Mechanics and Design

Chapter 8. Iso-Parametric Formulation

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CONTENTS

Outlines

Iso-Parametric Formulation

- It makes formulations for computer program simple
- It allows to create elements with a shape of a straight line or a curved surface. Make it possible to choose a variety of factors.
- We will derive the stiffness matrix of simple beam elements and rectangular elements using an iso-parametric formulation.
- Numerical integration: We will calculate the stiffness matrix of rectangular elements that is made using an iso-parametric formulation.
- Finally, we will consider several higher-order elements and shape functions.

1-D Iso-Parametric Formulation

Stiffness matrix of a beam element

The term of iso-parametric formulation comes from the usage of shape functions [N] which is used to determine an element shape for approximation of displacement.

- If a displacement function is $u=a_1+a_2s$, use a node $x=a_1+a_2s$ on an element.
- It is formulated using the natural (or intrinsic) coordinate system, *s*, defined by geometry of elements. A transformation mapping is used for the element formulation between natural coordinate system, *s*, and global coordinate system, *x*.

Step 1: Determination of element type

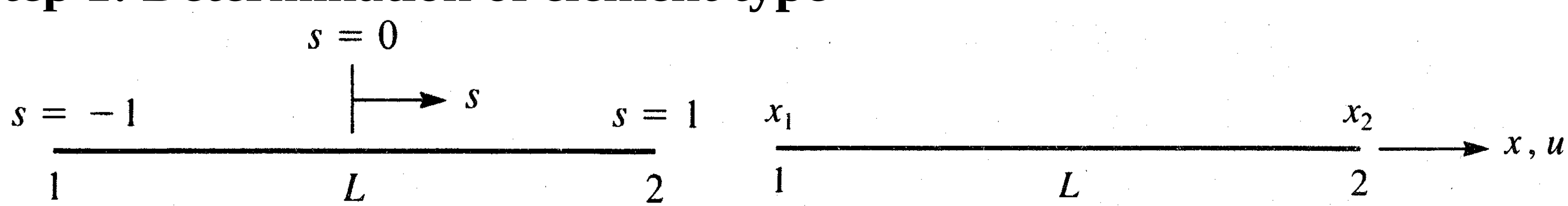


Fig. 1: Linear beam element at node x in
(a) natural coordinate system, *s* (b) global coordinate system, *x*.

1-D Iso-Parametric Formulation

Relation between *s* and *x* coordinate systems:

(when *s* and *x* coordinate systems are parallel)

$$x = x_c + \frac{L}{2}s \qquad x_c : center\ of\ element$$

x can be expressed as a function of x_1 and x_2

$$x = \frac{1}{2}[(1-s)x_1 + (1+s)x_2] = [N_1 \quad N_2]\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

Then shape functions are,

$$N_1 = \frac{1-s}{2} \qquad N_2 = \frac{1+s}{2}, \qquad Note : N_1 + N_2 = 1$$

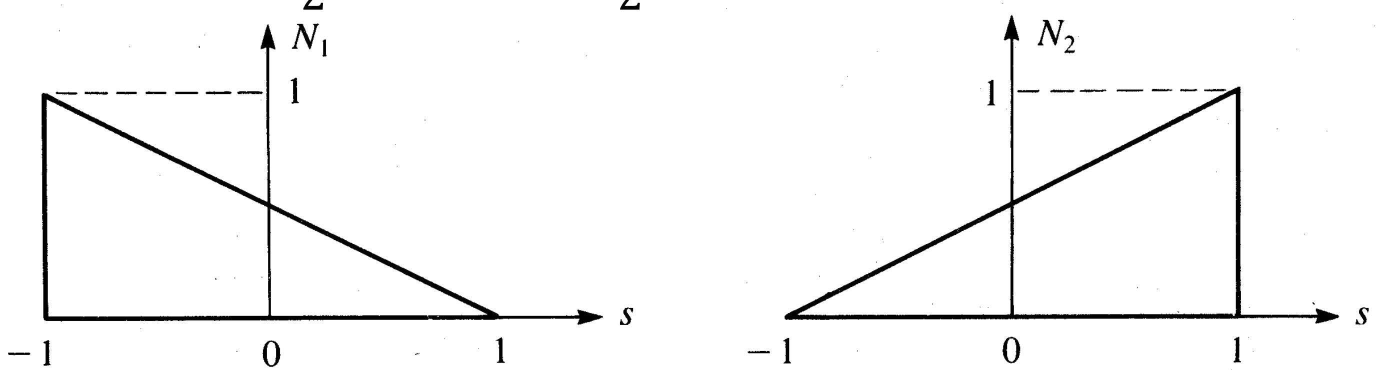


Fig. 2: Shape functions in natural coordinate system

1-D Iso-Parametric Formulation

Step 2: Determination of deformation function

$$\{u\} = [N_1 \quad N_2]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

u and *x* are called iso-parameter because they are defined by the same shape function at the same node.

Step 3: Definition of strain-displacement and stress-strain relations

Calculation of element matrix [B]:

- By chain rule

$$\frac{du}{ds} = \frac{du}{dx}\frac{dx}{ds}, \qquad \frac{du}{dx} = \frac{\left(\frac{du}{ds}\right)}{\left(\frac{dx}{ds}\right)} = \frac{[-\frac{1}{2}, \ \frac{1}{2}]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}}{\left(\frac{L}{2}\right)}$$

$$\therefore \{\varepsilon_x\} = \left[-\frac{1}{L} \quad \frac{1}{L}\right]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

- Therefore,

$$\{\varepsilon\} = [B]\{d\}, \qquad [B] = \left[-\frac{1}{L} \qquad \frac{1}{L}\right]$$

1-D Iso-Parametric Formulation

Step 4: Calculation of element stiffness matrix

Element stiffness matrix:

$$[k] = \int_0^L [B]^{\mathrm{T}}[D][B]A dx$$

- **In general, matrix [B] is a function of s:**

$$\int_0^L f(x)dx = \int_{-1}^{1} f(s)|\underline{J}|\, ds$$

where $\underline{J}$ is Jacobian,

In case of 1-D, $|\underline{J}| = \underline{J}$. In case of simple beam element:

$$|\underline{J}| = \frac{dx}{ds} = \frac{L}{2}$$

Ratio of element's length between global and natural coordinate systems

- **Stiffness matrix in a natural coordinate system:**

$$[k] = \frac{L}{2}\int_{-1}^{1} [B]^T E[B]A ds = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

