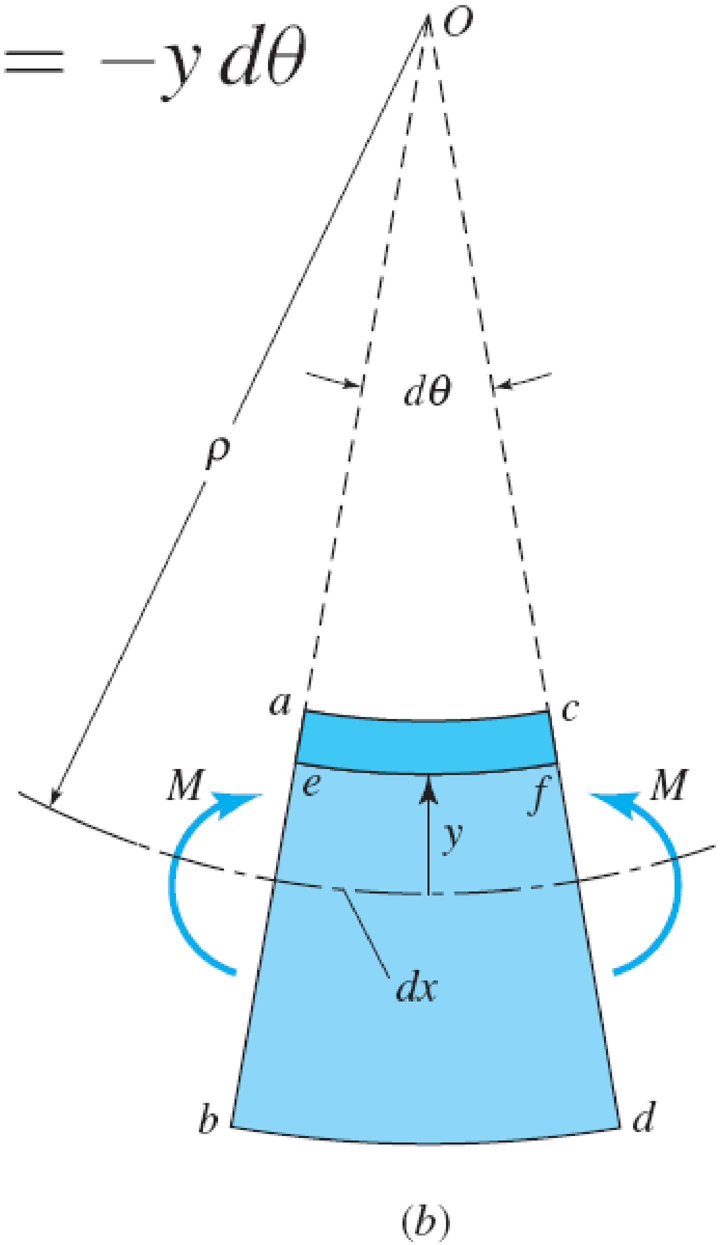



FIGURE 7.5 The geometry of an element in pure bending:
(b) after deformation.

# Transverse strains

$$\varepsilon_y = \varepsilon_z = \nu \kappa y = \frac{\nu y}{\rho}$$

"Poisson's effect"

Transverse radius of curvature is given by:

$$\boxed{\rho_1 = -\frac{\rho}{\nu} \qquad \kappa_1 = -\nu\kappa}$$

Anticlastic radius of curvature is given by:

$$\boxed{\kappa_1 = \frac{1}{\rho_1}}$$

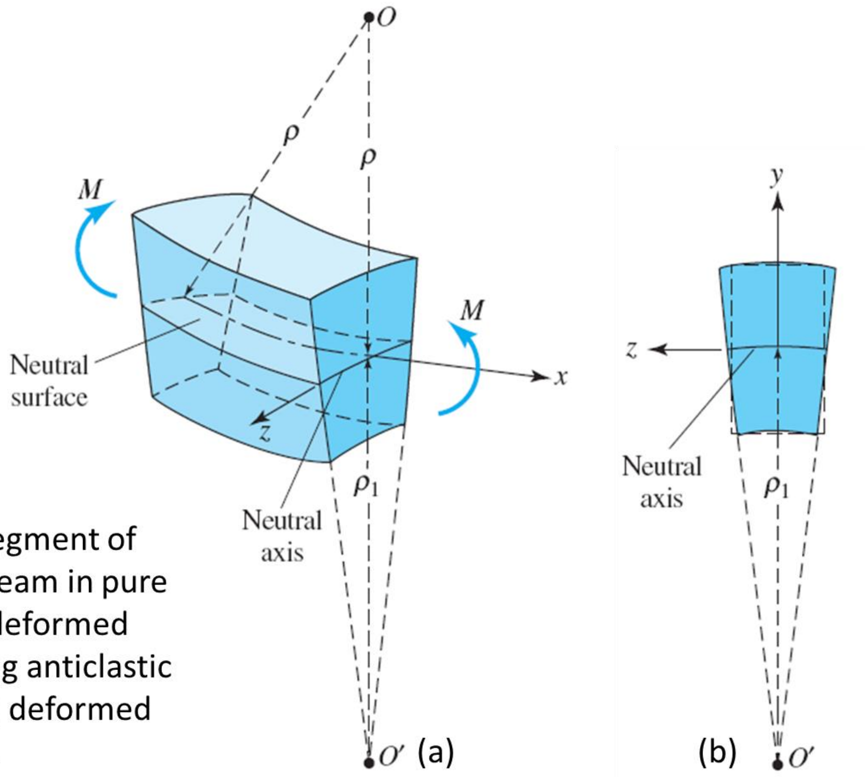



FIGURE 7.6 Segment of rectangular beam in pure bending: (a) deformed shape showing anticlastic curvature; (b) deformed cross section.

# Example 7.1: Curvature and deflection of cantilever beam

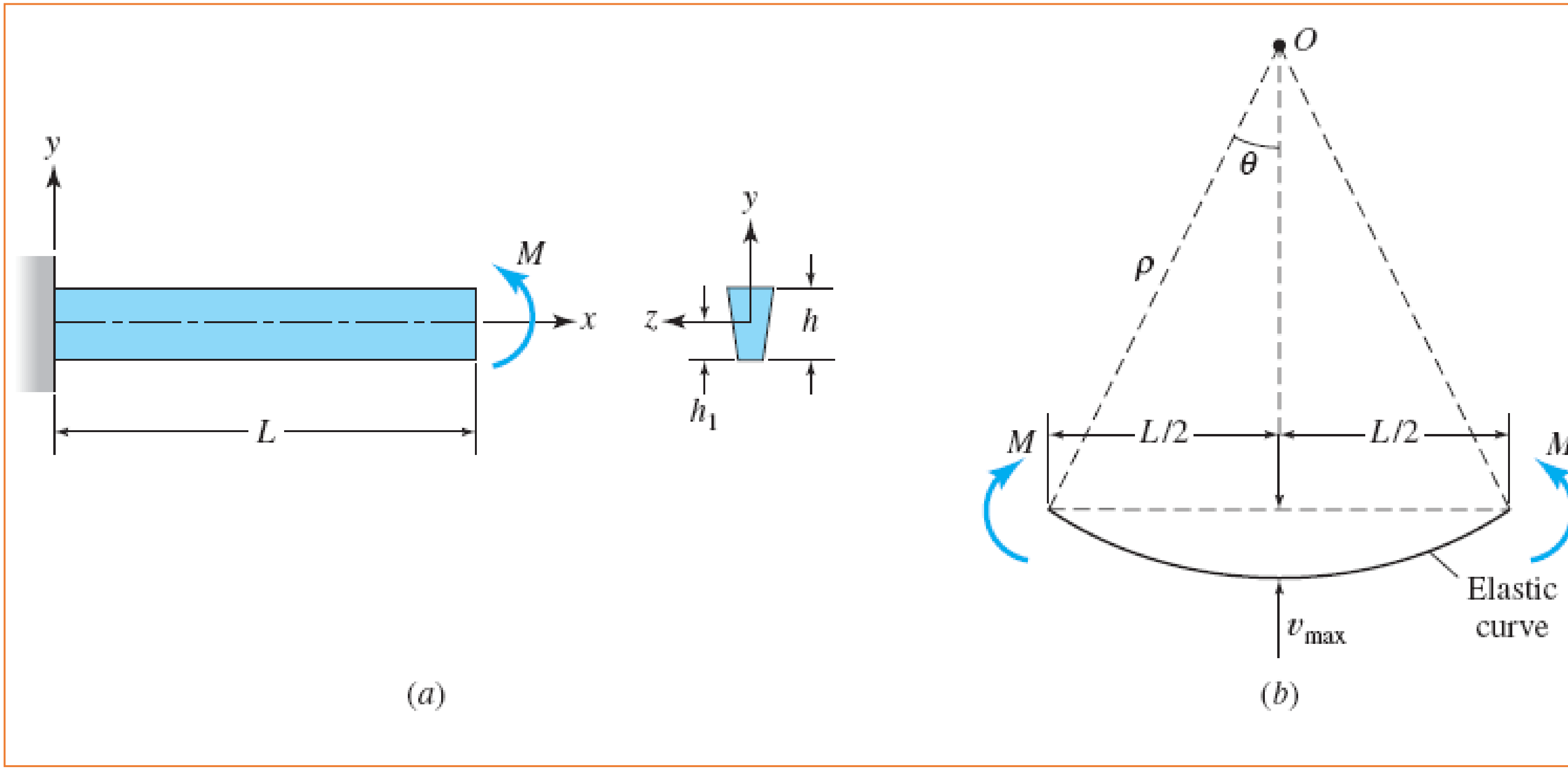
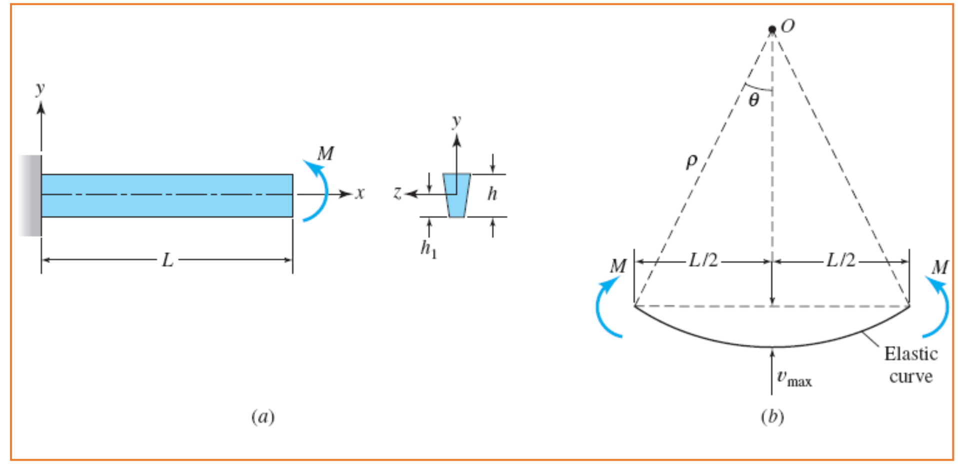



FIGURE 7.7 (a) Example of a  cantilever beam in pure bending; (b) deformed beam axis.




