Chap 8. Second-order Plastic Hinge Analysis

8.1 INTRODUCTION

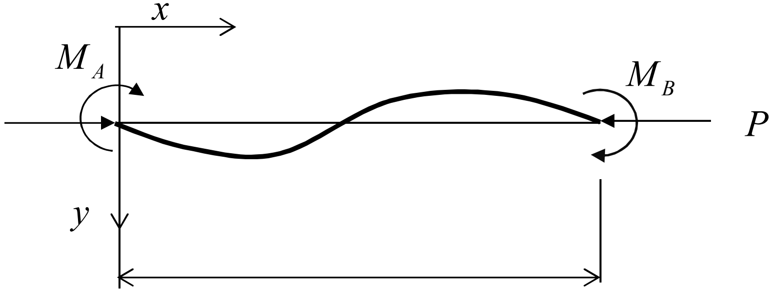
- Plastic Zone and Plastic Hinge
- Fiber Analysis
- Residual stresses, geometric imperfections, material strain hardening
- Zero length of plastic hinge
- Equilibrium based on deformed structural geometry

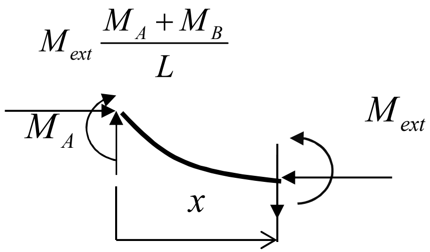
8.2 Modeling of Elastic Beam-Column Element

- 8.2.1 Assumptions

- initially straight and prismatic
- compact section
- sufficiently braced
- Small member deformation but large rigid body displacement
- Axial shortening due to member curvature bending is neglected

8.2.2 Stability Functions





$$EIy''+Py=\frac{M_A+M_B}{L}x-M_A$$

$$\rho=L\sqrt{\frac{P}{EI}}$$

$$y=\mathrm{A}\sin\rho x+\mathrm{B}\cos\rho x+\frac{M_A+M_B}{EIL\rho^2}x-\frac{M_A}{EI\rho^2}$$

Since $y(0)=y(L)=0$

$$\mathrm{A}=-\frac{1}{EI\rho^2\sin\rho L}\left(M_A\cos\rho L+M_B\right)$$

$$B=\frac{M_A}{EI\rho^2}$$

$$y=-\frac{1}{EI\rho^2}\left[\frac{\cos\rho L}{\sin\rho L}\sin\rho x-\cos\rho x-\frac{x}{L}+1\right]M_A-\frac{1}{EI\rho^2}\left[\frac{1}{\sin\rho L}\sin\rho x-\frac{x}{L}\right]M_B$$

$$\frac{dy}{dx}=-\frac{1}{EI\rho}\left[\frac{\cos\rho L}{\sin\rho L}\cos\rho x+\sin\rho x-\frac{1}{\rho L}\right]M_A-\frac{1}{EI\rho}\left[\frac{1}{\sin\rho L}\cos\rho x-\frac{1}{\rho L}\right]M_B$$

$$\begin{cases}\theta_A=\dfrac{dy}{dx}(0)\\[2mm]\theta_B=\dfrac{dy}{dx}(L)\end{cases}$$

$$P=\frac{EA}{L}e$$

$$\begin{Bmatrix}M_A\\M_B\\P\end{Bmatrix}=\frac{EI}{L}\begin{bmatrix}S_1&S_2&0\\S_2&S_1&0\\0&0&A/I\end{bmatrix}\begin{Bmatrix}\theta_A\\\theta_B\\e\end{Bmatrix}$$

$$S_1=\begin{cases}\dfrac{\rho\sin\rho-\rho^2\cos\rho}{2-2\cos\rho-\rho\sin\rho} & \text{for}\quad P<0\\[3mm]\dfrac{\rho^2\cosh\rho-\rho\sinh\rho}{2-2\cosh\rho} & \text{for}\quad P>0\end{cases}$$

$$S_2=\begin{cases}\dfrac{\rho^2-\rho\sin\rho}{2-2\cos\rho-\rho\sin\rho} & \text{for}\quad P<0\\[3mm]\dfrac{\rho\sinh\rho-\rho^2}{2-2\cosh\rho+\rho\sinh\rho} & \text{for}\quad P>0\end{cases}$$