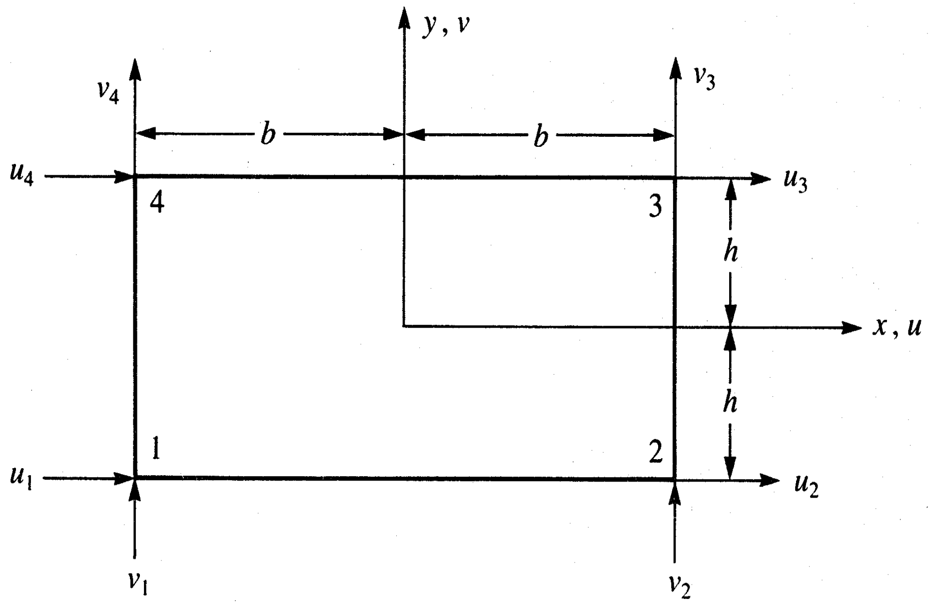
2-D Iso-Parametric Formulation

Rectangular plane stress element

Characteristics of rectangular element:

- It is easy to input data, and it is simple to calculate stress.
- Physical boundary conditions are not well approximated at the edge of rectangle.

Step 1: Determination of element type – using natural coordinate (x, y)



$$\{d\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

Fig. 3: Four node rectangular element and nodal displacement

2-D Iso-Parametric Formulation

Step 2: Determination of deformation function – element deformation functions,

u and *v* are linear along the rectangular corner.

$$u(x,y) = a_1 + a_2 x + a_3 y + a_4 xy$$
$$v(x,y) = a_5 + a_6 x + a_7 y + a_8 xy$$

➔

$$u(x,y) = \frac{1}{4bh}[(b-x)(h-y)u_1 + (b+x)(h-y)u_2 + (b+x)(h+y)u_3 + (b-x)(h+y)u_4]$$
$$v(x,y) = \frac{1}{4bh}[(b-x)(h-y)v_1 + (b+x)(h-y)v_2 + (b+x)(h+y)v_3 + (b-x)(h+y)v_4]$$

$$\therefore \{\psi\} = \begin{Bmatrix} u \\ v \end{Bmatrix} = [N]\{d\} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}\{d\}$$

where shape functions are

$$N_1 = \frac{(b-x)(h-y)}{4bh} \quad N_2 = \frac{(b+x)(h-y)}{4bh}$$
$$N_3 = \frac{(b+x)(h+y)}{4bh} \quad N_4 = \frac{(b-x)(h+y)}{4bh}$$

2-D Iso-Parametric Formulation

Step 3: Definition of strain-displacement and stress-strain relationships

Element strain in a 2-D stress state:

$$\{\varepsilon\} \equiv \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = [B]\{d\}$$

where

$$[B] = \frac{1}{4bh}\begin{bmatrix} -(h-y) & 0 & (h-y) & 0 & (h+y) & 0 & -(h+y) & 0 \\ 0 & -(b-x) & 0 & -(b+x) & 0 & (b+x) & 0 & (b-x) \\ -(b-x) & -(h-y) & -(b+x) & (h-y) & (b+x) & (h+y) & (b-x) & -(h+y) \end{bmatrix}$$

2-D Iso-Parametric Formulation

Step 4: Calculation of element stiffness matrix and element equation

Element stiffness matrix:

$$[k] = \int_{-h}^{h}\int_{-b}^{b}[B]^T[D][B]t\,dx\,dy$$

Element force matrix:

$$\{f\} = \iiint_V [N]^T\{X\}dV + \{P\} + \iint_S [N]^T\{T\}dS$$

Element equation:

$$\{f\} = [k]\{d\}$$

Step 5,6, and 7

Step 5, 6, and 7 are constitution of global stiffness matrix, determinant of unknown deformation, calculation of stress. However, stress in each element varies in all directions of x and y.

2-D Iso-Parametric Formulation

Stiffness matrix of a plane element

A process of iso-parametric formulation is same in all elements

Step 1: Determination of element type

It is possible to numerically integrate the rectangular element defined in natural coordinate system *s*- *t*.

Transformation equation: $x = x_c + b_s$, $y = y_c + h_t$

$$x = x_c + \frac{L}{2}s \qquad x_c : center\ of\ element$$

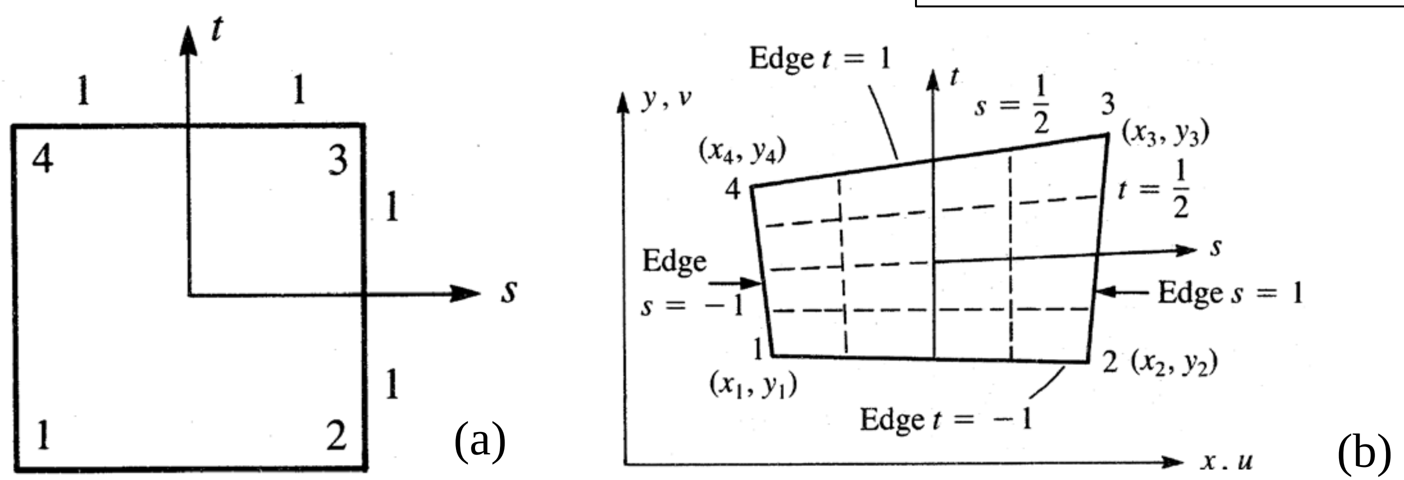


Fig. 4: (a) A linear rectangular element in a coordinate system, *s*-*t* (b) A rectangular element in a coordinate system, The size and shape of the rectangular element are defined by coordinates of four nodes