(1)

· $\rho = -\dfrac{y}{\varepsilon_x}$

$\varepsilon_x = 1200 \times 10^{-6}$

$y = -h = -3$ in

· $\kappa = \dfrac{1}{\rho}$ —

(2) $\theta$ ?

$\sin\theta = \dfrac{L}{2\rho}$

$\therefore v_{max} = \rho - \rho \cdot \cos\theta$

# Normal stresses (or bending stress)

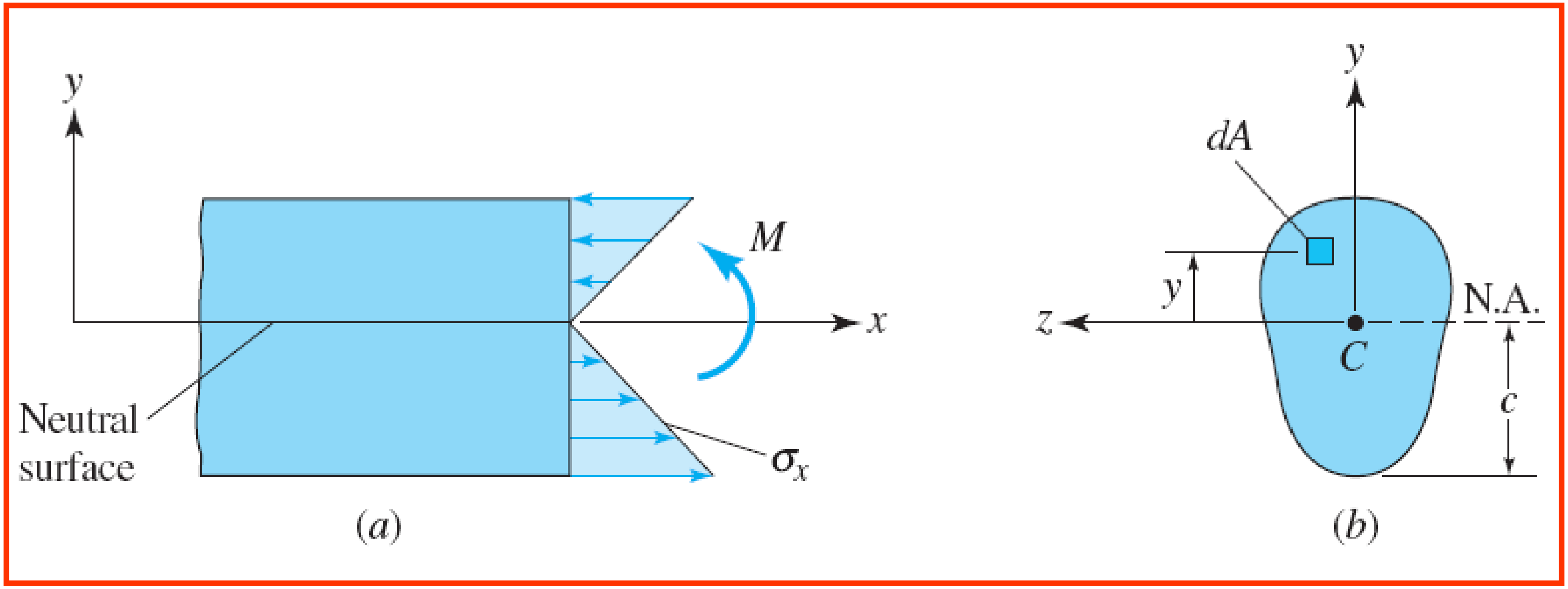



FIGURE 7.8 Distribution of bending stress in a beam.

# Equilibriums

$$\int \sigma_x dA = 0$$

force eq.

$$\int (-\sigma_x dA) y = M$$

Moment eq.

$$\sigma_x = E \cdot \varepsilon_y \overset{(-)}{=} E \cdot \kappa \cdot y$$

$$-E\kappa \int y \, dA = 0$$

$$E\kappa \int y^2 dA = M$$

$$\boxed{\int y\, dA = 0}$$

* From force eq.

- This term is the first moment of area that requires the neutral axis to pass through the centroidal axis

$$\boxed{I = \int y^2 dA}$$

- This is the second moment of area or **<u>moment of inertia</u>** equation which is a geometric property of the cross section

$$M = \kappa EI = \frac{EI}{\rho}$$

- This is the moment equation with the product $EI$ known as **flexural rigidity**

$\sigma_x = E \cdot \varepsilon_x$

$\varepsilon_x = -\kappa \cdot y$

$$\sigma_x = -\frac{My}{I}$$

$$\sigma_{\max} = \frac{M}{S}$$

- This is the **flexure formula** to calculate the bending stress
- In this formula, $S$ is called the **section modulus**

($S$ = I/c)

# Moment of inertia

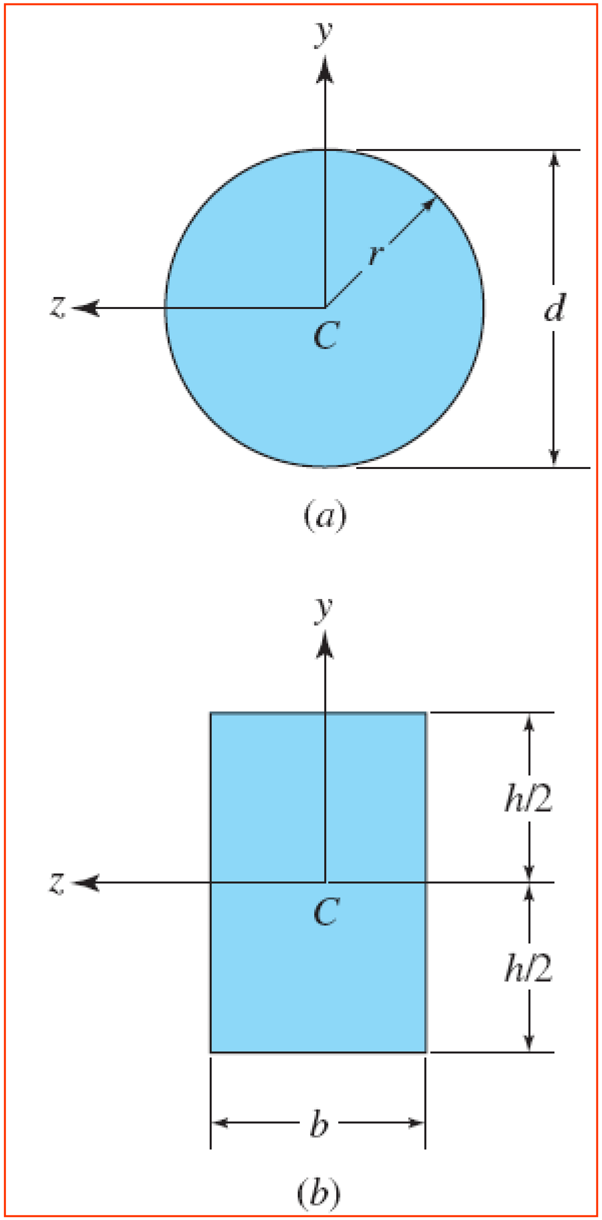



FIGURE 7.9 Doubly symmetric cross-sectional shapes.

The moment of inertia $I$, and section modulus $S$ for a rectangular section with the neutral axis parallel to the base *b* is:

$$I_z = \frac{bh^3}{12} \qquad S = \frac{bh^2}{6}$$

$S = \frac{I}{c}$

$c = \frac{h}{2}$

$I$ and $S$ for a solid circular section with radius *r* and diameter *d* are given by:

$$I = \frac{\pi r^4}{4} = \frac{\pi d^4}{64} \qquad S = \frac{\pi r^3}{4} = \frac{\pi d^3}{32}$$

# Dreterminination of the bending stress

- Draw free body diagram(s) and determine the support reaction forces if necessary
- Draw the bending moment diagram to determine the magnitude and location of the maximum bending moment
- Locate the centroid of the cross section using the principles of statics
- Determine the moment of inertia using the parallel axis theorem if necessary
- Determine the maximum value (+ or -) of the bending stress using the flexure formula

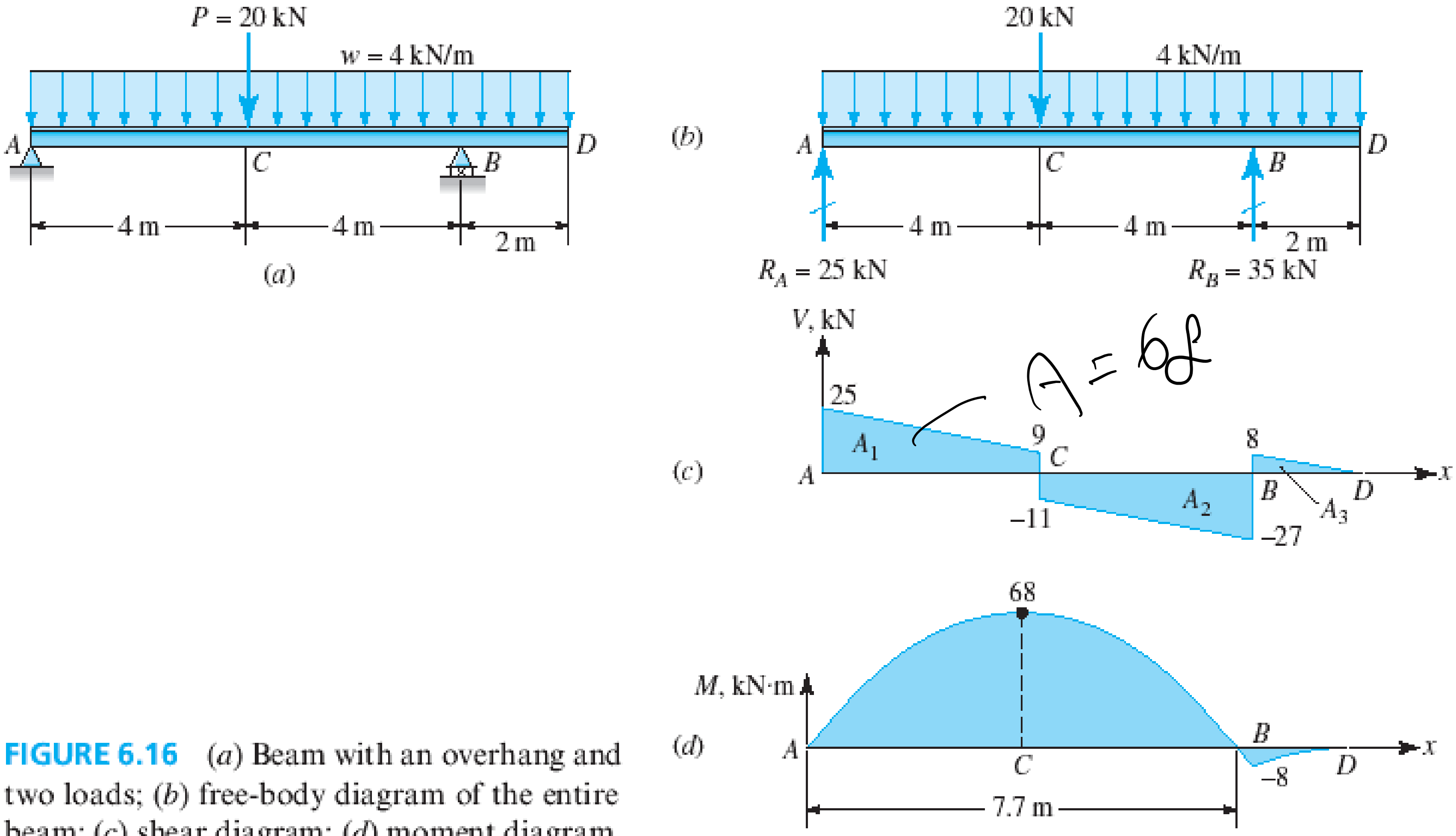



**FIGURE 6.16** (*a*) Beam with an overhang and two loads; (*b*) free-body diagram of the entire beam; (*c*) shear diagram; (*d*) moment diagram.