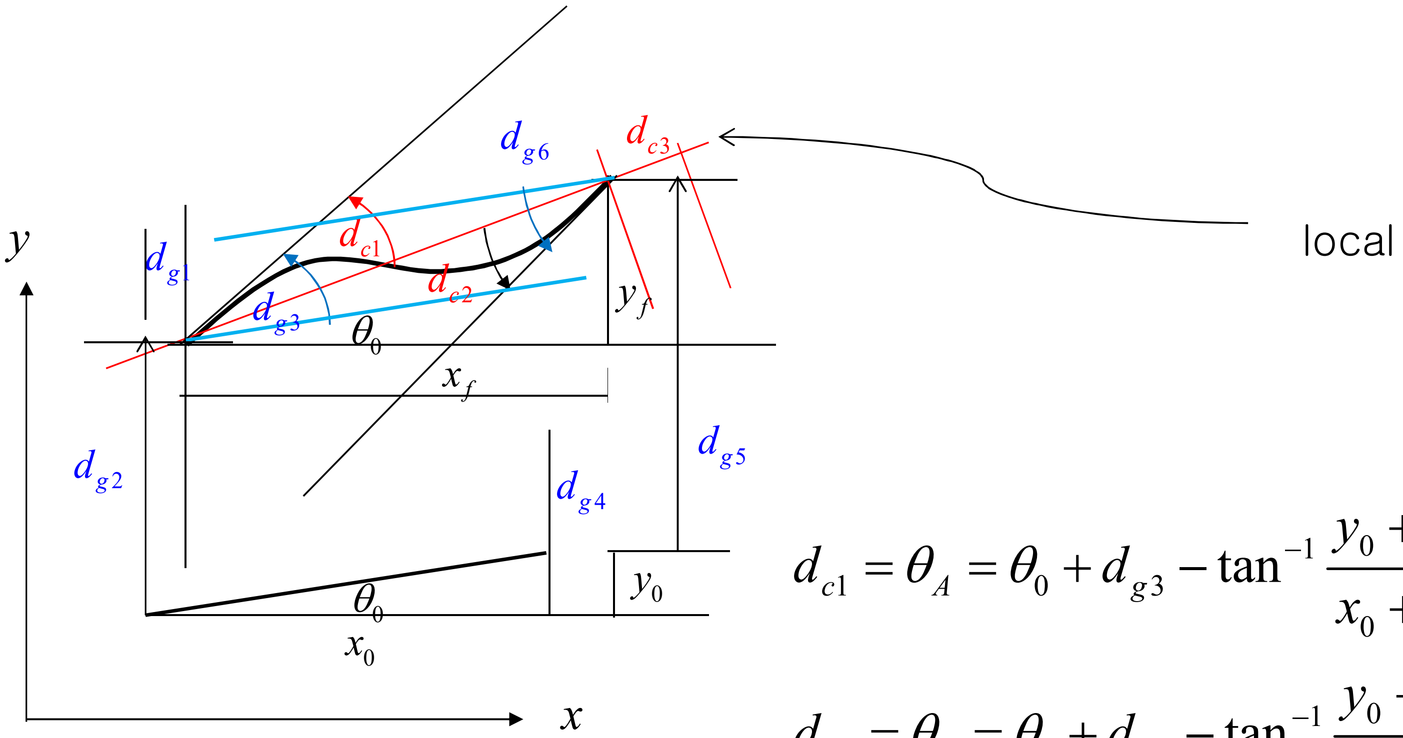
8.2.3 Tangent Stiffness Relationship

Element force-displacement equation

$$\begin{Bmatrix} \dot{M}_A \\ \dot{M}_B \\ \dot{P} \end{Bmatrix} = \frac{EI}{L} \begin{bmatrix} S_1 & S_2 & 0 \\ S_2 & S_1 & 0 \\ 0 & 0 & A/I \end{bmatrix} \begin{Bmatrix} \dot{\theta}_A \\ \dot{\theta}_B \\ \dot{e} \end{Bmatrix}$$

Vector notation

$$\dot{\mathbf{f}}_c = \mathbf{k}_c \dot{\mathbf{d}}_c$$



$$d_{c1} = \theta_A = \theta_0 + d_{g3} - \tan^{-1}\frac{y_0 + d_{g5} - d_{g2}}{x_0 + d_{g4} - d_{g1}}$$

$$d_{c2} = \theta_B = \theta_0 + d_{g6} - \tan^{-1}\frac{y_0 + d_{g5} - d_{g2}}{x_0 + d_{g4} - d_{g1}}$$

$$d_{c3} = \frac{\left(2x_0 + d_{g4} - d_{g1}\right)\left(d_{g4} - d_{g1}\right) + \left(2y_0 + d_{g5} - d_{g2}\right)\left(d_{g5} - d_{g2}\right)}{L_f + L}$$

