Mechanics and Design

Chapter 9. Plastic and Anisotropic Behavior

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Idealizations of Stress-Strain Curve

- Different materials often have quite dissimilar stress-strain relations and no simple mathematical equation can fit the entire stress-strain curve of one of the materials.
- Because we wish the mathematical part of our analysis to be as simple as possible, consistent with physical reality, we shall idealize the stress-strain curves into forms which can be described by simple equations.

For instance,

Springs must accommodate the desired deformations repeatedly and reproducibly.

→ linear approximation to the material: the stress-strain curve.

Bumpers should deform plastically in case of an accident.

→ approximation is needed for plastic region as well as elastic region.

Shear pins are intended to fracture completely at certain loads.

→ the elastic deformation may of no importance at all.

Idealizations of Stress-Strain Curve

Failure Classification

Fracture

brittle structures – fracture with little plastic deformation

ductile structures – there are difficulties in predicting fracture (*read the Text* p.275)

→ nonlinear relation between stress and strain

Fatigue

occurs even if the stresses are below the yield strength

→ linear relation between stress and strain

in case of plastic yielding at the tip of the growth of fatigue cracks

→ relations between stress and strain which take plasticity into account

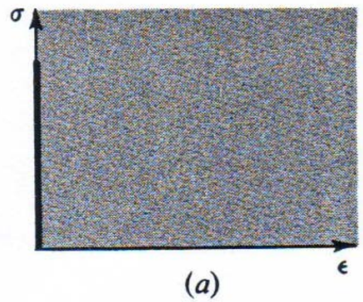
Corrosion

can be greatly accelerated in the presence of stress

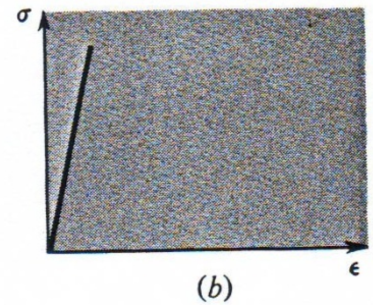
→ the elastic stress-strain assumptions are of practical use

Idealizations of Stress-Strain Curve

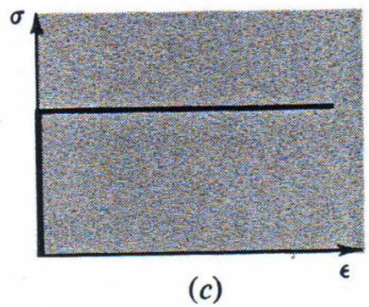
Idealized Models of Stress-Strain Curve



Rigid material

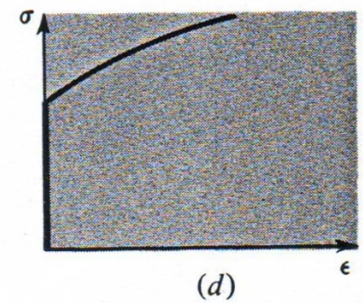


Linearly elastic material



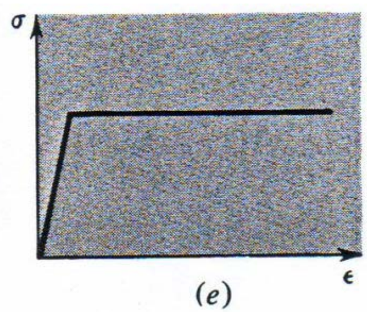
Perfectly plastic material

(non-strain-hardening)



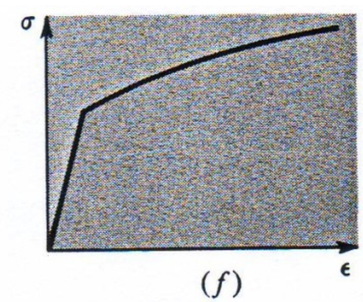
Rigid-plastic material

(strain-hardening)



Elastic-perfectly plastic material

(non-strain-hardening)



Elastic-plastic material

(strain-hardening)

Fig. 9.1 Stress-strain curve*

* Lardner, Thomas J. *An introduction to the mechanics of solids*. McGraw-Hill College, 1972.

Idealizations of Stress-Strain Curve

Material Classification Based Stress-Strain Relation

Rigid material

is one which has no strain regardless of the applied stress. This idealization is useful in studying the gross motions and forces on machine parts to provide for adequate power and for resistance to wear.

Linearly elastic material

is one in which the strain is proportional to the stress. This idealization is useful when we are designing for small deformations, for stiffness, or to prevent fatigue or fracture in brittle structures.

Fig. 9.2 Rigid material

Fig. 9.3 Linearly elastic material

Idealizations of Stress-Strain Curve

Material Classification Based Stress-Strain Relation

Rigid plastic material

is one in which elastic and time-dependent deformations are neglected. If the stress is released, the deformation remains. Strain-hardening may be neglected, or a relation for the strain-hardening may be assumed; in former case, the material is termed *perfectly plastic*. Such idealizations are useful in designing structures for their maximum loads, in studying many machining and metal-forming problems, and in some detailed studies of fracture.

Elastic-plastic material

is one which both elastic and plastic strains are present; strain-hardening may or may not be assumed to be negligible. This idealization is useful in designing against moderate deformations and carrying out detailed studies of the mechanisms of fracture, wear, and friction.