2-D Iso-Parametric Formulation

Transformation equation between a local coordinate system, s-t, and a global coordinate system, x-y:

$$x = a_1 + a_2 s + a_3 t + a_4 st$$
$$y = a_5 + a_6 s + a_7 t + a_8 st$$

$$\rightarrow$$

$$x = \frac{1}{4}[(1-s)(1-t)x_1 + (1+s)(1-t)x_2 + (1+s)(1+t)x_3 + (1-s)(1+t)x_4]$$
$$y = \frac{1}{4}[(1-s)(1-t)y_1 + (1+s)(1-t)y_2 + (1+s)(1+t)y_3 + (1-s)(1+t)y_4]$$

In a matrix form:

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix},$$

$$N_1 = \frac{(1-s)(1-t)}{4}$$
$$N_2 = \frac{(1+s)(1-t)}{4}$$
$$N_3 = \frac{(1+s)(1+t)}{4}$$
$$N_4 = \frac{(1-s)(1+t)}{4}$$

2-D Iso-Parametric Formulation

1. **Shape function is linear.**
2. **Any point in rectangular element (s, t) can be mapped to the quadrilateral element point (x, y) in Fig. 4(b).**
3. **Note that for all values of s and t, $N_1+N_2+N_3+N_4=1$**
4. **N_i (i=1, 2, 3, 4) is 1 for node i , and 0 for the other nodes.**

Step 5,6, and 7

Step 5, 6, and 7 are constitution of global stiffness matrix, determinant of unknown deformation, calculation of stress. However, stress in each element varies in all directions of x and y.

1. $\sum_{i=1}^{n} N_i = 1 \quad (i = 1, 2, \ldots, n)$

2. $N_i = 1$ for node i, and $N_i = 0$ for the other nodes.

Additional conditions:
3. Continuity of deformation --- Lagrangian Interpolation
4. Continuity of slope --- Hermitian Interpolation

2-D Iso-Parametric Formulation

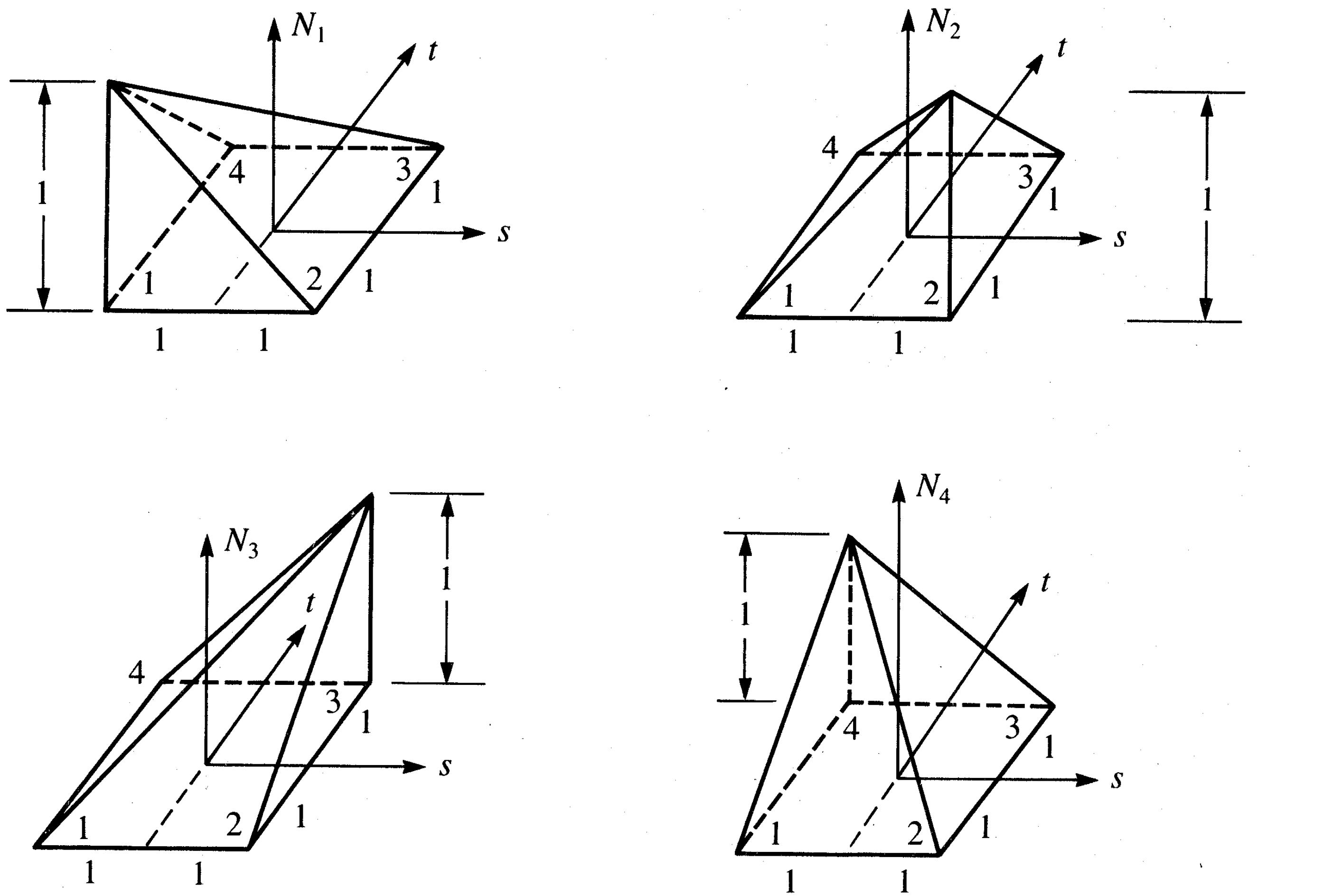


Fig. 5: Change of shape functions in a linear rectangular element

2-D Iso-Parametric Formulation

Reference: chain rule of *f*

$$\frac{\partial f}{\partial s}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial f}{\partial t}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

Calculating $(\partial f/\partial x)$ and $(\partial f/\partial y)$ using Cramer's rule (Appendix. B).

$$\frac{\partial f}{\partial x}=\frac{1}{|\underline{J}|}\begin{vmatrix}\frac{\partial f}{\partial s} & \frac{\partial y}{\partial s}\\ \frac{\partial f}{\partial t} & \frac{\partial y}{\partial t}\end{vmatrix},\qquad \frac{\partial f}{\partial y}=\frac{1}{|\underline{J}|}\begin{vmatrix}\frac{\partial x}{\partial s} & \frac{\partial f}{\partial s}\\ \frac{\partial x}{\partial t} & \frac{\partial f}{\partial t}\end{vmatrix},\qquad \text{where, } |\underline{J}|=\begin{vmatrix}\frac{\partial x}{\partial s} & \frac{\partial y}{\partial s}\\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t}\end{vmatrix}\ (*)$$

Element strain:

$$\underline{\varepsilon}\equiv\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}=\begin{bmatrix}\frac{\partial()}{\partial x} & 0\\ 0 & \frac{\partial()}{\partial y}\\ \frac{\partial()}{\partial y} & \frac{\partial()}{\partial x}\end{bmatrix}\begin{Bmatrix}u\\ v\end{Bmatrix}=\begin{bmatrix}\frac{\partial()}{\partial x} & 0\\ 0 & \frac{\partial()}{\partial y}\\ \frac{\partial()}{\partial y} & \frac{\partial()}{\partial x}\end{bmatrix}\underline{N}\underline{d}=\underline{B}\underline{d}$$

A formulation to obtain B is required.