Based on the given $M_{max}$, find the diameter of the beam ACBD assuming that its cross section is solid round

# Centroid, area moment of inertia (appendix a)

$$\bar{x} = \frac{\Sigma A_i \bar{x}_i}{\Sigma A_i}$$

$$\bar{y} = \frac{\Sigma A_i \bar{y}_i}{\Sigma A_i}$$

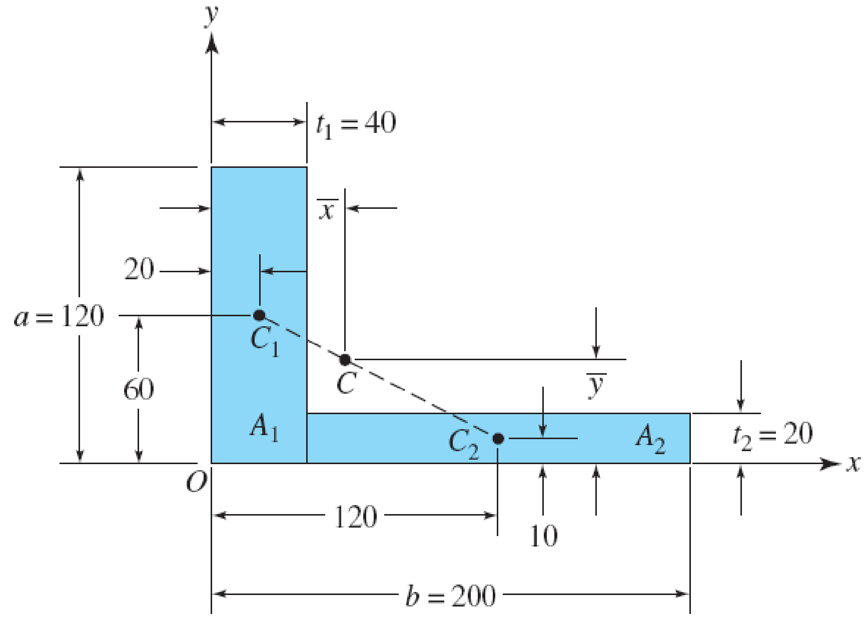



FIGURE A.5 An unequal-leg section.

**Composite areas**

# Centroid, area moment of inertia (appendix a)

## A.3 PARALLEL-AXIS THEOREM

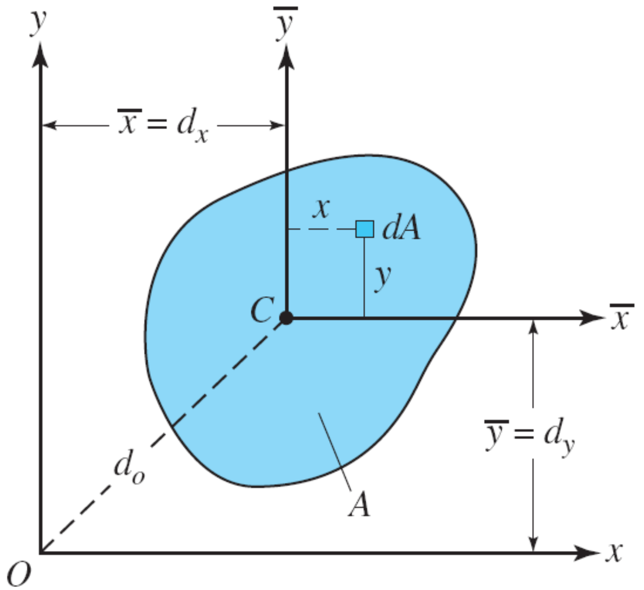



**FIGURE A.8** Parallel axes.

$$I_x = \int_A (y + \overline{y})^2 dA = \int_A y^2\, dA + 2\overline{y} \int_A y\, dA + \overline{y}^2 \int_A dA$$

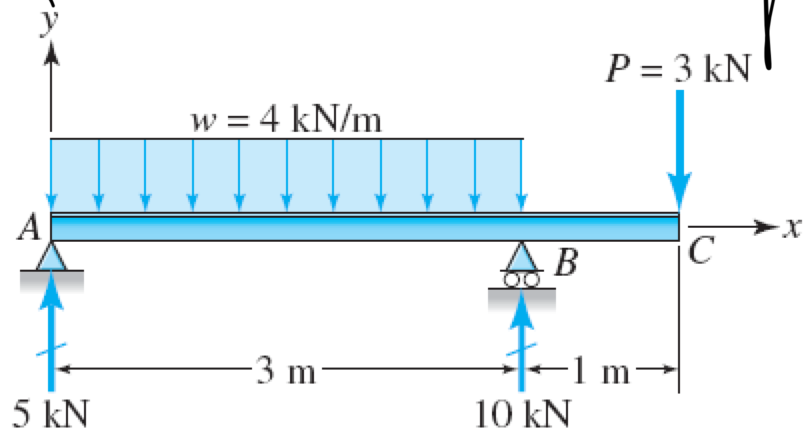
$$I_x = I_{\overline{x}} + A\overline{y}^2 = I_{\overline{x}} + A\, d_y^2$$

$$I_y = I_{\overline{y}} + A\overline{x}^2 = I_{\overline{y}} + A\, d_x^2$$

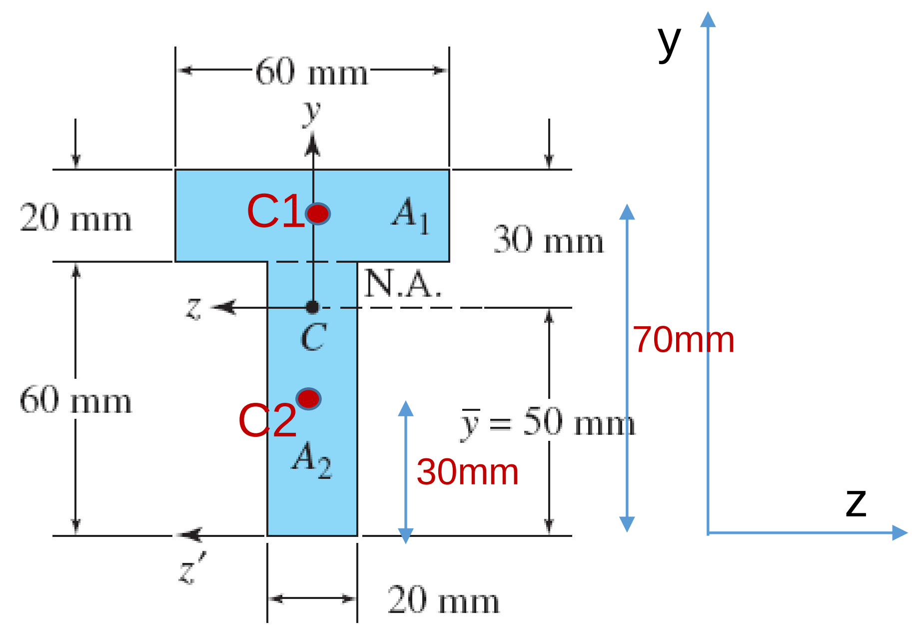
# Example:

Q: Max. Tensile & Comp. stress

Single- overhang beam with distributed load and T-cross section




(a)




(b)

$$\bar{y} = \frac{A_1\bar{y}_1 + A_2\bar{y}_2}{A_1 + A_2}$$

$A_1 = 20 \times 60$

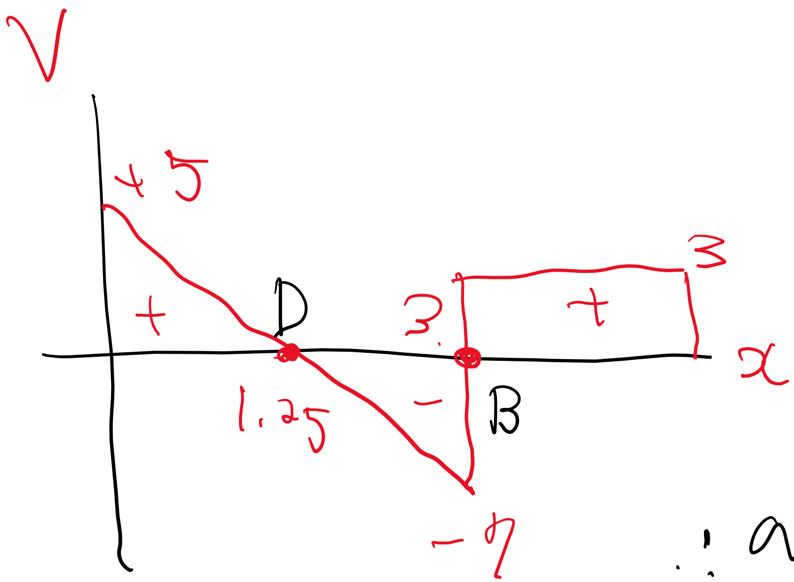
$A_2 = 60 \times 20$

$\bar{y}_1 = 70, \bar{y}_2 = 30$

$$I = \sum_i \left( \frac{1}{12} b_i h_i^3 + A_i d_i^2 \right) \qquad \bar{y} = 50$$

$b_1 = 60,\ h_1 = 20,\ A_1 = 60 \times 20,\ d_1 = 20$

$b_2 = 20,\ h_2 = 60,\ A_2 = 20 \times 60,\ d_2 = 20$




$M_D = 3.125\ kN \cdot m$

$M_B = -3\ kN \cdot m$

$\therefore\ \sigma = -My/I$

$\therefore$ at $x = 1.25\ m$

$\sigma_c = ?\ (y = 0.03) \qquad \sigma_T = ?\ (y = -0.05)$

at $x = 3.0$?