Fig. 9.4 Rigid plastic material

Fig. 9.5 Elastic-Plastic material

Idealizations of Stress-Strain Curve

Example 9.1*

Two coaxial tubes, the inner one of 1020 CR steel and cross sectional area A_s, and the outer one of 2024-T4 aluminum alloy and of area A_a, are compressed between heavy, flat end plates. Determine the load-deflection curve of the assembly as it is compressed into the plastic region by an axial force P

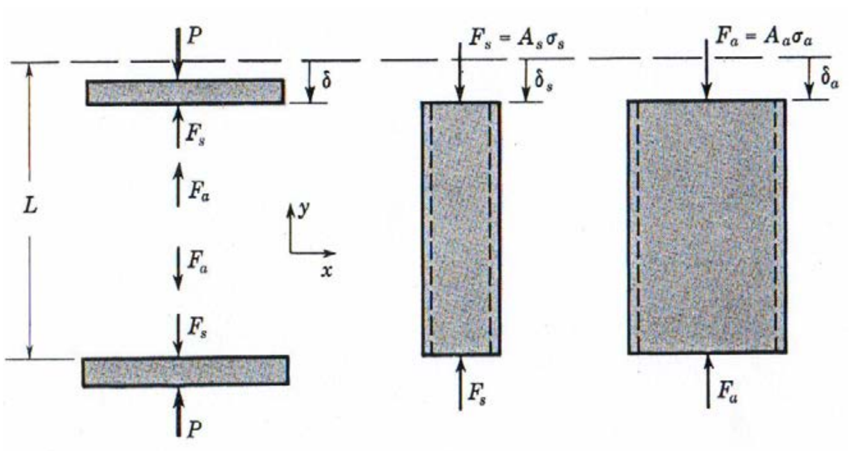


Fig. 9.6 Example 5.1*

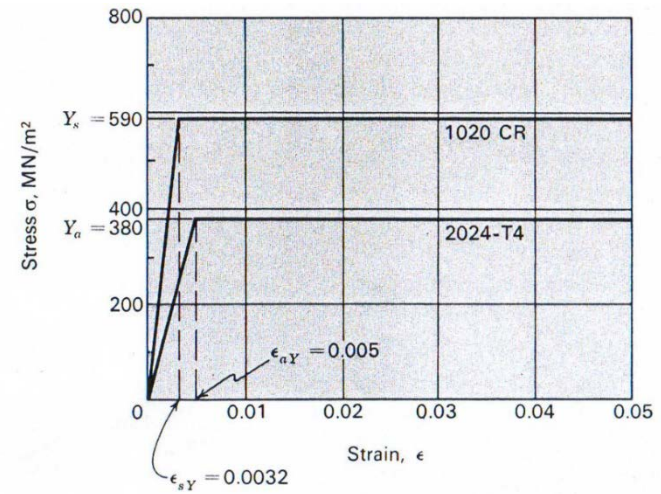


Fig. 9.7 Example 5.1*

* Lardner, Thomas J. *An introduction to the mechanics of solids*. McGraw-Hill College, 1972.

Idealizations of Stress-Strain Curve

Example 9.1*

Geometric Compatibility

$$\varepsilon_s = \varepsilon_a = \varepsilon = \frac{\delta}{L}$$

Stress-Strain Relations

$0 \leq \varepsilon \leq 0.0032$	$0.0032 \leq \varepsilon \leq 0.005$	$0.005 \leq \varepsilon$
$\sigma_s = E_s \varepsilon_s = E_s \varepsilon$ $\sigma_a = E_a \varepsilon_a = E_a \varepsilon$	$\sigma_s = Y_s = 590 \text{ MN/m}^2$ $\sigma_a = E_a \varepsilon_a = E_a \varepsilon$	$\sigma_s = Y_s = 590 \text{ MN/m}^2$ $\sigma_a = Y_a = 380 \text{ MN/m}^2$

Equilibrium

$$\sum F_y = \sigma_s A_s + \sigma_a A_a - P = 0$$

where

$$E_s = \frac{590}{0.0032} = 184 GN/m^2$$

$$E_a = \frac{380}{0.005} = 76 GN/m^2$$

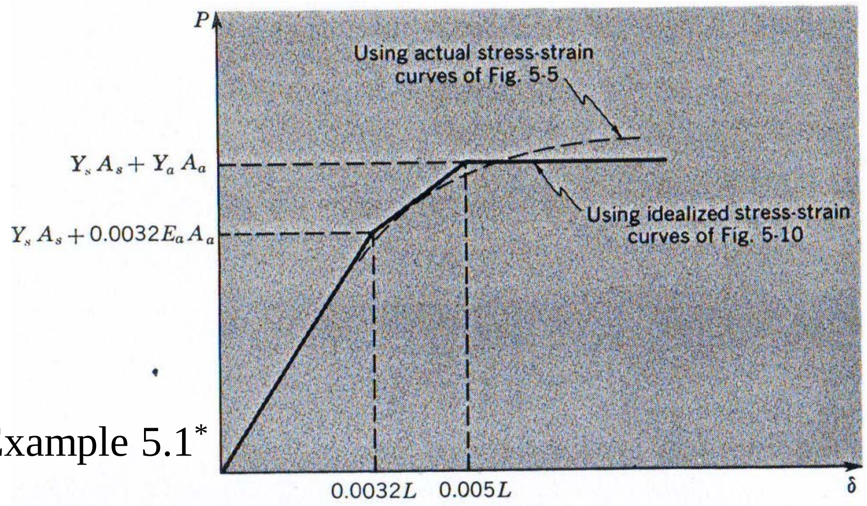


Fig. 9.8 Example 5.1*

* Lardner, Thomas J. *An introduction to the mechanics of solids*. McGraw-Hill College, 1972.