2-D Iso-Parametric Formulation

Using the equation (*) in previous page (use *u* or *v* instead of *f*)

$$\frac{\partial f}{\partial x} = \frac{1}{|\underline{J}|}\begin{vmatrix} \frac{\partial f}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial f}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix} \qquad \frac{\partial f}{\partial y} = \frac{1}{|\underline{J}|}\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial f}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial f}{\partial t} \end{vmatrix}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{|\underline{J}|}\begin{bmatrix} \frac{\partial y}{\partial t}\frac{\partial ()}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial ()}{\partial t} & 0 \\ 0 & \frac{\partial x}{\partial s}\frac{\partial ()}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial ()}{\partial s} \\ \frac{\partial x}{\partial s}\frac{\partial ()}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial ()}{\partial s} & \frac{\partial y}{\partial t}\frac{\partial ()}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial ()}{\partial t} \end{bmatrix}\begin{Bmatrix} u \\ v \end{Bmatrix}$$

Or

$$\underline{\varepsilon} = \underline{D'}\underline{N}\underline{d} = \underline{B}\underline{d}$$

$$where\,\underline{D'} = \frac{1}{|\underline{J}|}\begin{bmatrix} \frac{\partial y}{\partial t}\frac{\partial ()}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial ()}{\partial t} & 0 \\ 0 & \frac{\partial x}{\partial s}\frac{\partial ()}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial ()}{\partial s} \\ \frac{\partial x}{\partial s}\frac{\partial ()}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial ()}{\partial s} & \frac{\partial y}{\partial t}\frac{\partial ()}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial ()}{\partial t} \end{bmatrix}$$

Thus, $$\underset{(3\times 8)}{\underline{B}} = \underset{(3\times 2)}{\underline{D'}}\ \underset{(2\times 8)}{\underline{N}}$$

2-D Iso-Parametric Formulation

Step 4: Calculation of element stiffness matrix and element equation

Element stiffness matrix:

$$[k] = \int_{-h}^{h}\int_{-b}^{b}[B]^T[D][B]t\,dx\,dy$$

Element force matrix:

$$\{f\} = \iiint_V [N]^T\{X\}dV + \{P\} + \iint_S [N]^T\{T\}dS$$

Element equation:

$$\{f\} = [k]\{d\}$$

Step 5,6, and 7

Step 5, 6, and 7 are constitution of global stiffness matrix, determinant of unknown deformation, calculation of stress. However, stress in each element varies in all directions of x and y.

2-D Iso-Parametric Formulation

Step 4: Derivation of element stiffness matrix and equation

Stiffness matrix in a coordinate system, s-t :

$$[k] = \iint_A [B]^T [D]\,[B] t dxdy$$

Converge the integral region from x-y to s-t :

$$[k] = \int_{-1}^{1}\int_{-1}^{1} [B]^T\,[D][B]t|J|dsdt$$

Determinant $|\underline{J}|$ is

$$|\underline{J}| = \frac{1}{8}\{X_c\}^T \begin{bmatrix} 0 & 1-t & t-s & s-1 \\ t-1 & 0 & s+1 & -s-t \\ s-t & -s-1 & 0 & t+1 \\ 1-s & s+t & -t-1 & 0 \end{bmatrix} \{Y_c\}$$

$$where, \{X_c\}^T = [x_1\ x_2\ x_3\ x_4], \qquad \{Y_c\} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$

$|\underline{J}|$ is a function of s, t in natural coordinate system, and x1, x2, …, y4 in the known global coordinate system.

2-D Iso-Parametric Formulation

Calculation of $\underline{B}$:

$$\underline{B}(s,t) = \frac{1}{|\underline{J}|}[\underline{B}_1 \;\; \underline{B}_2 \;\; \underline{B}_3 \;\; \underline{B}_4]$$

where,

$$\underline{B}_i = \begin{bmatrix} aN_{i,s} - bN_{i,t} & 0 \\ 0 & cN_{i,t} - dN_{i,s} \\ cN_{i,t} - dN_{i,s} & aN_{i,s} - bN_{i,t} \end{bmatrix}, \qquad i = 1,2,3,4$$

And

$$a = \frac{1}{4}[y_1(s-1) + y_2(-1-s) + y_3(1+s) + y_4(1-s)]$$

$$b = \frac{1}{4}[y_1(t-1) + y_2(1-t) + y_3(1+t) + y_4(-1-t)]$$

$$c = \frac{1}{4}[x_1(t-1) + x_2(1-t) + x_3(1+t) + x_4(-1-t)]$$

$$d = \frac{1}{4}[x_1(s-1) + x_2(-1-s) + x_3(1+s) + x_4(1-s)]$$

For example, $N_{1,s} = \frac{1}{4}(t-1) \quad N_{1,t} = \frac{1}{4}(s-1)$

2-D Iso-Parametric Formulation

Element body force matrix:

$$\underset{(8\times 1)}{\{f_b\}} = \int_{-1}^{1}\int_{-1}^{1} \underset{(8\times 2)}{[N]^T}\underset{(2\times 1)}{\{X\}} t|\underline{J}|\, ds dt$$

Element surface force matrix: Length is *L*, an edge *t=1* (See Fig. 4(b))

$$\{f_s\} = \int_{-1}^{1} [N]^T\{T\}t\frac{L}{2}ds\,, \qquad \begin{Bmatrix} f_{s3s} \\ f_{s3t} \\ f_{s4s} \\ f_{s4r} \end{Bmatrix} = \int_{-1}^{1} \begin{bmatrix} N_3 & 0 & N_4 & 0 \\ 0 & N_3 & 0 & N_4 \end{bmatrix}^T \begin{Bmatrix} p_s \\ p_t \end{Bmatrix} t\frac{L}{2}ds$$

For $N_1=0$ and $N_2=0$ along the edge $t=1$, the nodal force is zero at node 1 and 2.

Gaussian Quadrature (Numerical integration)

One node Gaussian quadrature

$$I = \int_{-1}^{1} y dx \approx y_1 * \{(1) - (-1)\}$$
$$= 2y_1$$

If function *y* is straight line,
it has exact solution.