# Example: Two-plane bending problem

Calculate the diameter of the shaft based on maximum bending stress

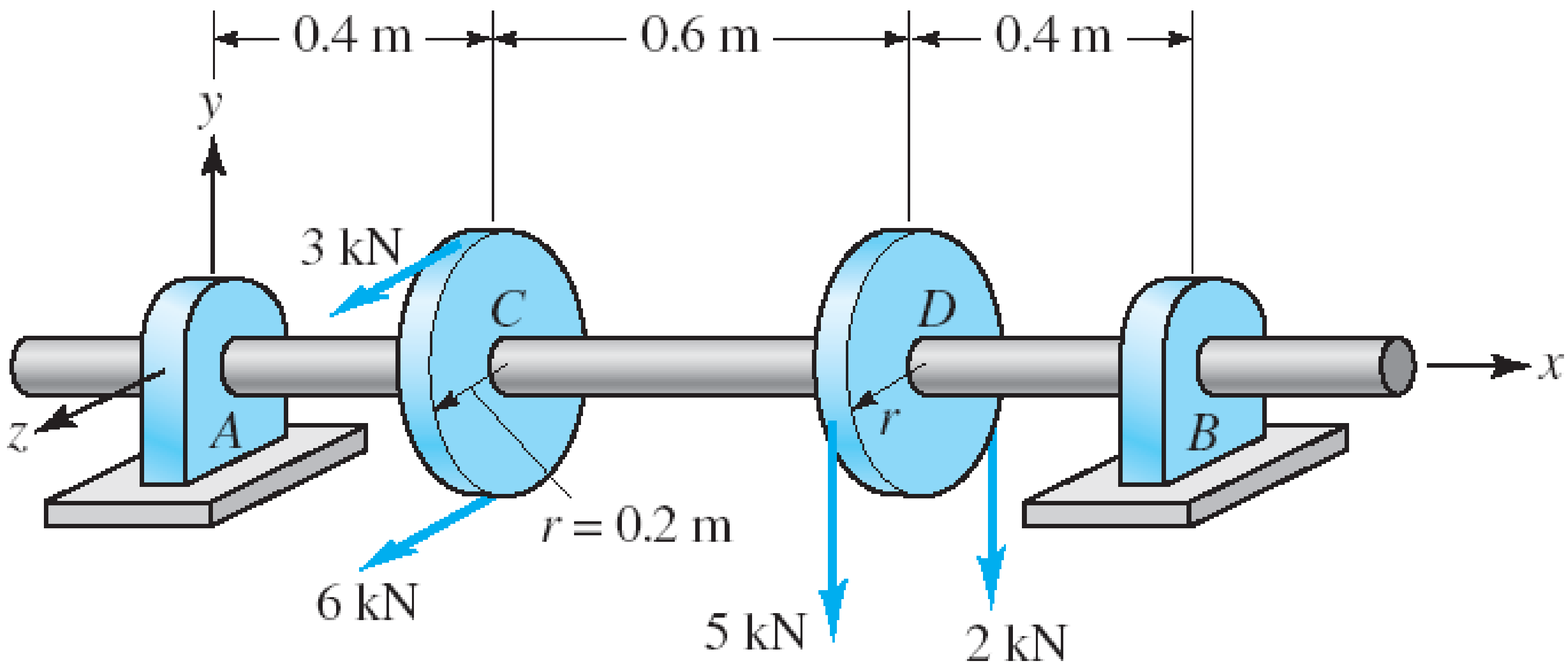



Figure P6.56

# Stress concentration in bending

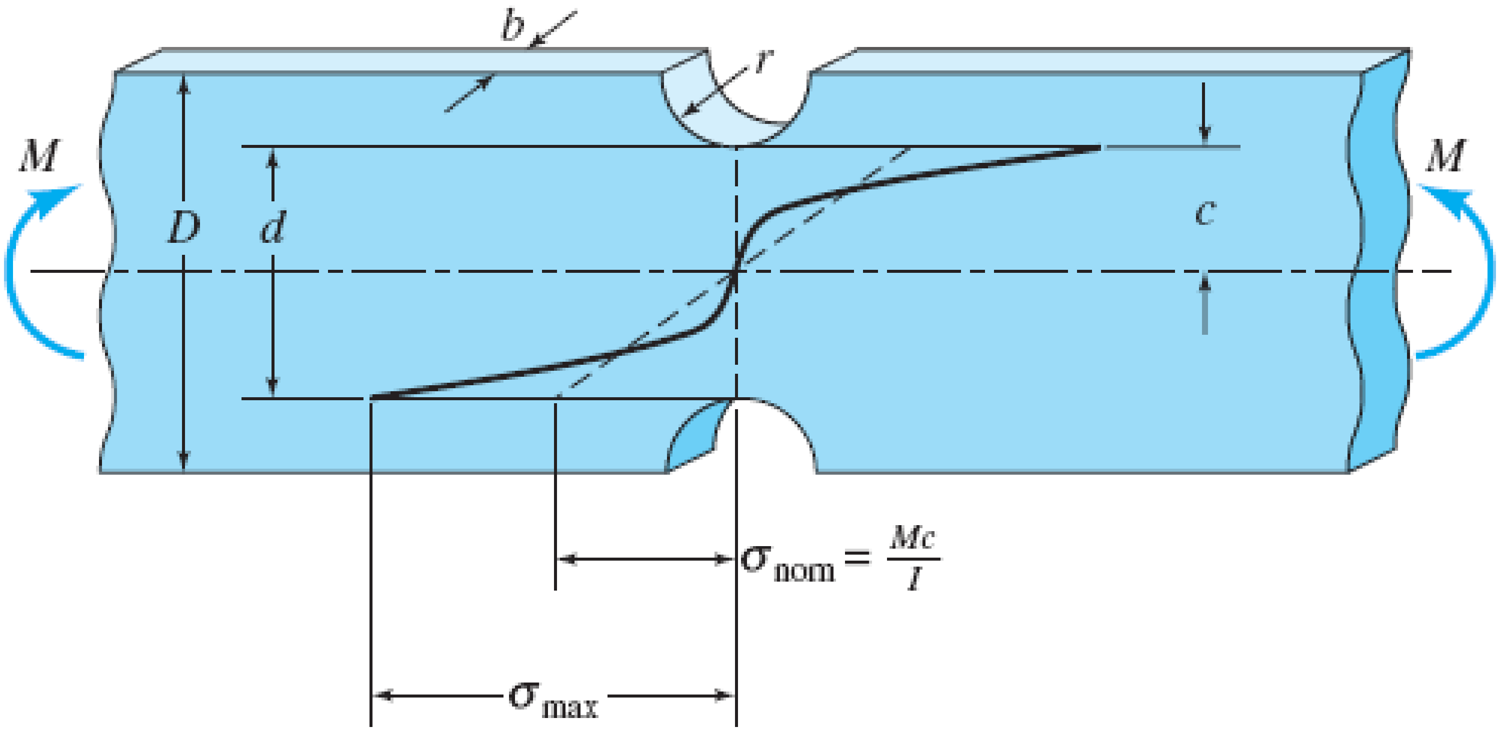



FIGURE 7.12 Stress distribution near grooves in a beam under pure bending.