Composite Materials and Anisotropic Elasticity

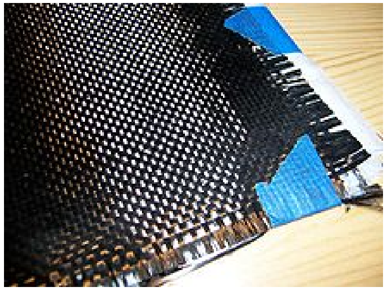
Glass – least cost, but low stiffness

Boron filaments – commercially available, but expensive

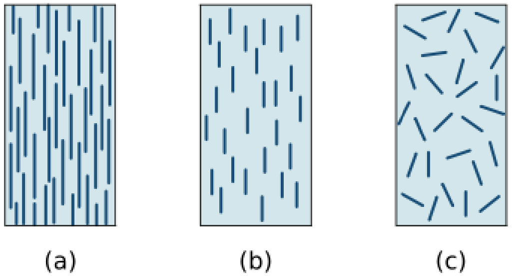
Graphite filaments – high stiffness and strength and very costly

Applications – machine tools, circular saws, printing presses and textile

machinery (or springs, bearings, and pressure vessels)



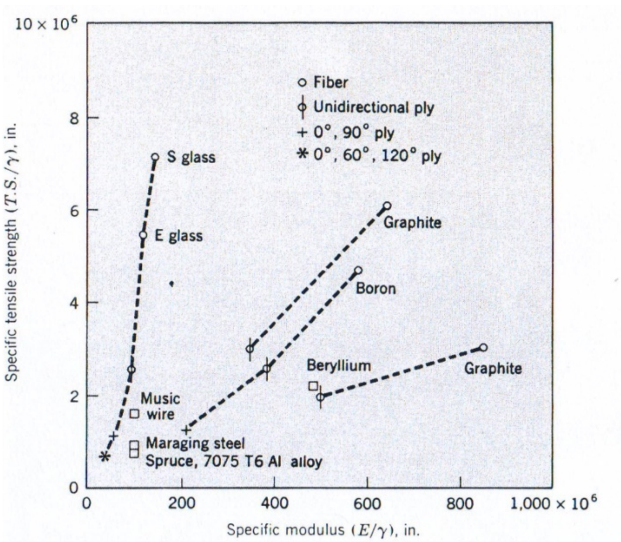
Property	Glass		Boron[2]	Graphite	
	Type E	Type S		Stiff	Strong
Diameter, in.	0.0002~0.0008		0.004	0.00027~0.00035	
Specific gravity	2.54	2.50	2.63	1.96	1.74
Modulus of elasticity, 10^6 psi	10.5	12.6	55	>55	>35
Tensile strength, 10^3 psi	450-550	650	450	>200	>350
Cost per pound in epoxy tape 1968				$595	
Cost per pound in epoxy tape 1969				$495	
Cost per pound in epoxy tape 1970	$5		$150	$380	



Typologies of fibre-reinforced composite materials:
a) continuous fibre-reinforced
b) discontinuous aligned fibre-reinforced
c) discontinuous random-oriented fibre-reinforced.

Composite Materials and Anisotropic Elasticity

Fibers require a matrix to hold them in place during normal handling. Much of the advantage of the fibers themselves is lost in this process. In general the compressive modulus should be same as the tensile value. The compressive strength is limited by buckling of fibers within the matrix, whereas the tensile strength is determined by fracture from flaws.



Specific tensile strength (tensile strength per unit weight density) and specific modulus of high-strength materials. Materials are in sheet form and isotropic.

Uniaxial properties of fiber laminates. (Parallel fibers in epoxy matrix)

Property		*Glass*	*Boron*	*Graphite*	
Volume, % fiber		64 (wt.)	50	55	
Specific gravity		1.8		1.5–1.6	
Tensile modulus, 10^6 psi,	72°F	5.7	30–32	25–33	18–22
	300°F			25–32	18–20
Tensile strength, 10^3 psi,	72°F	160	190–230	105–120	150–200
	300°F	50		105–115	150–190
Compressive modulus, 10^6 psi,	72°F		35–36		
Compressive strength, 10^3 psi,	72°F		440–460	75–110	120–160
Flexural modulus, 10^6 psi,	72°F	5.3	28	23–32	18–21
	300°F	1.0	24	20–30	17–18
Flexural strength, 10^3 psi,	72°F	165	245	120–140	185–250
	300°F	30	215	95–105	155–220
Short beam, shear strength, 10^3 psi,	72°F		17	7–9	13–16
	300°F		9	5	8–9

Composite Materials and Anisotropic Elasticity

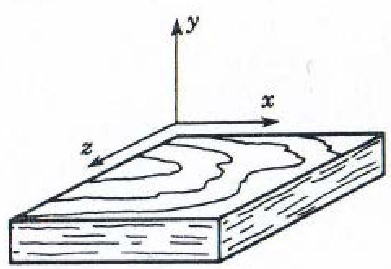
Anisotropic materials:

Materials with different properties in different directions

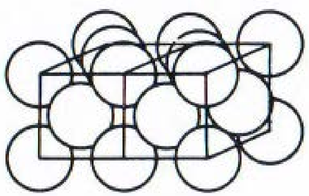
Orthotropic materials: http://en.wikipedia.org/wiki/Orthotropic_material

Materials having the structures all appear identical after a 180° rotation about any one of the three orthogonal coordinate axes.

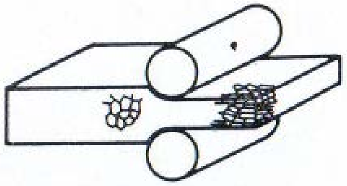
(An orthogonal material is a special case of an anisotropic material.)