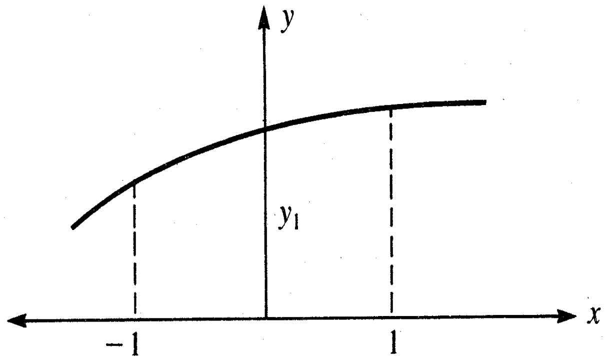


Fig. 6: Gaussian quadrature with one point

General equation:

$$I = \int_{-1}^{1} y dx = \sum_{i=1}^{n} W_i y_i$$

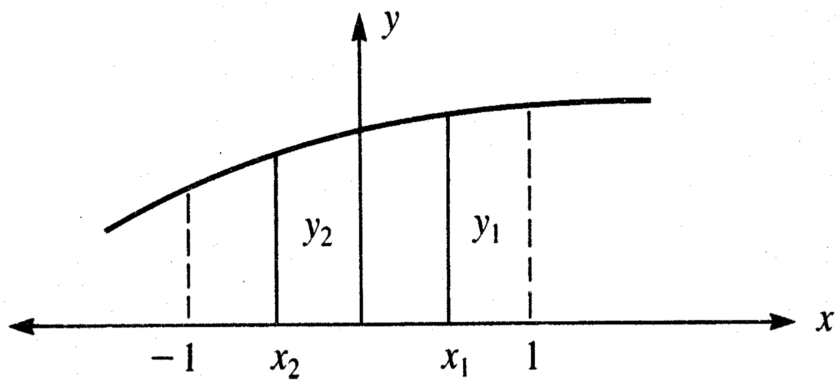
Gaussian quadrature using n nodes (Gaussian point) can exactly calculate polynomial equation which has the integral terms under *(2n-1)* order.

When function *f(x)* is not a polynomial, Gaussian quadrature is inaccurate. However, the more Gaussian points are used, the more accurate solution is. In general, the ratio of two polynomials is not a polynomial.

Gaussian Quadrature (Numerical integration)

Table 1. Gaussian points for integration from –1 to +1

Number of Points	*Locations, x_i*	*Associated Weights, W_i*
1	$x_1 = 0.000\ldots$	2.000
2	$x_1, x_2 = \pm\, 0.57735026918962$	1.000
3	$x_1, x_3 = \pm\, 0.77459666924148$ $x_2 = 0.000\ldots$	5/9 = 0.555.. 8/9 = 0.888..



$x_1 = +0.5773\ldots$
$x_2 = -0.5773\ldots$

Fig. 7: Gaussian quadrature with two extraction points

Gaussian Quadrature (Numerical integration)

2-D problem: Integrate about second coordinate after integrate about first coordinate.

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\,ds\,dt = \int_{-1}^{1}\left[\sum_i W_i f(s_i,t)\right]$$

$$= \sum_j W_j\left[\sum_i W_i f(s_i,t_j)\right] = \sum_i \sum_j W_i W_j f(s_{i,} t_j)$$

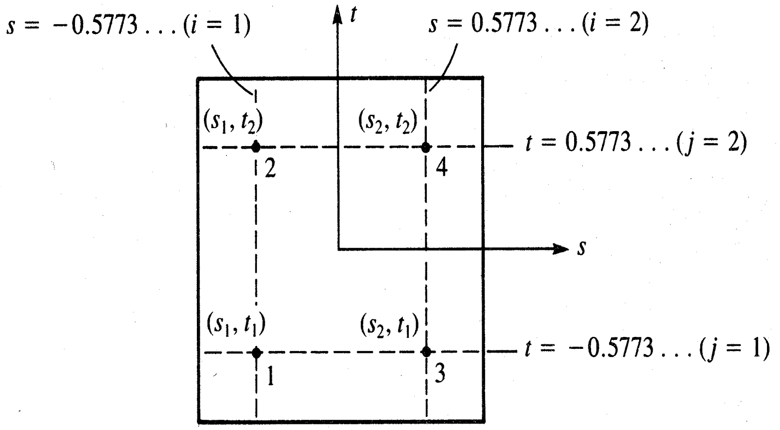
For 2x2:

$$I = W_1 W_1 f(s_1,t_1) + W_1 W_2 f(s_1,t_2) + W_2 W_1 f(s_2,t_1) + W_2 W_2 f(s_2,t_2)$$

where the sample four points are located at

$$s_{i,}\, t_i = \pm\, 0.5773\ldots = \pm 1/\sqrt{3}$$

And the all weight factors are 1.000. Thus, the two summation marks can be interpreted as one summation mark for four points of the rectangle.



Gaussian Quadrature (Numerical integration)

3-D problem:

$$I=\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(s,t,z)\,ds\,dt\,dz=\sum_i\sum_j\sum_k W_iW_jW_k f(s_i,t_j,z_k)$$

NOTE: If the integration limit is $\int_0^1 f(x)dx=\sum_{i=1}^{n} W_i f(x_i)$, the weight factor Wi, and the location x_i are different from that of the integration limit which is between -1 and 1 (See table 2).

Table 2. Gaussian points of the four node gaussian integration (integration from 0 to 1)

Locations, x_i	*Associated Weights,* W_i
0.0693185	0.1739274
0.3300095	0.3260725
0.6699905	0.3260725
0.9305682	0.1739274

Gaussian Quadrature (Numerical integration)

Example 1: Calculate the integration of $\sin\pi x$ using numerical integration

$$I = \int_0^1 sin\pi x dx$$

Using table 2, the following can be obtained.

$$I = \sum_{i=1}^{4} W_i \sin\pi x_i = W_1 \sin\pi x_1 + W_2 \sin\pi x_2 + W_3 \sin\pi x_3 + W_4 \sin\pi x_4$$
$$= 0.1739\sin\pi(0.0694) + 0.3261\sin\pi(0.3300)$$
$$+0.3261\sin\pi(0.6700) + 0.1739\sin\pi(0.9306) = \mathbf{0.6366}$$

Use four decimal places. The exact value of direct integration is 0.6366. Note that location x_i and weight factor W_i are different from that in table 2 if we use the 3-points Gaussian integration.

Calculation of Stiffness Matrix by Gaussian Integration

Element stiffness matrix in 2-D:

$$\iint_A \underline{B}^T(x,y)\underline{D}\underline{B}(x,y)t\,dx\,dy = \int_{-1}^{1}\int_{-1}^{1} \underline{B}^T(s,t)\underline{D}\underline{B}(s,t)|\underline{J}|t\,ds\,dt$$

The integral term, $\underline{B}^T\underline{D}\underline{B}|\underline{J}|$ which is a function of *(s, t)* is calculated by the numerical integration.