$$\boxed{\sigma_{\max} = K\sigma_{\text{nom}} = K\frac{Mc}{I}}$$

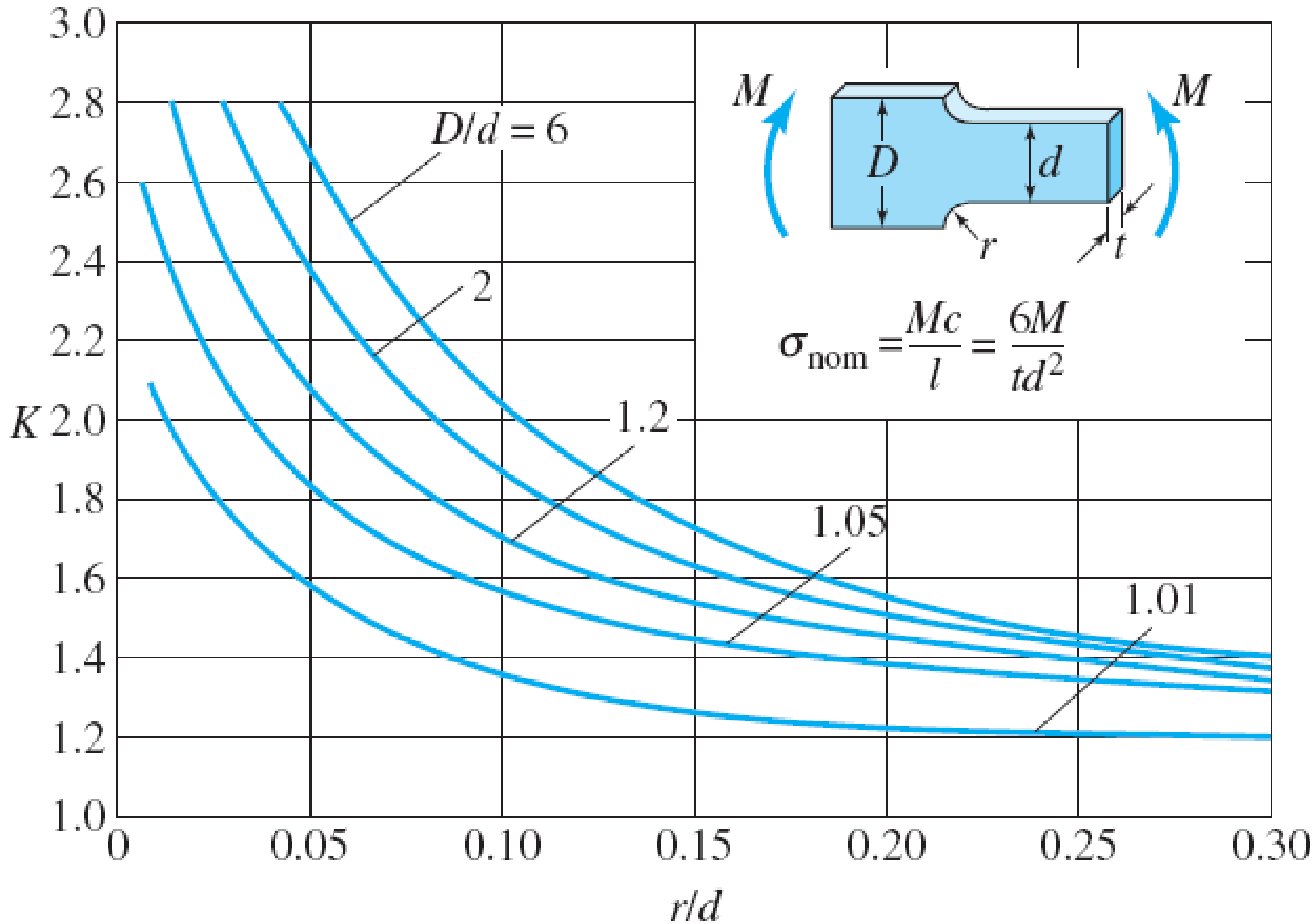



**FIGURE 7.13** Stress-concentration factor $K$ for a filleted flat bar in bending (Refs. 7.5 to 7.7).

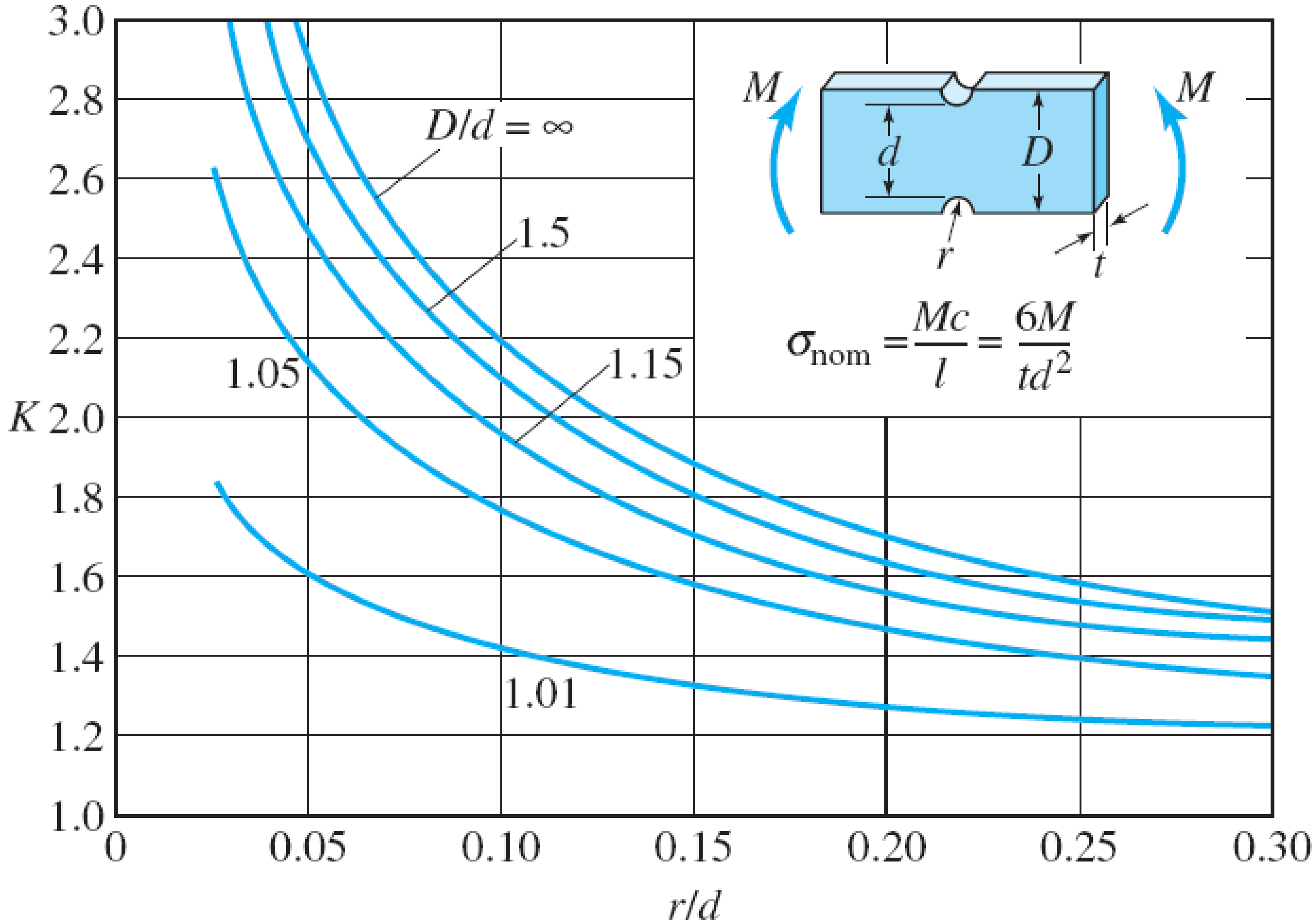
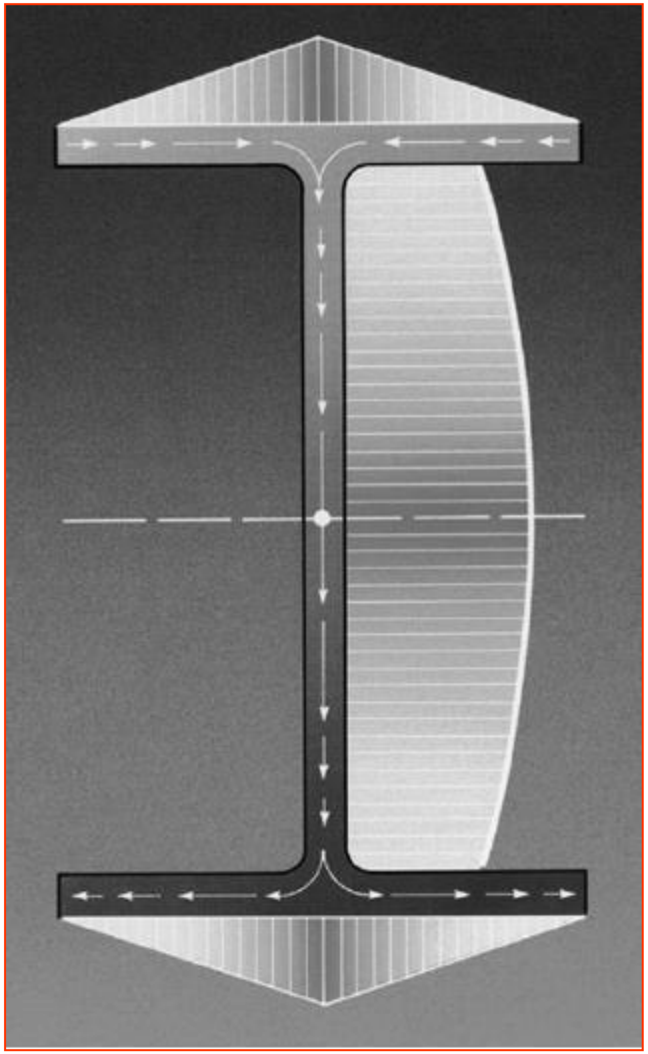



**FIGURE 7.14** Stress concentration factor $K$ for a grooved flat bar in bending (Refs. 7.5 to 7.7).

# Part B

## Shear and Bending
## (Bending under shear and moment)

# Shear Stresses in Beams: introduction



- This chapter deals with the distribution of the shear stresses and compare the magnitudes of the shear and bending stresses, and related design of beams

**Shear flow in a wide-flange beam**

- Beams subjected to non-uniform bending, transverse (or vertical shear) forces V, in addition to bending moments M, are associated to the beam cross section.

- The normal stresses with the bending moments (along beam axis) can be determined by the flexure formula (Part A).

- The vertical shearing stress $\tau_{xy}$ at any point on the cross section must be equal to the horizontal shear stress $\tau_{xy}$ at the same point: $\tau_{xy} = \tau_{yx}$.

- Therefore, the horizontal shear stresses must also exist in any beam subjected to a transverse loading