Wood

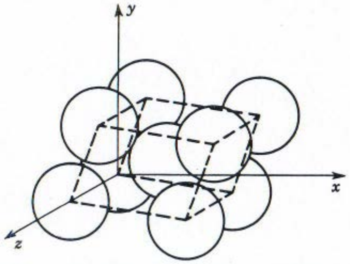


Single Crystal



Rolled metal

Plastic-impregnated cloth laminate



Composite Materials and Anisotropic Elasticity

If we assume that the strains in an elastic anisotropic material are linearly related to the stresses, then the stress-strain relations are given by

$$\begin{aligned}
\varepsilon_x &= S_{11}\sigma_x + S_{12}\sigma_y + S_{13}\sigma_z + S_{14}\tau_{xy} + S_{15}\tau_{yz} + S_{16}\tau_{zx} \\
\varepsilon_y &= S_{21}\sigma_x + S_{22}\sigma_y + S_{23}\sigma_z + S_{24}\tau_{xy} + S_{25}\tau_{yz} + S_{26}\tau_{zx} \\
\varepsilon_z &= S_{31}\sigma_x + S_{32}\sigma_y + S_{33}\sigma_z + S_{34}\tau_{xy} + S_{35}\tau_{yz} + S_{36}\tau_{zx} \\
\gamma_{xy} &= S_{41}\sigma_x + S_{42}\sigma_y + S_{43}\sigma_z + S_{44}\tau_{xy} + S_{45}\tau_{yz} + S_{46}\tau_{zx} \\
\gamma_{yz} &= S_{51}\sigma_x + S_{52}\sigma_y + S_{53}\sigma_z + S_{54}\tau_{xy} + S_{55}\tau_{yz} + S_{56}\tau_{zx} \\
\gamma_{zx} &= S_{61}\sigma_x + S_{62}\sigma_y + S_{63}\sigma_z + S_{64}\tau_{xy} + S_{65}\tau_{yz} + S_{66}\tau_{zx}
\end{aligned}$$

Actually, the elastic constants with unequal subscripts are the same when the order of the subscripts is reversed; $S_{12} = S_{21}, S_{45} = S_{54}$, etc.

The symmetry of an orthotropic material requires that there be no interaction between the various shear components or the shear and normal components when the *x*, *y*, *z* axes are chosen parallel to the axes of structural symmetry.

$$\begin{aligned}
\varepsilon_x &= S_{11}\sigma_x + S_{12}\sigma_y + S_{13}\sigma_z & \qquad \gamma_{xy} &= S_{44}\tau_{xy} \\
\varepsilon_y &= S_{21}\sigma_x + S_{22}\sigma_y + S_{23}\sigma_z & \qquad \gamma_{yz} &= S_{55}\tau_{yz} \\
\varepsilon_z &= S_{31}\sigma_x + S_{32}\sigma_y + S_{33}\sigma_z & \qquad \gamma_{zx} &= S_{66}\tau_{zx}
\end{aligned}$$

Composite Materials and Anisotropic Elasticity

Orthotropic elastic constants for fiber-epoxy materials

Fiber	*Directions*	S_{11}, 10^{-6} in.^{2}/lb	$\frac{S_{22}}{S_{11}}$	$\frac{S_{12}}{S_{11}}$	$\frac{S_{44}}{S_{11}}$	*Coeff. of linear expansion*, 10^{-6}/°F α_x	α_y
Isotropic		$1/E$	1	$-\nu$	$2(1+\nu)$		
Glass A[1]	0°	0.178	3.25	−(0.28–0.30)	15.0		
Glass B[1]	0°	0.175				4.8	12.3
	0°, 90°	0.27	1			7.1	7.1
	±45°	0.62	1			7.1	7.1
	0°, 60°, 120°	0.35–0.38	1			8.4	8.4
Boron	0°	0.031–33	8.2–9.4	−(0.17–0.20)	16.2–17.6		
	0°, 90°	0.055–58	1	−0.05			
	±45°	0.26	1	−0.85			

[1] Glasses A and B are from two different manufacturers.

Modulus of elasticity for orthotropic materials in sheet form

Material	*Principal structural direction*	*Angle* 0° E, 10^6 psi	45° E, 10^6 psi	90° E, 10^6 psi
Cold-rolled iron[1]	Direction of rolling	32.8	29.3	39.1
Cold-rolled copper[2]	Direction of rolling	19.8	15.5	20.0
Cold-rolled copper, recrystallized[2]	Direction of rolling	10.0	17.5	9.5
Glass-fiber-reinforced polyester[3]	Direction of warp	2.0–2.7	1.2–1.8	1.7–2.4

Composite Materials and Anisotropic Elasticity

Cubic materials:

If a material has equal properties in three orthogonal directions, it is said to have a *cubic structure*. In this case many of the elastic constants are identical, and those equation reduce to:

$$\varepsilon_x = S_{11}\sigma_x + S_{12}(\sigma_y + \sigma_z) \qquad \gamma_{xy} = S_{44}\tau_{xy}$$
$$\varepsilon_y = S_{11}\sigma_y + S_{12}(\sigma_z + \sigma_x) \qquad \gamma_{yz} = S_{44}\tau_{yz}$$
$$\varepsilon_z = S_{11}\sigma_z + S_{12}(\sigma_x + \sigma_y) \qquad \gamma_{zx} = S_{44}\tau_{zx}$$

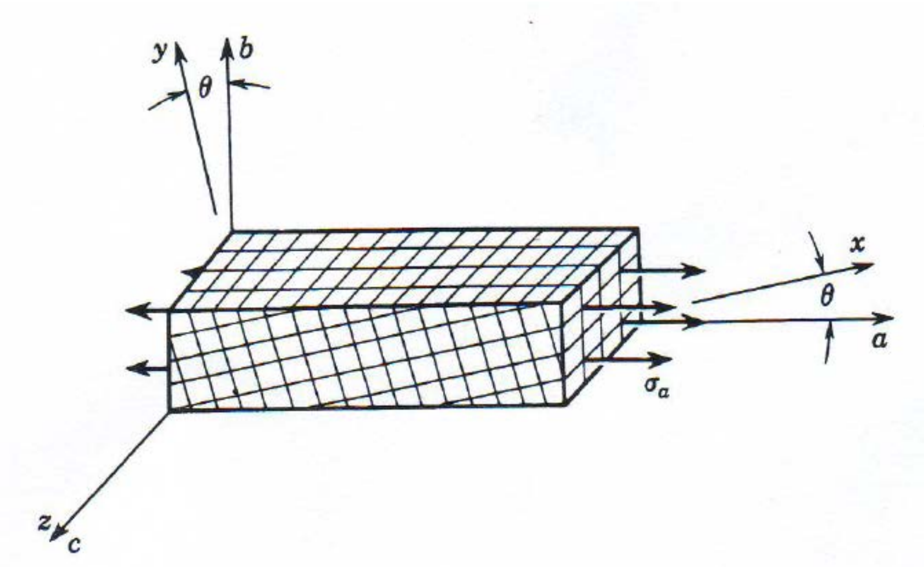
The isotropy condition (equal properties in all directions) is not in general satisfied, so there remain three independent elastic constants.