Global and local displacement Relationship

$$\begin{Bmatrix} \dot{\theta}_A \\ \dot{\theta}_B \\ \dot{e} \end{Bmatrix} = \begin{bmatrix} -\sin\theta/L & \cos\theta/L & 1 & \sin\theta/L & -\cos\theta/L & 0 \\ -\sin\theta/L & \cos\theta/L & 0 & \sin\theta/L & -\cos\theta/L & 1 \\ -\cos\theta & -\sin\theta & 0 & \cos\theta & \sin\theta & 0 \end{bmatrix} \begin{Bmatrix} \dot{d}_{g1} \\ \dot{d}_{g2} \\ \dot{d}_{g3} \\ \dot{d}_{g4} \\ \dot{d}_{g5} \\ \dot{d}_{g6} \end{Bmatrix}$$

$$\dot{\mathbf{d}}_c = \mathbf{T}_{cg}\dot{\mathbf{d}}_g$$

Relationship between global and local displacement

$$\begin{Bmatrix} f_{g1} \\ f_{g2} \\ f_{g3} \\ f_{g4} \\ f_{g5} \\ f_{g6} \end{Bmatrix} = \begin{bmatrix} -\sin\theta / L & -\sin\theta / L & -\cos\theta \\ \cos\theta / L & \cos\theta / L & -\sin\theta \\ 1 & 0 & 0 \\ \sin\theta / L & \sin\theta / L & \cos\theta \\ -\cos\theta / L & -\cos\theta / L & \sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} M_A \\ M_B \\ P \end{Bmatrix}$$

$$\mathbf{f}_g = \mathbf{T}^T{}_{cg} \mathbf{f}_c$$

Taking derivative

$$\dot{\mathbf{f}}_g = \mathbf{T}^T{}_{cg} \dot{\mathbf{f}}_c + \dot{\mathbf{T}}^T_{cg} \mathbf{f}_c$$

$$\dot{\mathbf{f}}_c = \mathbf{k}_c \dot{\mathbf{d}}_c$$

$$\dot{\mathbf{f}}_g = \mathbf{T}^T{}_{cg} \dot{\mathbf{f}}_c + \dot{\mathbf{T}}^T_{cg} \mathbf{f}_c$$

$$\dot{\mathbf{f}}_g = \mathbf{T}^T{}_{cg} \mathbf{k}_c \mathbf{T}_{cg} \dot{\mathbf{d}}_g + \dot{\mathbf{T}}^T_{cg} \mathbf{f}_c$$

$$\dot{\mathbf{T}}^T_{cg} = \left[\frac{\partial \dot{\mathbf{T}}^T_{cg}}{\partial d_{gk}} \right] \dot{\mathbf{d}}_{gk}, k = 1, 2, \cdot\cdot, 6.$$

$$\dot{\mathbf{T}}_{cg}^{T} = \left[\frac{\partial^2 \mathbf{d}_{ci}}{\partial d_{gj} \partial d_{gk}} \right]^T \dot{\mathbf{d}}_{gk}, i = 1, 2, \cdot\cdot, 6. \quad j = 1, 2, \cdot\cdot, 6 \quad k = 1, 2, \cdot\cdot, 6$$

$$\dot{\mathbf{T}}_{cg}^{T} = \left[\frac{\partial^2 \mathbf{d}_{c1}}{\partial d_{gj} \partial d_{gk}} \middle| \frac{\partial^2 \mathbf{d}_{c2}}{\partial d_{gj} \partial d_{gk}} \middle| \frac{\partial^2 \mathbf{d}_{c3}}{\partial d_{gj} \partial d_{gk}} \right]^T \dot{\mathbf{d}}_{gk} = \left[\mathbf{T}_1 \middle| \mathbf{T}_2 \middle| \mathbf{T}_3 \right] \dot{\mathbf{d}}_{gk}$$

$$\mathbf{T}_1 = \mathbf{T}_2 = \frac{1}{L^2} \begin{bmatrix} -2sc & c^2 - s^2 & 0 & 2sc & -\left(c^2 - s^2\right) & 0 \\ & 2sc & 0 & -\left(c^2 - s^2\right) & -2sc & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & -2sc & c^2 - s^2 & 0 \\ & Sym & & & 2sc & 0 \\ & & & & & 0 \end{bmatrix}$$

$$\mathbf{T}_3 = \frac{1}{L^2}\begin{bmatrix} s^2 & -sc & 0 & -s^2 & sc & 0 \\ & c^2 & 0 & sc & -c^2 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & s^2 & -sc & 0 \\ & Sym & & & c^2 & 0 \\ & & & & & 0 \end{bmatrix}$$

$$\dot{\mathbf{f}}_g = \mathbf{T}^T{}_{cg}\mathbf{k}_c\mathbf{T}_{cg}\dot{\mathbf{d}}_g + \dot{\mathbf{T}}^T_{cg}\mathbf{f}_c$$