Using four-points Gaussian integration,

$$\begin{aligned}\underline{k} = {} & \underline{B}^T(s_1,t_1)\underline{D}\underline{B}(s_1,t_1)|\underline{J}(s_1,t_1)|tW_1W_1 \\ & +\underline{B}^T(s_2,t_2)\underline{D}\underline{B}(s_2,t_2)|\underline{J}(s_2,t_2)|tW_2W_2 \\ & +\underline{B}^T(s_3,t_3)\underline{D}\underline{B}(s_3,t_3)|\underline{J}(s_3,t_3)|tW_3W_3 \\ & +\underline{B}^T(s_4,t_4)\underline{D}\underline{B}(s_4,t_4)|\underline{J}(s_4,t_4)|tW_4W_4\end{aligned}$$

where,
$s_1 = t_1 = -0.5773$, $s_2 = -0.5773$, $t_2 = 0.5773$, $s_3 = 0.5773$, $t_3 = -0.5773$,
$s_4 = t_4 = 0.5773$ and $W_1 = W_2 = W_3 = W_4 = 1.000$

Calculation of Stiffness Matrix by Gaussian Integration

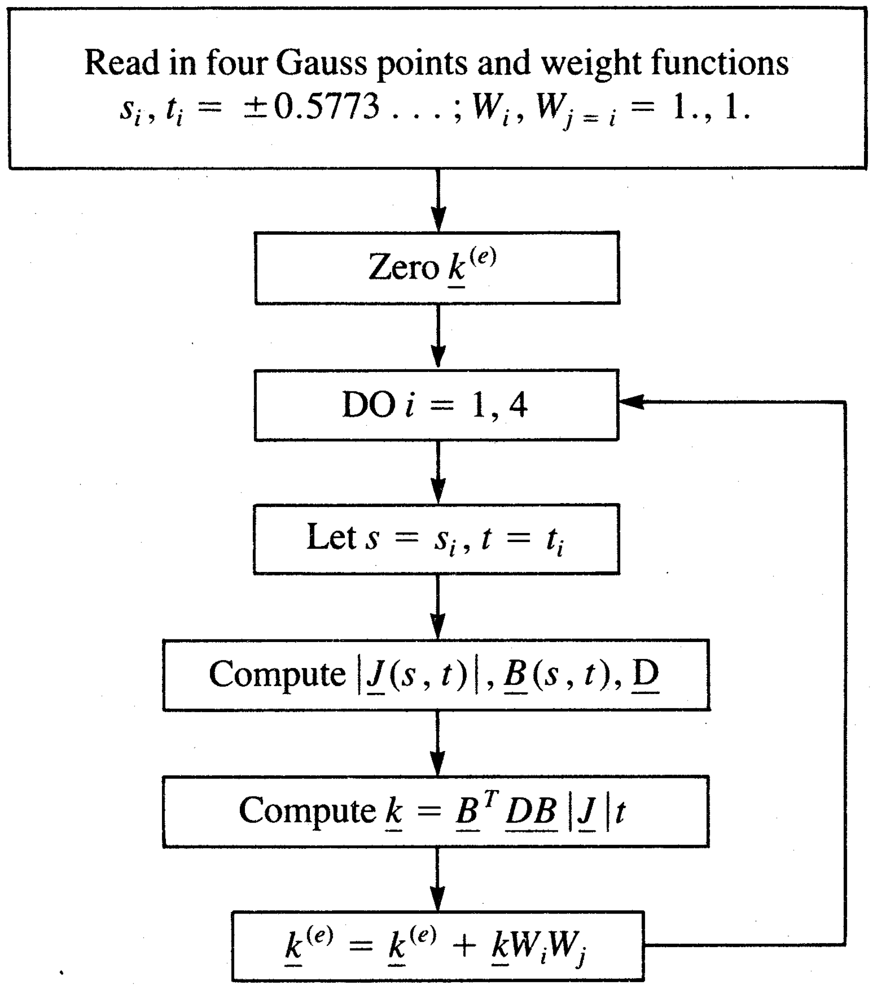


Fig. 9: Flow chart for obtaining using Gaussian integration

Calculation of Stiffness Matrix by Gaussian Integration

Example 2

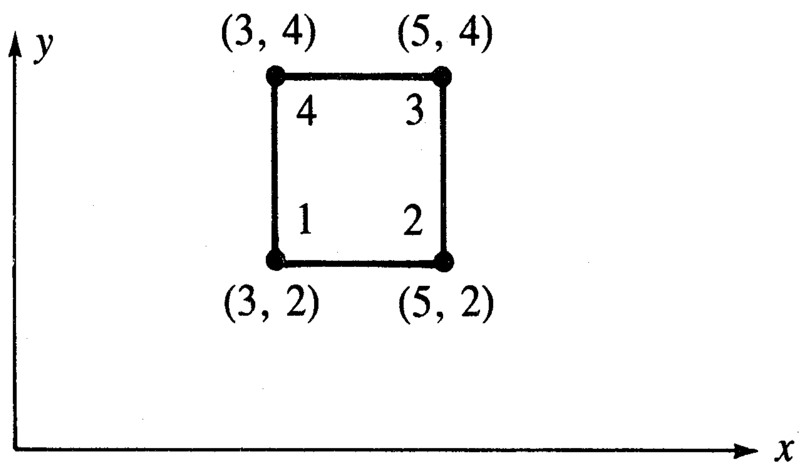


Fig. 10: Quadrilateral elements for calculation of stiffness

Calculate the stiffness matrix of rectangular element using four-point Gaussian integration.
***E*=30x10⁶*psi*, *v*=0.25**

The unit length in global coordinate system is inch, and ***t*=1 *in***

Using 4-points rule:

$$(s_1, t_1) = (-0.5733, -0.5773) \quad W_1 = 1.0$$
$$(s_2, t_2) = (-0.5733, 0.5773) \quad W_2 = 1.0$$
$$(s_3, t_3) = (0.5733, -0.5773) \quad W_3 = 1.0$$
$$(s_4, t_4) = (0.5733, 0.5773) \quad W_4 = 1.0$$

Calculation of Stiffness Matrix by Gaussian Integration

Calculation of stiffness matrix:

$$\begin{aligned}
\underline{k} = {} & \underline{B}^T(-0.5773,-0.5733)\underline{D}\,\underline{B}(-0.5773,-0.5773) \\
& \times |\underline{J}(-0.5773,-0.5773)|(1)(1.000)(1.000) \\
& +\underline{B}^T(-0.5773,0.5773)\underline{D}\,\underline{B}(-0.573,0.5773) \\
& \times |\underline{J}(-0.5773,0.5773)|(1)(1.000)(1.000) \\
& +\underline{B}^T(0.5773,-0.5773)\underline{D}\,\underline{B}(0.573,-0.5773) \\
& \times |\underline{J}(0.5773,-0.5773)|(1)(1.000)(1.000) \\
& +\underline{B}^T(0.5773,0.5773)\underline{D}\,\underline{B}(0.573,0.5773) \\
& \times |\underline{J}(0.5773,0.5773)|(1)(1.000)(1.000)
\end{aligned}$$

We need to calculate $|\underline{J}|$ and $\underline{B}$ at Gaussian points

$$(s_1,t_1) = (-0.5733,-0.5773),(s_2,t_2) = (-0.5733,0.5773)$$
$$(s_3,t_3) = (0.5733,-0.5773),(s_4,t_4) = (0.5733,0.5773)$$