Elastic constants for cubic materials

Material	S_{11}, 10^{-7} in.^{2}/lb	S_{12}, 10^{-7} in.^{2}/lb	S_{44} 10^{-7} in.^{2}/lb	$S_{11} - S_{12} - \frac{1}{2}S_{44}$, 10^{-7} in.^{2}/lb
Al	1.10	−0.40	2.43	0.28
Cu	1.03	−0.43	0.92	1.00
Fe	0.522	−0.195	0.595	0.419
Pb	6.43	−0.29	4.80	4.32
W	0.178	−0.050	0.455	0.000
95% Al, 5% Cu	1.04	−0.48	2.56	0.024
72% Cu, 28% Zn	1.34	−0.58	0.96	0.96

Composite Materials and Anisotropic Elasticity

Cubic materials:



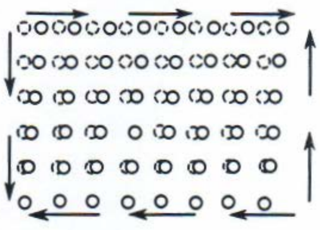
Even with cubic symmetry, the stiffness of a crystal depends markedly on the orientation.

The ratio between the normal stress component σ_a and the normal strain component ε_a gives the modulus of elasticity in the *a* direction.

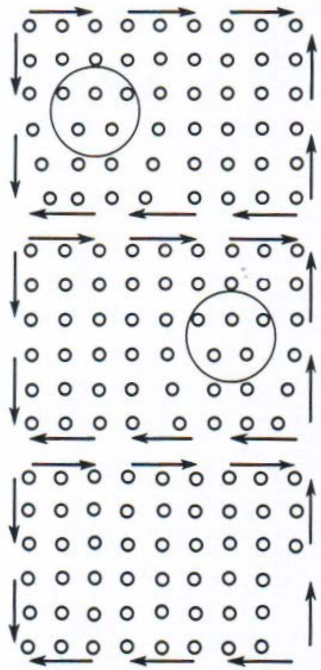
This modulus of elasticity may differ from that in one of the crystallographic directions by a large amount.

In orthotropic materials the coefficients of thermal expansion will, in general, be different in the different crystallographic directions.

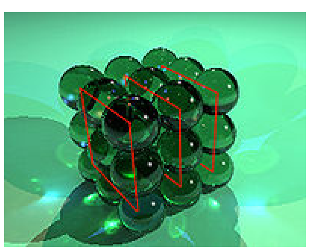
Criteria for Initial Yielding

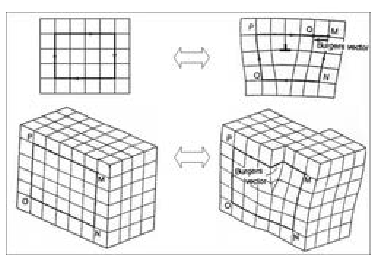


(*a*)



(*b*)





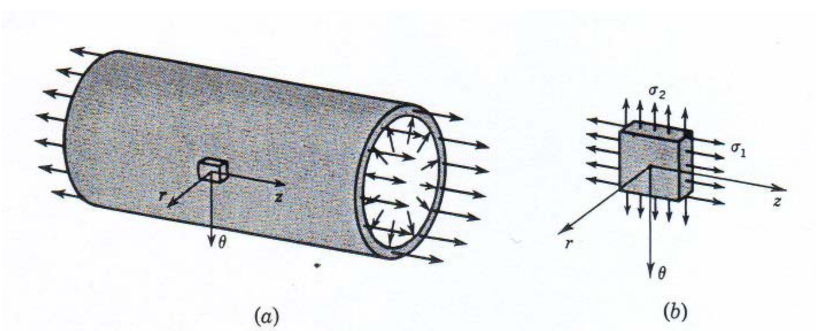
- During **elastic deformation** of a crystal, there is a uniform shifting of whole planes of atoms relative to each other (Fig. *a*).
- **Plastic deformation** depends on the motion of individual imperfections (edge dislocation) in the crystal structure (Fig. *b*).

Under the presence of a shear stress the dislocation will tend to migrate. These dislocations can move in a variety of directions on a number of crystallographic planes. By a combination of such motions, plastic strain can be produced.

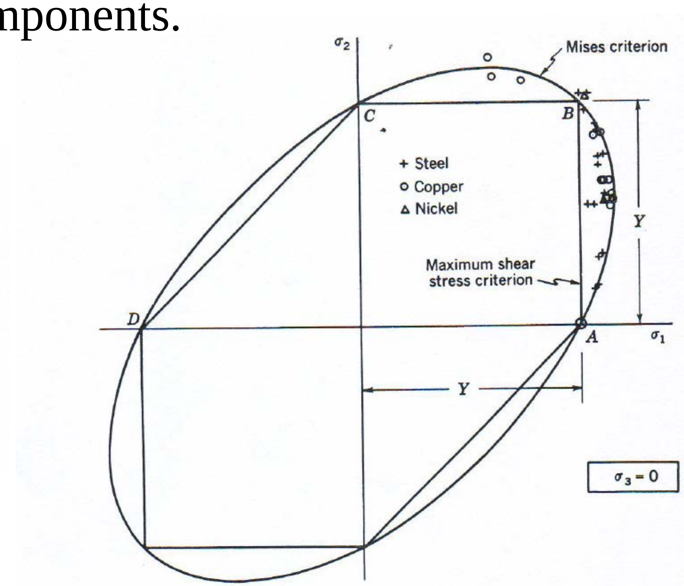
(A hydrostatic state of stress would not tend to move the dislocation.)