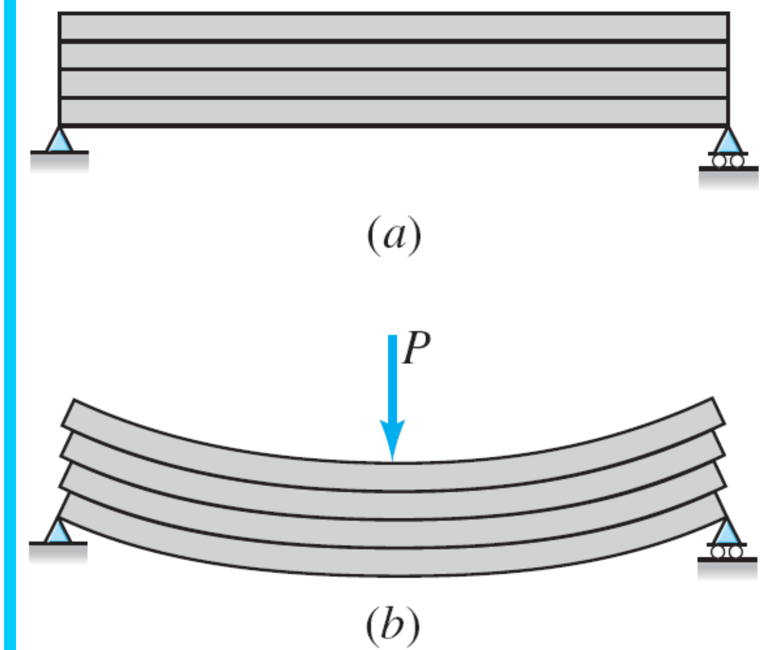
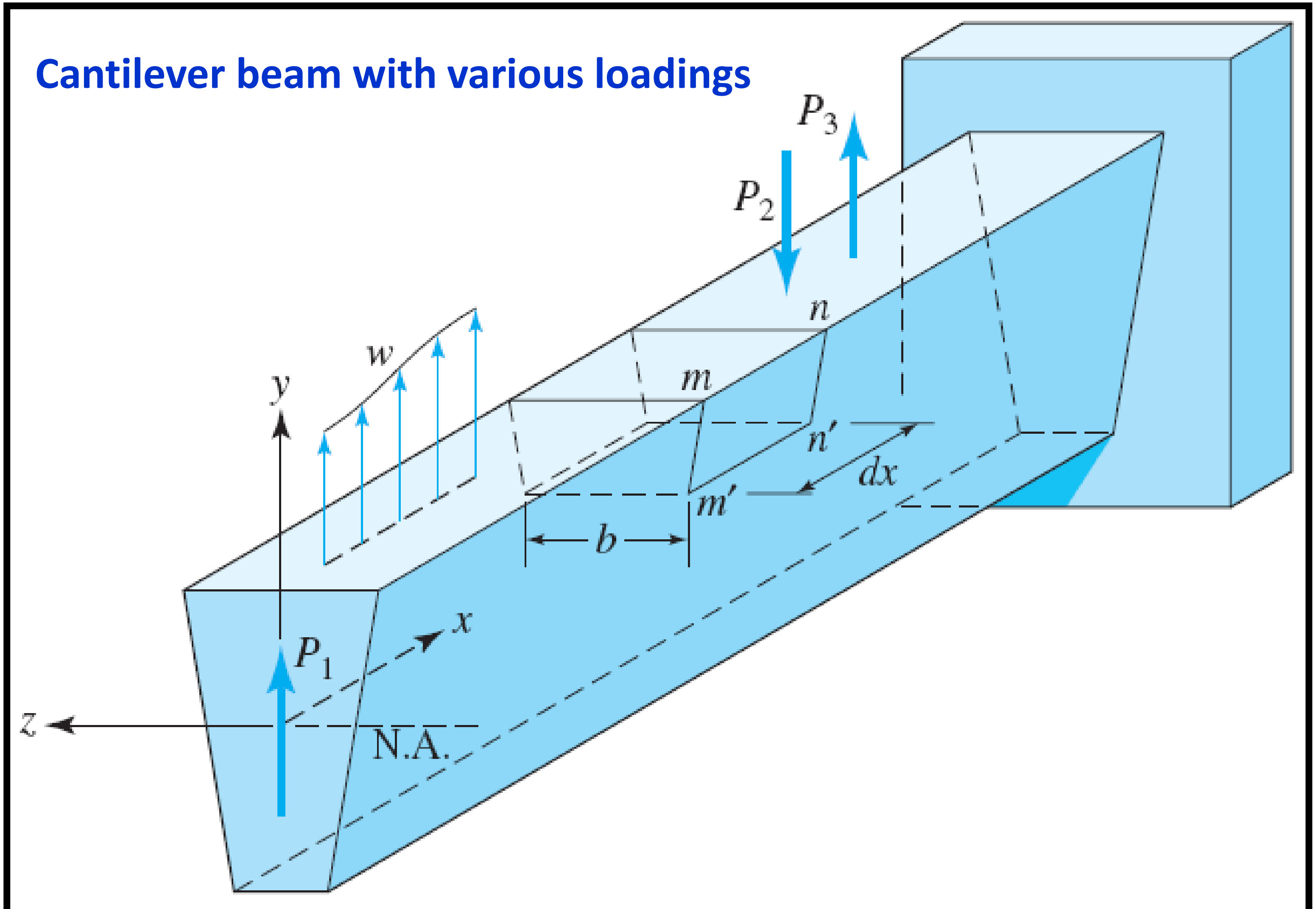
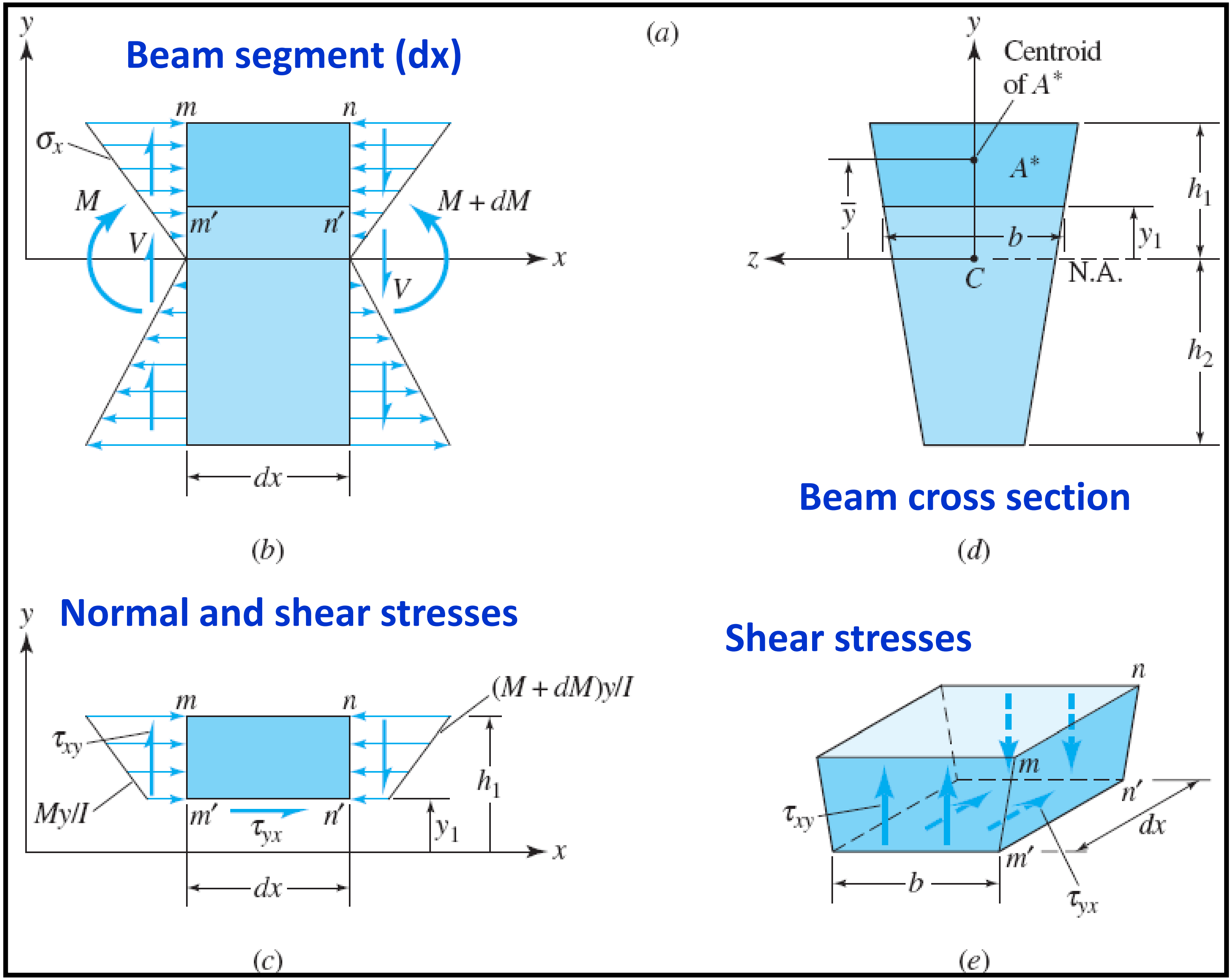


**FIGURE 7.16** Bending of four nonbonded planks.

# Derivation of shear stresses





Beam segment (dx)
(a)
y
m
n
$\sigma_x$
M
M + dM
V
m′
n′
V
x
dx
(b)
y
Centroid of $A^*$
$A^*$
$h_1$
$\bar{y}$
b
$y_1$
z
C
N.A.
$h_2$
Beam cross section
(d)
Normal and shear stresses
y
m
n
(M + dM)y/I
$\tau_{xy}$
$h_1$
My/I
m′
$\tau_{yx}$
n′
$y_1$
x
dx
(c)
Shear stresses
n
m
n′
$\tau_{xy}$
dx
m′
b
$\tau_{yx}$
(e)

Equil. of horizontal forces

$$\int_{A^*} \frac{My}{I}\,dA - \int_{A^*} \frac{(M+dM)\,y}{I}\,dA + \tau_{yx}\, b \cdot dy = 0$$

$$\therefore \quad \tau_{yx} = \frac{1}{Ib} \int_{A^*} \frac{dM}{dx}\, y\, dA$$

$$\frac{dM}{dx} = V, \quad \tau_{yx} = \tau_{xy}$$

$$\therefore \quad \tau_{xy} = \frac{V}{Ib} \underline{\underline{\int_{A^*} y\, dA}} \quad Q$$

# Derivation of shear stresses – important formula

$$\tau_{xy} = \frac{V}{Ib}\int_{A^*} y\,dA = \frac{VQ}{Ib}$$

**Shear formula**

$$Q = \int_{A^*} y\,dA = A^*\overline{y}$$

**First moment of area**

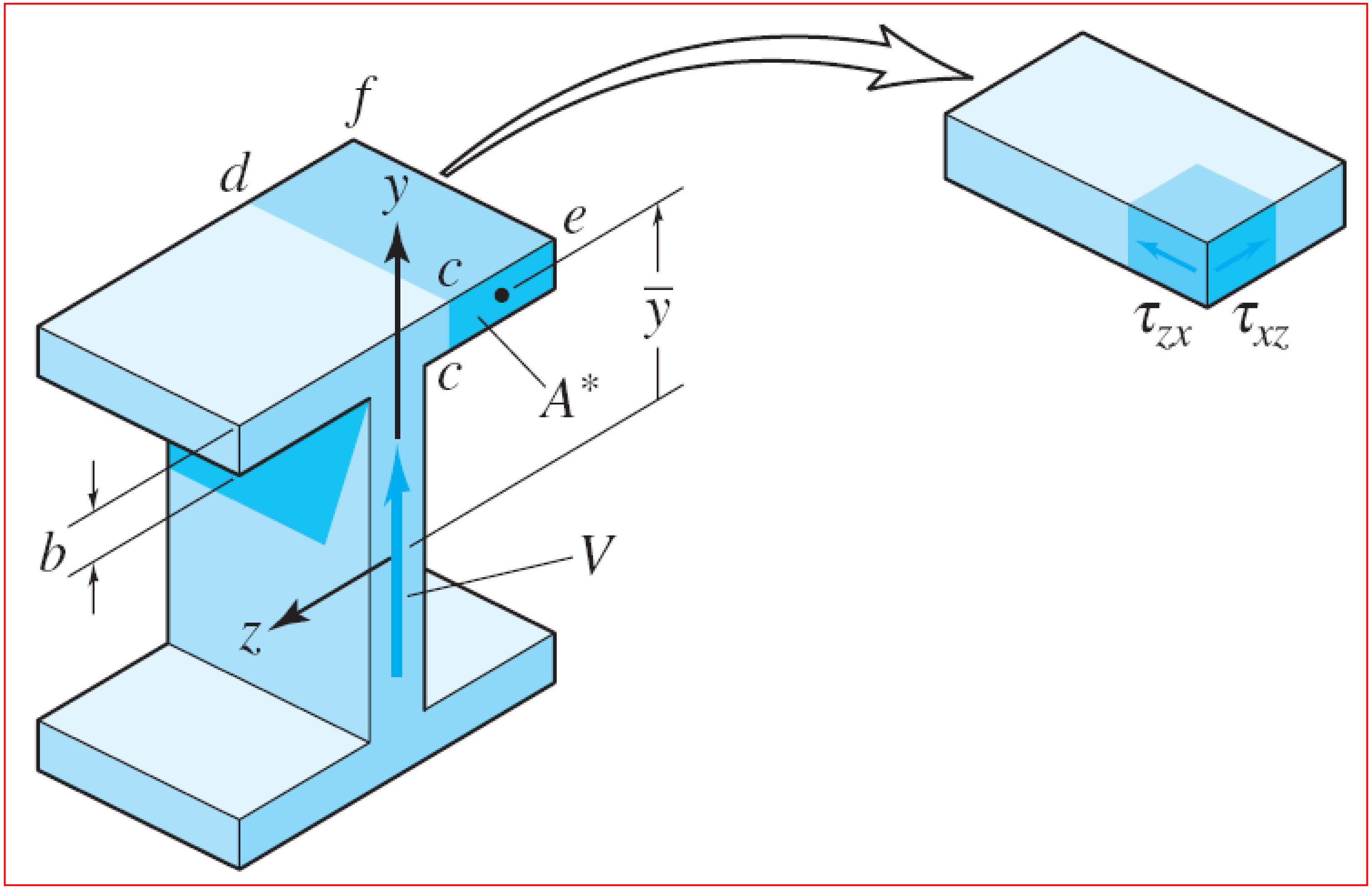
$$q = \frac{VQ}{I}$$

**Shear flow**

$\leftarrow$ $\underline{q = \tau_{xy} \cdot b}$

(per unit length)