Calculation of Stiffness Matrix by Gaussian Integration

Calculation of $|\underline{J}|$:

$$|\underline{J}(-0.5773,-0.5773)| = \frac{1}{8}[3 \;\; 5 \;\; 5 \;\; 3]$$

$$\times \begin{bmatrix} 0 & 1-(-0.5773) & -0.5773-(-0.5773) & -0.5773-1 \\ -0.5773-1 & 0 & -0.5773+1 & -0.5773-(-0.5773) \\ -0.5773-(-0.5773) & -0.5773-1 & 0 & -0.5773+1 \\ 1-(-0.5773) & -0.5773+(-0.5773) & -0.5773-1 & 0 \end{bmatrix}$$

$$\times \begin{Bmatrix} 2 \\ 2 \\ 4 \\ 4 \end{Bmatrix} = 1.000$$

Similarly,

$$|\underline{J}(-0.5733,-0.5733)| = 1.000$$
$$|\underline{J}(0.5733,-0.5733)| = 1.000$$
$$|\underline{J}(0.5733,0.5733)| = 1.000$$

Generally $|\underline{J}| \neq 1$, and it changes within the element.

Calculation of Stiffness Matrix by Gaussian Integration

Calculation of |$\underline{B}$| :

$$\underline{B}(-0.5733, -0.5733) = \frac{1}{|\underline{J}(-0.5733, -0.5733)|}[\underline{B}_1 \quad \underline{B}_2 \quad \underline{B}_3 \quad \underline{B}_4]$$

Calculation of $\underline{B_1}$:

$$\underline{B}_1 = \begin{bmatrix} aN_{1,s} - bN_{1,t} & 0 \\ 0 & cN_{1,t} - dN_{1,s} \\ cN_{1,t} - dN_{1,s} & aN_{1,s} - bN_{1,t} \end{bmatrix}$$

Where,

$$a = \frac{1}{4}[y_1(s-1) + y_2(-1-s) + y_3(1+s) + y_4(1-s)]$$

$$= \frac{1}{4}[2(-0.5773 - 1) + 2(-1 - 0.5773))$$

$$+4(1 + (-0.5773)) + 4(1 - (- -0.5773))]$$

$$= 1.00$$

The same calculation can be used to obtain *b*, *c*, *d*

Calculation of Stiffness Matrix by Gaussian Integration

Also,

$$N_{1,s} = \frac{1}{4}(t-1) = \frac{1}{4}(-0.5773-1) = -0.3943$$

$$N_{1,t} = \frac{1}{4}(s-1) = \frac{1}{4}(-0.5773-1) = -0.3943$$

Similarly $\underline{B}_2, \underline{B}_3, \underline{B}_4$, can be calculated at other Gaussian points.

Generally a computer program is used to calculate $\underline{B}$ and $\underline{k}$.
Final form of $\underline{B}$ is,

$$\underline{B} = \begin{bmatrix} -0.1057 & 0 & 0.1057 & 0 & 0 & -0.1057 & 0 & -0.3943 \\ -0.1057 & -0.1057 & -0.3743 & 0.1057 & 0.3943 & 0 & -0.3943 & 0 \\ 0 & 0.3943 & 0 & 0.1057 & 0.3943 & 0.3943 & 0.1057 & -0.3943 \end{bmatrix}$$

Calculation of Stiffness Matrix by Gaussian Integration

Matrix $\underline{D}$:

$$\underline{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} = \begin{bmatrix} 32 & 8 & 0 \\ 8 & 32 & 0 \\ 0 & 0 & 12 \end{bmatrix} \times 10^6 psi$$

Finally the stiffness matrix $\underline{k}$:

$$\underline{k} = 10^4 \begin{bmatrix} 1466 & 500 & -866 & -99 & -733 & -500 & 133 & 99 \\ 500 & 1466 & 99 & 133 & -500 & -733 & -99 & -866 \\ -866 & 99 & 1466 & -500 & 133 & -99 & -733 & 500 \\ -99 & 133 & -500 & 1466 & 99 & -866 & 500 & -733 \\ -733 & -500 & 133 & 99 & 1466 & 500 & -866 & -99 \\ -500 & -733 & -99 & -866 & 500 & 1466 & 99 & 133 \\ 133 & -99 & -733 & 500 & -866 & 99 & 1466 & -500 \\ 99 & -866 & 500 & -733 & -99 & 133 & -500 & 1466 \end{bmatrix}$$

Higher Order Shape Function

- **Matrix D :Higher order shape function can be obtained by adding additional nodes to the each side of the linear element.**

- **It has higher order strain distribution in element, and it converges to the exact solution rapidly with few elements.**

- **It can more accurately approximate the irregular boundary shape.**

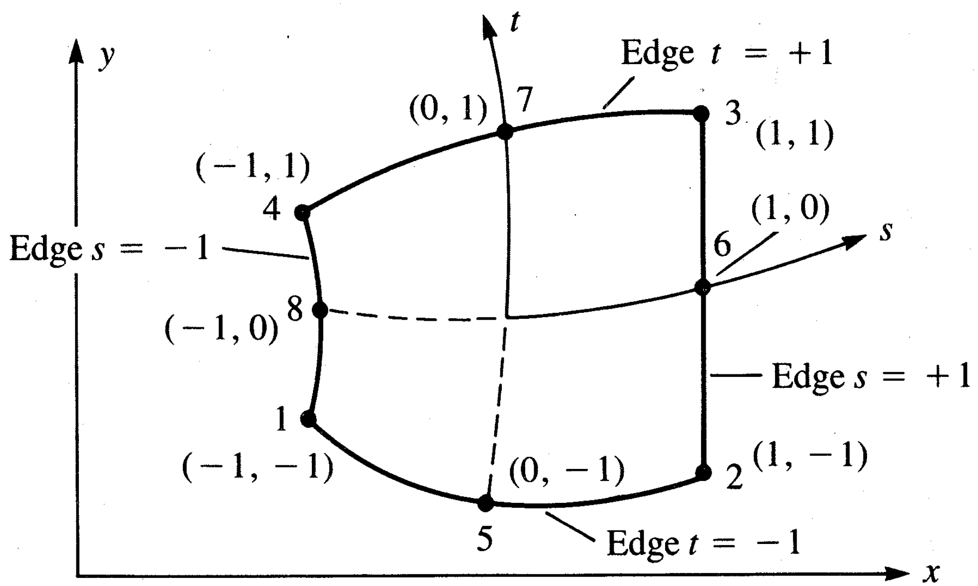


Fig. 11: 2nd order iso-parametric element

Higher Order Shape Function

Second order Iso-parametric element

$$x = a_1 + a_2 s + a_3 t + a_4 st + a_5 s^2 + a_6 t^2 + a_7 s^2 t + a_8 st^2$$
$$y = a_9 + a_{10} s + a_{11} t + a_{12} st + a_{13} s^2 + a_{14} t^2 + a_{15} s^2 t + a_{16} st^2$$