Criteria for Initial Yielding

- In the uniaxial tensile test, the condition for the beginning of plastic flow was described by the yield strength, giving the axial normal component of stress at which practically important plastic deformation was observed.
- When several components of stress are present, yielding must depend on some particular combination of these components.



A thin-walled cylinder of internal radius r and wall thickness t with an internal pressure and axial load (*a*). The radial stress is small compared with the tangential stress, and thus we may consider a small element of this shell as being in plane stress with the principal stress components in (*b*)



Experiments have been carried out on such thin-walled tubes with various amounts of axial load applied to determine under what combinations of these two normal components of stress the material will yield.

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Criteria for Initial Yielding

Two empirical equations:

1.Criteria for yielding are based only on the magnitude of the principal stresses (isotropic material).

2.Since experimental work has substantiated the expectation from dislocation theory that a hydrostatic state of stress does not affect yielding, the two criteria are based on the *differences* between the principal stresses.

The first of the criteria assumes that the yielding can occur in a three-dimensional state of stress when the root mean square of the differences between the principal stresses reached the same value which it has when yielding occurs in the tensile test ($\sigma_1 = Y, \sigma_2 = 0, \sigma_3 = 0$). When Y is the stress at which yielding begins in the tensile test,

$$\sqrt{1/3\,[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]} = \sqrt{2/3}\,Y$$

$$Y = \sqrt{1/2\,[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]} \qquad (9.1)$$

→ ***Mises yield criterion***

Criteria for Initial Yielding

When stress state is known in terms of stress components with respect to non-principal axes

$$Y = \sqrt{1/2\,\{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2\} + 3\tau_{xy}^2 + 3\tau_{yz}^2 + 3\tau_{zx}^2} \qquad (9.2)$$

The second empirical criterion assumes that yielding occurs whenever the maximum shear stress reaches the values it has when yielding occurs in the tensile test.

In the tensile test the maximum shear stress is $Y/2$, so this criterion says that yielding occurs when

$$\tau_{\max} = \frac{\sigma_{\max} - \sigma_{\min}}{2} = \frac{Y}{2} \qquad (9.3)$$

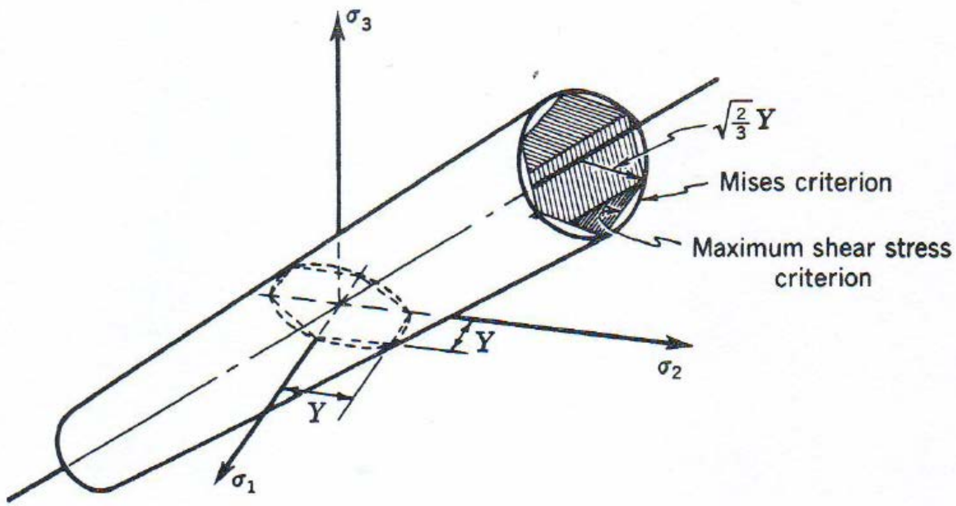
→ ***maximum shear-stress criterion***

Criteria for Initial Yielding

A geometrical interpretation of the criteria (5.23) and (5.25)

(9.1): Right-circular cylinder of radius $\sqrt{2/_3}\,Y$. Yielding occurs for any state of stress which lies on the surface of this circular cylinder. When we have a state of plane stress ($\sigma_3 = 0$), the Mises criterion is represented by an ellipse.

(9.3): Hexagonal cylinder inscribed within the right-circular cylinder of the Mises criterion. Yielding occurs for any state of stress which lies on the surface of this hexagonal cylinder. For plane stress ($\sigma_3 = 0$) the maximum shear-stress criterion is represented by six-sided polygon.