# The shear stress in thin-walled beams



# Shear Stress Distribution in Rectangular Beams

$$\tau_{xy} = \frac{V}{2I}\left(\frac{h^2}{4} - y_1^2\right)$$

$$\tau_{max} = \frac{Vh^2}{8I} = \frac{Vh^2}{8bh^3/12} = \frac{3}{2}\frac{V}{A}$$

$$\text{"}\frac{V}{A} = \tau_{avg}\text{"}$$

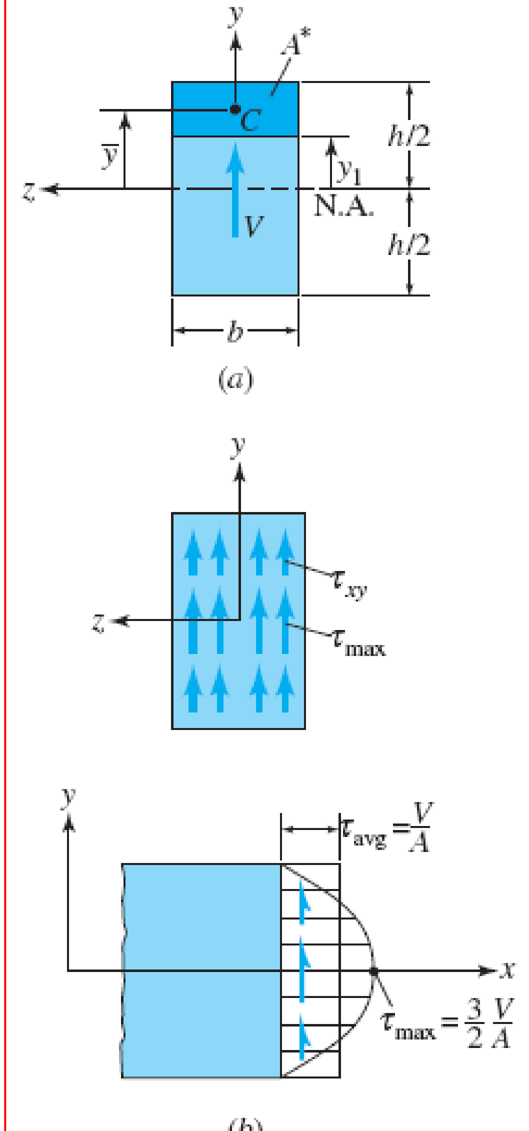



**FIGURE 7.20** Shear stresses in a beam of rectangular cross section.

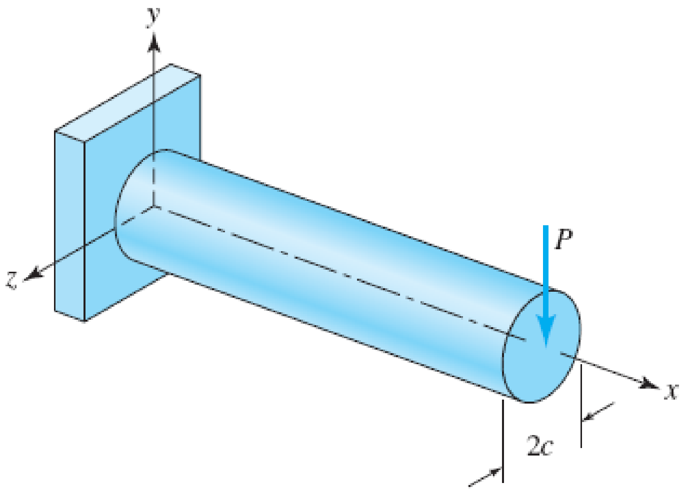
$$\tau_{xy} = \frac{VQ}{Ib} = \frac{V}{Ib}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{y_1}^{\frac{h}{2}} y\,dy\,dz$$

$$= \frac{V}{2I}\left(\frac{h^2}{4} - y_1^2\right)$$

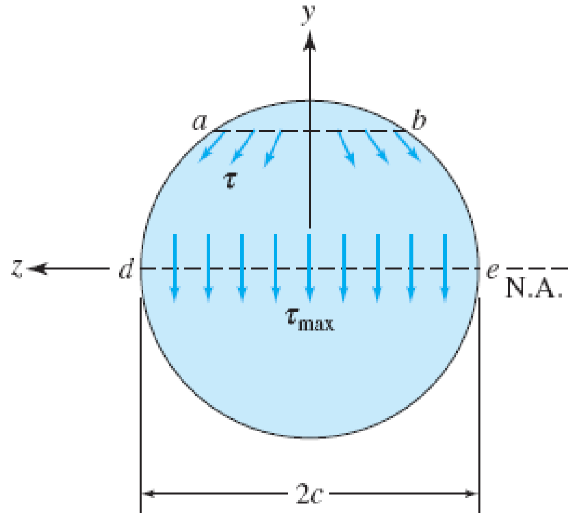
$$\tau_{xy,max}\,(y_1 = 0)$$

$$\text{" } I = \frac{1}{12} b h^3 \text{ "}$$

# Shear Stresses in Beams of Circular Cross Section




(a)

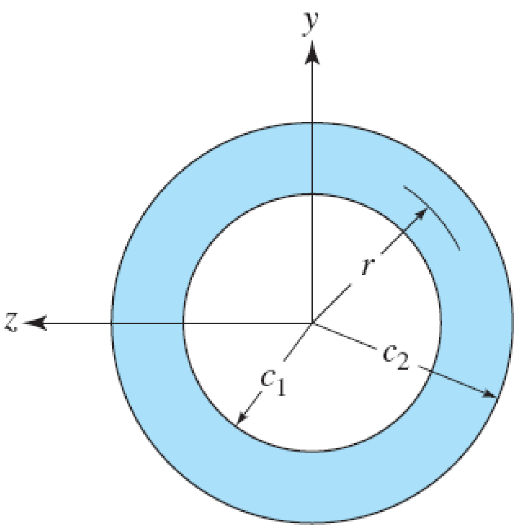



"Approximate solution"

$$\tau_{max} = \frac{VQ}{Ib} = \frac{4V}{3\pi c^2} = \frac{4V}{3A}$$

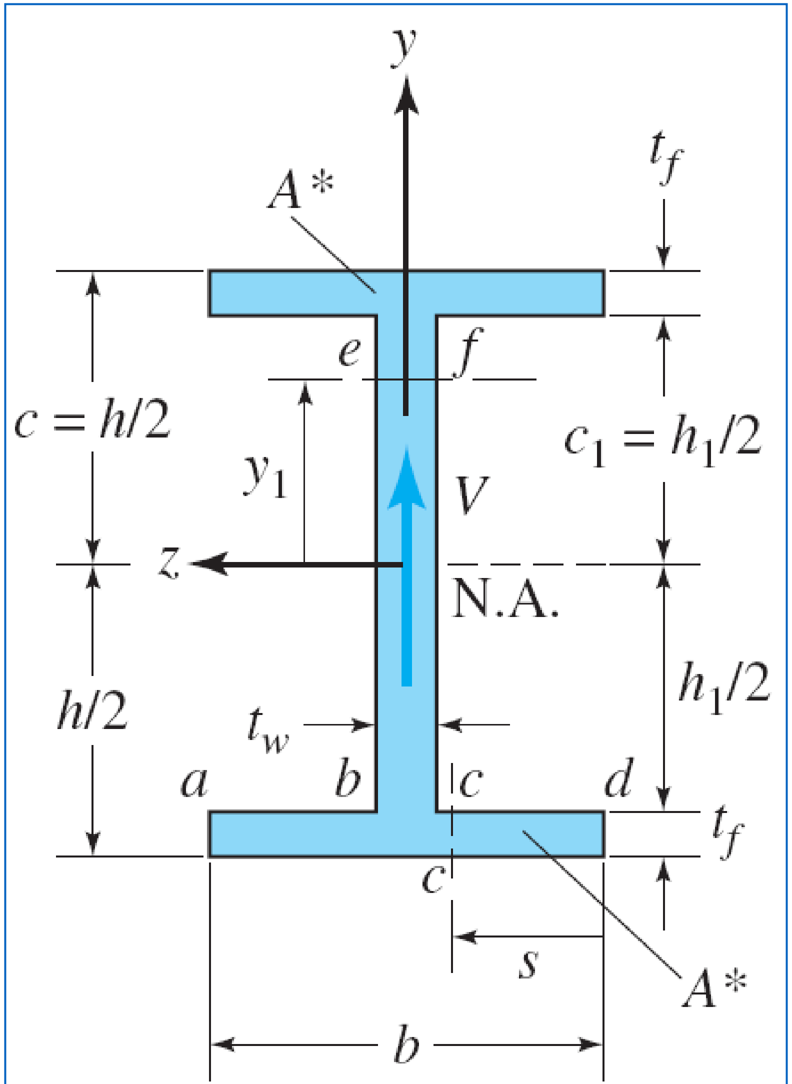
$Q = A^* \bar{y} = \frac{\pi c^2}{2}\left(\frac{4c}{3\pi}\right)$

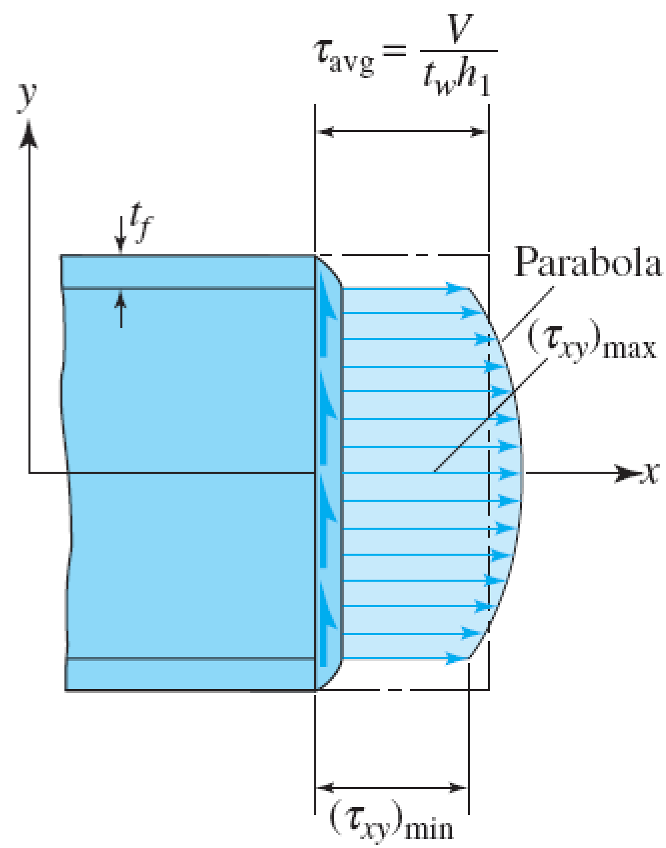
$I = \frac{1}{4}\pi c^4$




$$\tau_{max} = \frac{VQ}{Ib} = \frac{4V}{3A}\,\frac{c_2^2 + c_2c_1 + c_1^2}{c_2^2 + c_1^2}$$

# Shear Stress Distribution in Flanged Beams




(*b*)

**FIGURE 7.24** (*a*) Shear force in a wide-flange beam; (*b*) shear stress distribution at a cross section.

- For the wide flange beam, $I$ for the entire cross section about the neutral axis is given by:

$$I = \frac{b(2c)^3}{12} - \frac{(b - t_w)(2c)^3}{12} = \frac{2}{3}\left(bc^3 - bc_1^3 + t_w c_1^3\right)$$