For the corner node (i = 1, 2, 3, 4)

$$N_1 = \frac{1}{4}(1-s)(1-t)(-s-t-1)$$
$$N_2 = \frac{1}{4}(1+s)(1-t)(s-t-1)$$
$$N_3 = \frac{1}{4}(1+s)(1+t)(s+t-1)$$
$$N_4 = \frac{1}{4}(1-s)(1+t)(-s+t-1)$$

or

$$N_i = \frac{1}{4}(1+ss_i)(1+tt_i)(ss_i+tt_i-1)$$
$$s_i = -1,1,1,-1 \quad for\ i = 1,2,3,4$$
$$t_i = -1,-1,1,1 \quad for\ i = 1,2,3,4$$

Higher Order Shape Function

For the middle node (i =5, 6, 7, 8)

$$N_5 = \frac{1}{2}(1-t)(1+s)(1-s)$$
$$N_6 = \frac{1}{2}(1+s)(1+t)(1-t)$$
$$N_7 = \frac{1}{2}(1+t)(1+s)(1-s)$$
$$N_8 = \frac{1}{2}(1-s)(1+t)(1-t)$$

or

$$N_i = \frac{1}{2}(1-s^2)(1+tt_i) \;\; t_i = -1{,}1 \; for \, i = 5{,}7$$
$$N_i = \frac{1}{2}(1-ss_i)(1-t^2) \;\; s_i = -1{,}1 \; for \, i = 5{,}7$$

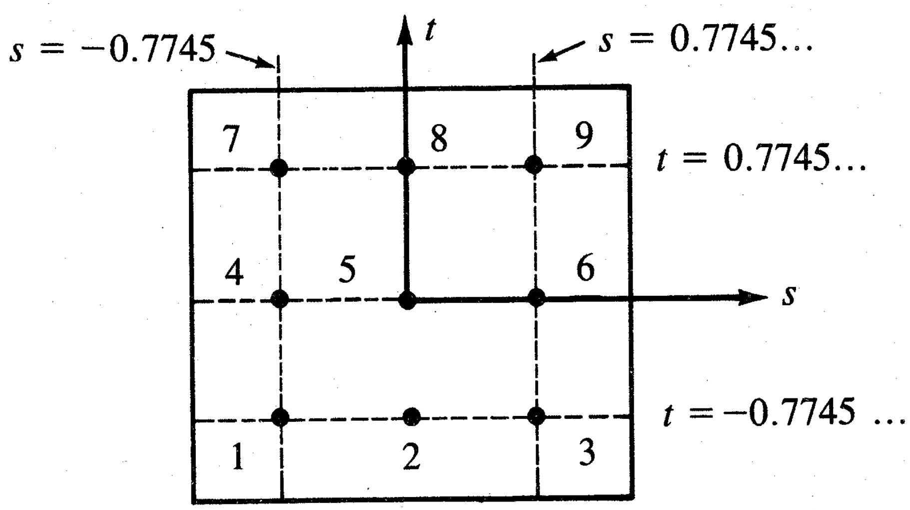
When edge shape and displacement are function of s^2 (if t is constant) or t^2 (if s is constant), it satisfies the general shape function conditions.

Higher Order Shape Function

Deformation function:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 & N_7 & 0 & N_8 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 & N_7 & 0 & N_8 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ v_8 \end{Bmatrix}$$

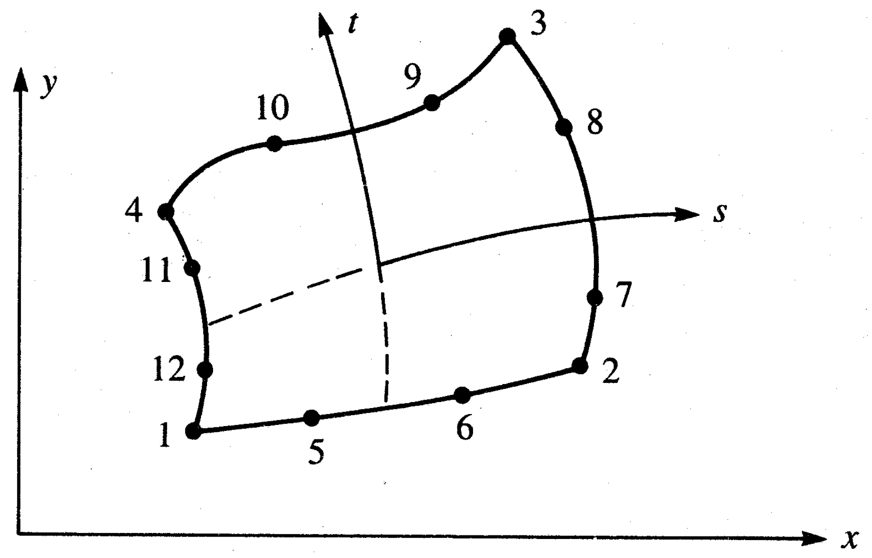
Strain matrix: $\varepsilon = \underline{B}d = \underline{D}'\underline{N}\underline{d}$



2^{nd} order iso-parameter with 8 nodes
For the calculation of $\underline{B}$ and $\underline{k}$,
9-points Gaussian rule is used
(3×3 rule). There is large difference between 2×2 and 3×3 rule, and 3×3 rule is generally recommended.
(Bathe and Wilson[7])

Higher Order Shape Function

3rd order Iso-parametric element:



Shape function of a 3rd order element is based on incomplete 4th order polynomial (see reference [3]).

$$x = a_1 + a_2 s + a_3 t + a_4 st + a_5 s^2 + a_6 t^2 + a_7 s^2 t + a_8 st^2$$
$$+ a_9 s^3 + a_{10} t^3 + a_{11} s^3 t + a_{12} st^3$$

y also has same polynomial equation.

Higher Order Shape Function

For the corner nodes (i = 1, 2, 3, 4):

$$N_i = \frac{1}{32}(1+ss_i)(1+tt_i)[9(s^2+t^2)-10]$$

$$where, \quad \begin{array}{l} s_i = -1,1,1,-1 \;\; for \;\; i = 1,2,3,4 \\ t_i = -1,-1,1,1 \;\; for \;\; i = 1,2,3,4 \end{array}$$

For the nodes (i = 7, 8, 11, 12) when $s = \pm 1$:

$$N_i = \frac{9}{32}(1+ss_i)(1+9tt_i)(1-t^2)$$

$$where, \qquad s = \pm 1, \qquad t_i = \pm 1/3$$

For the nodes (i = 5, 6, 9, 10) when $t = \pm 1$:

$$N_i = \frac{9}{32}(1+tt_i)(1+9ss_i)(1-s^2)$$

$$where, \qquad t = \pm 1, \qquad s_i = \pm 1/3$$

When the shape function of coordinates has lower order than that of deformation, it is called Subparametric formulation (For example, x is linear, u is is 2nd order function). The opposite way is called Superparametric formulation.

THANK YOU
FOR LISTENING