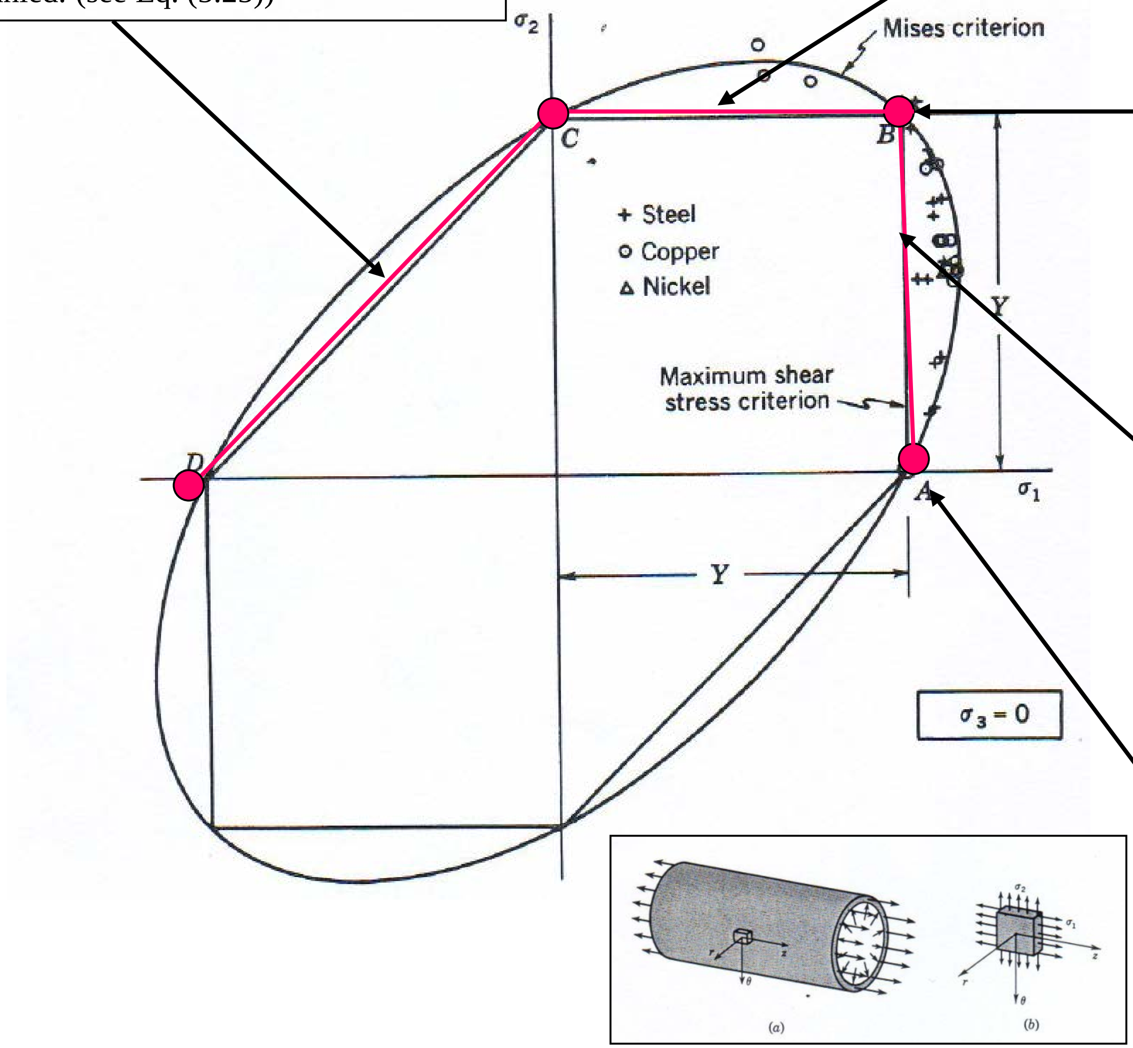


Criteria for Initial Yielding

When the tube is compressed in the axial direction, an equal decrease in σ_θ must be accompanied. (see Eq. (5.25))

When the internal pressure is constant and the axial load is decreased .

When internal pressure is great so that the σ_θ equals the σ_z. The shear stress on the plane at 45° to the θ and r axes becomes equal to the shear stress on the plane at 45° to the z and r axes.

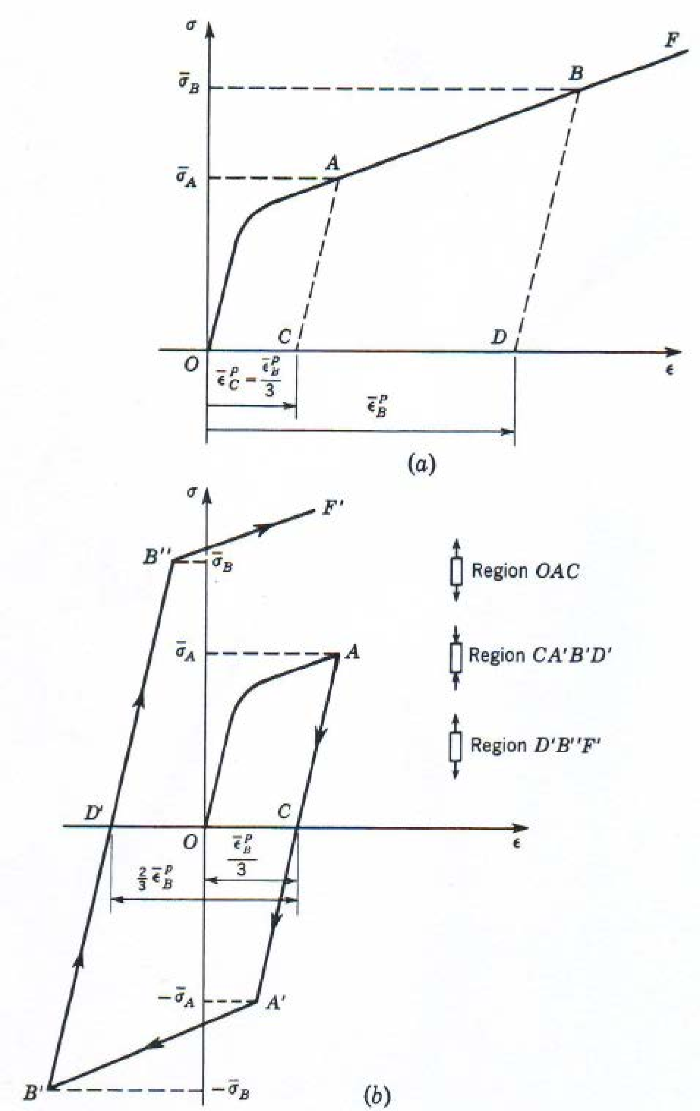


Increasing the internal pressure does not change either the σ_z or the σ_r , so the addition of the internal pressure does not increase the tendency to yield.