$$\dot{\mathbf{f}}_g = \left(\mathbf{T}^T{}_{cg}\mathbf{k}_c\mathbf{T}_{cg} + \mathbf{T}_1 M_A + \mathbf{T}_2 M_B + \mathbf{T}_3 P\right)\dot{\mathbf{d}}_g$$

$$\dot{\mathbf{f}}_g = \mathbf{k}_g\dot{\mathbf{d}}_g$$

8.3 Modeling of Truss Elements

$$\dot{\mathbf{f}}_g = \left(\mathbf{T}^T{}_{cg} \mathbf{k}_c \mathbf{T}_{cg} + \cancel{\mathbf{T}_1 M_A} + \cancel{\mathbf{T}_2 M_B} + \mathbf{T}_3 P \right) \dot{\mathbf{d}}_g$$

$$\dot{\mathbf{f}}_g = \left(\mathbf{T}^T{}_{cg} \mathbf{k}_c \mathbf{T}_{cg} + \mathbf{T}_3 P \right) \dot{\mathbf{d}}_g$$

$$\dot{\mathbf{f}}_g = \begin{Bmatrix} \dot{f}_{g1} \\ \dot{f}_{g2} \\ \dot{f}_{g3} \\ \dot{f}_{g4} \end{Bmatrix} \quad \dot{\mathbf{d}}_g = \begin{Bmatrix} \dot{d}_{g1} \\ \dot{d}_{g2} \\ \dot{d}_{g3} \\ \dot{d}_{g4} \end{Bmatrix}$$

$$\mathbf{T}_{cg} = \begin{bmatrix} -\cos\theta & -\sin\theta & \cos\theta & \sin\theta \end{bmatrix}^T$$

$$\mathbf{T}_3 = \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta \\ & \cos^2\theta & \sin\theta\cos\theta & \cos^2\theta \\ & & \sin^2\theta & -\sin\theta\cos\theta \\ sym & & & \cos^2\theta \end{bmatrix}$$

8.4.2 Modifies Tangent Stiffness Relationship

$$\begin{Bmatrix} \Delta\dot{M}_{pcA} \\ \dot{M}_B \\ \dot{P} \end{Bmatrix} = \frac{EI}{L}\begin{bmatrix} S_1 & S_2 & 0 \\ S_2 & S_1 & 0 \\ 0 & 0 & A/I \end{bmatrix}\begin{Bmatrix} \dot{\theta}_A \\ \dot{\theta}_B \\ \dot{e} \end{Bmatrix}$$

$$\dot{\theta}_A = \frac{L\Delta M_{pcA}}{EIS_1} - \frac{S_2}{S_1}\dot{\theta}_B$$

$$\begin{Bmatrix} \dot{M}_A \\ \dot{M}_B \\ \dot{P} \end{Bmatrix} = \frac{EI}{L}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \left(S_1 - S_2^2\right)/S_1 & 0 \\ 0 & 0 & A/I \end{bmatrix}\begin{Bmatrix} \dot{\theta}_A \\ \dot{\theta}_B \\ \dot{e} \end{Bmatrix} + \begin{Bmatrix} 1 \\ S_2/S_1 \\ 0 \end{Bmatrix}\Delta M_{pcA}$$

$$\begin{Bmatrix} \dot{M}_A \\ \dot{M}_B \\ \dot{P} \end{Bmatrix} = \frac{EI}{L} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A/I \end{bmatrix} \begin{Bmatrix} \dot{\theta}_A \\ \dot{\theta}_B \\ \dot{e} \end{Bmatrix} + \begin{Bmatrix} \Delta M_{pcA} \\ \Delta M_{pcB} \\ 0 \end{Bmatrix}$$

$$\dot{\mathbf{f}}_c = \mathbf{k}_{ch} \dot{\mathbf{d}}_c + \dot{\mathbf{f}}_{cp}$$

$$\dot{\mathbf{f}}_g = \left(\mathbf{T}^T{}_{cg} \mathbf{k}_{ch} \mathbf{T}_{cg} + \mathbf{T}_1 M_A + \mathbf{T}_2 M_B + \mathbf{T}_3 P \right) \dot{\mathbf{d}}_g + \mathbf{T}^T{}_{cg} \dot{\mathbf{f}}_{cp}$$

$$\dot{\mathbf{f}}_g = \mathbf{k}_{gh} \dot{\mathbf{d}}_g + \dot{\mathbf{f}}_{gp}$$

$$\mathbf{k}_{gh} = \mathbf{T}^T{}_{cg} \mathbf{k}_{ch} \mathbf{T}_{cg} + \mathbf{T}_1 M_A + \mathbf{T}_2 M_B + \mathbf{T}_3 P$$

$$\dot{\mathbf{f}}_{gp} = \mathbf{T}^T{}_{cg} \dot{\mathbf{f}}_{cp}$$