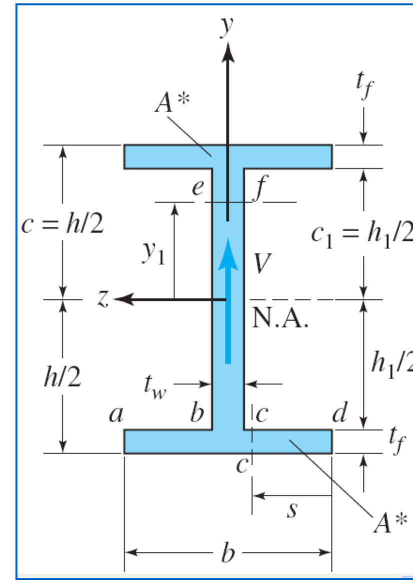
- Shear stress in the web sections is given by:

$$(\tau_{xy})_{\max} = \frac{V}{2It_w}\left(bc^2 - bc_1^2 + t_w c_1^2\right)$$

$$(\tau_{xy})_{\min} = \frac{Vb}{2It_w}\left(c^2 - c_1^2\right)$$

$0 \le y_1 \le c_1$

$$\tau_{xy} = \frac{VQ}{Ib} = \frac{V}{I t_w}\left(A_f^* \bar{y}_f + A_w^* \bar{y}_w\right) \quad — (\star)$$




$$\left(\begin{array}{l} A_f^* = b(c - c_1), \quad A_w^* = t_w(c_1 - y_1) \\ \bar{y}_f = c_1 + \dfrac{c - c_1}{2}, \quad \bar{y}_w = y_1 + \dfrac{c_1 - y_1}{2} \end{array}\right.$$

$$(\star) \quad \tau_{xy} = \frac{V}{2 I t_w}\left[b(c^2 - c_1^2) + t_w(c_1^2 - y_1^2)\right]$$

$$\tau_{xy,max} \leftarrow y_1 = 0, \quad \tau_{xy,min} \leftarrow y_1 = \pm c_1$$

$$\underline{c_1 \le y_1 \le c}$$

$$\tau_{xy} = \frac{V}{Ib}\left[b(c-y_1)\left(y_1 + \frac{c-y_1}{2}\right)\right]$$

$$= \frac{V}{2I}(c^2 - y_1^2) \ll \underline{\underline{\tau_{xy,web}}}$$

$$\therefore \tau_{xy} \doteq \frac{V}{A_{web}}$$

(cf)

$$\tau_{xz} = \frac{VQ}{It_f} = \frac{V}{I}\left(c - \frac{t_f}{2}\right) \cdot s$$

$$A^* = s t_f$$

$$\bar{y} = c - \frac{t_f}{2}$$

$$\tau_{xz} = \frac{V}{I}\left(c - \frac{t_f}{2}\right) \cdot s$$

- Shear stress in the flange sections

$$\boxed{\tau_{xy} = \frac{V}{2I}\left(c^2 - y_1^2\right)}$$

$$\boxed{\tau_{avg} = \frac{V}{A_{web}}}$$

$s = \frac{b}{2}$

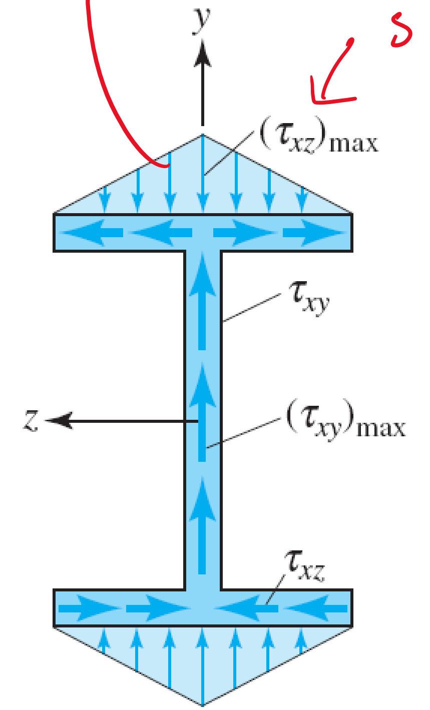
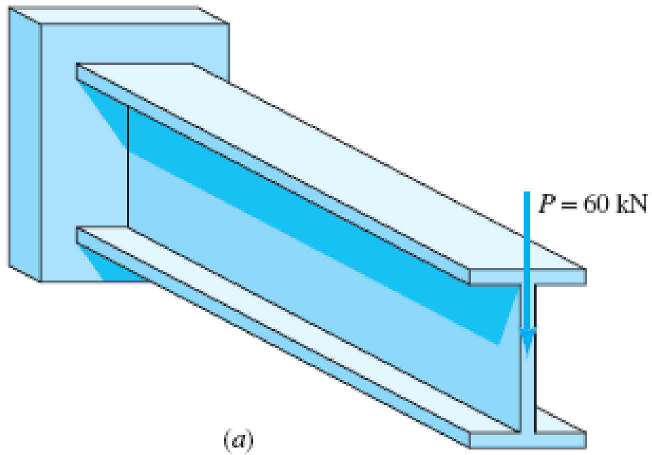



**FIGURE 7.25** Shear flow in a flanged beam.



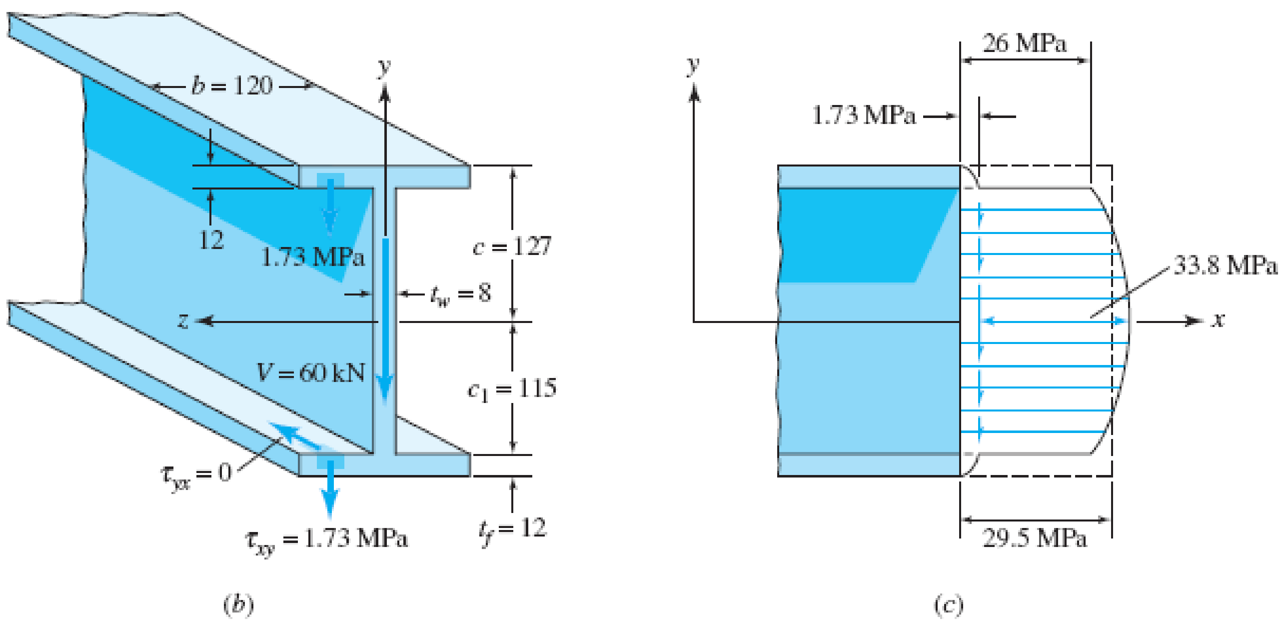



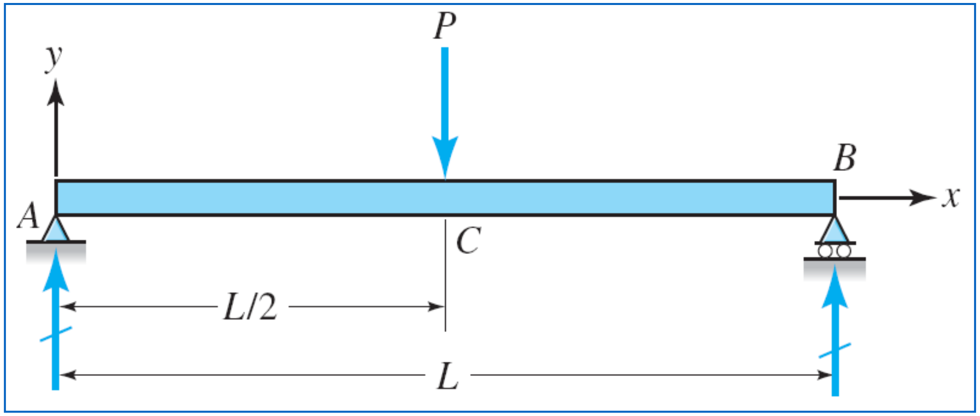

FIGURE 7.26 (*a*) A cantilever beam with a vertical load $P$; (*b*) shear force and shear stresses at a cross section; (*c*) shear stress distribution at a cross section.

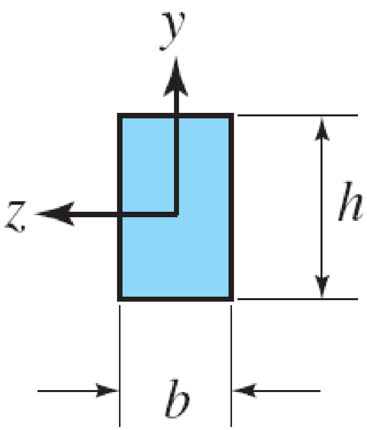
# Comparison of Shear and Bending Stresses

$$\tau_{\max} = \frac{3}{2}\frac{V}{A} = \frac{3}{2}\frac{P/2}{bh} = \frac{3}{4}\frac{P}{bh}$$

$$\frac{\tau_{\max}}{\sigma_{\max}} = \frac{1}{2}\left(\frac{h}{L}\right)$$

$$\sigma_{\max} = \frac{Mc}{I} = \frac{(PL/4)(h/2)}{bh^3/12} = \frac{3}{2}\frac{PL}{bh^2}$$




- If $L = 10h$ ("long" beam), then this ratio is only 1/20, which means that $\tau_{max}$ is only 5% of $\sigma_{max}$

# Design of Prismatic Beams

1. *Evaluate the modes of possible failure*. It is assumed that failure results from yielding or from fracture, and flexural stress is considered to be most closely associated with structural damage.
2. *Determine the relationships between load and stress*. The significant value of the bending stress is $\sigma = M_{max}/S$.
3. *Determine the maximum usable value of stress*. The maximum usable value of $\sigma$ to avoid failure, $\sigma_{max}$, is the yield strength $\sigma_y$ or the ultimate strength $\sigma_u$.
4. *Select the factor of safety*. A factor of safety $n_s$ is applied to $\sigma_{max}$ to obtain the allowable stress: $\sigma_{all} = \sigma_{max}/n_s$. The required **section modulus** of a beam is then

$$\boxed{S = \frac{M_{max}}{\sigma_{all}}} \tag{7.28}$$