The tube is under primarily axial tension with just a little internal pressure.

$\sigma_{\max} = \sigma_z, \sigma_\theta = 0, \sigma_r = 0,$

Behavior Beyond Initial Yielding in the Tensile Test



OA: specimen is stretched in tension. (plastic extension strain)

ACA’: the load is released (*C*), and then reapplied as compression (*A*’).

A’*B*’: as compressive load increases, yielding continues (same shape as the curve *AB* and compressive plastic strain).

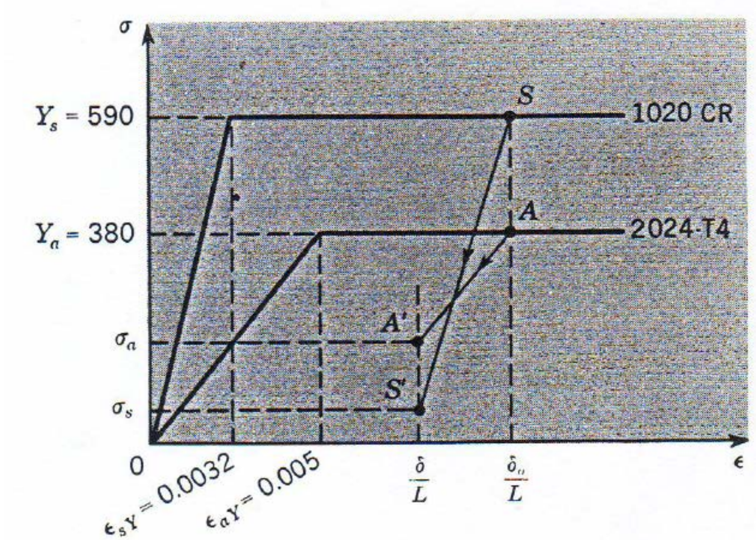
B’*D*’: the load is released,

D’*B*”*F*’: reapplication of the tensile load (the same shape of *DBF*).

Behavior Beyond Initial Yielding in the Tensile Test

Example 9.2*

Returning to Example 5.1, we ask, what will happen if we remove the load *P* after we have strained the combined assembly so that both the steel and the aluminum are in the plastic range, that is, beyond a strain of 0.005?



Stress-Strain relations

S and *A*: the states of the steel and aluminum under the load *P*,

S' and *A*': the states after the load has been removed.

From the unloading curve *SS*' and *AA*',

$$\sigma_s = Y_s - E_s \frac{\delta_o - \delta}{L} \quad \text{(a)}$$

$$\sigma_a = Y_a - E_a \frac{\delta_o - \delta}{L}$$

* Lardner, Thomas J. *An introduction to the mechanics of solids*. McGraw-Hill College, 1972.

Behavior Beyond Initial Yielding in the Tensile Test

Example 9.2*

Substituting (a) into equilibrium equation of Example 5.1 and setting $P = 0$

$$A_s\left(Y_s - E_s\frac{\delta_o - \delta}{L}\right) + A_a\left(Y_a - E_a\frac{\delta_o - \delta}{L}\right) = 0 \qquad \text{(b)}$$

$$\frac{\delta_o - \delta}{L} = \frac{A_s Y_s + A_a Y_a}{A_s E_s + A_a E_a} \qquad \text{(c)}$$

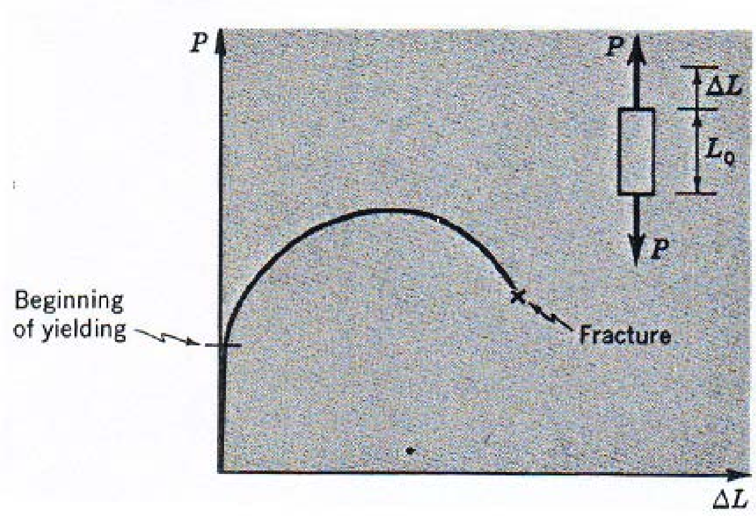
Substituting (c) into (b)

$$\begin{aligned} \sigma_{s_{residual}} &= Y_s \frac{1 - \dfrac{Y_a/E_a}{Y_s/E_s}}{1 + E_s\,A_s/E_a\,A_a} = Y_s \frac{1 - \dfrac{\varepsilon_{aY}}{\varepsilon_{sY}}}{1 + E_s\,A_s/E_a\,A_a} \\ \sigma_{a_{residual}} &= Y_a \frac{1 - \dfrac{Y_s/E_s}{Y_a/E_a}}{1 + E_a\,A_a/E_s\,A_s} = Y_a \frac{1 - \dfrac{\varepsilon_{sY}}{\varepsilon_{aY}}}{1 + E_a\,A_a/E_s\,A_s} \end{aligned} \qquad \text{(d)}$$

* Lardner, Thomas J. *An introduction to the mechanics of solids*. McGraw-Hill College, 1972.

Behavior Beyond Initial Yielding in the Tensile Test

A tensile test of a ductile material



True stress: the intensity of load per unit of actual area

Engineering stress: the intensity of load per unit of original area

- At the maximum load: Tensile strength (for the true stress, at this point is already higher than the tensile strength.)

- Due to avoidable variations one particular section of the specimen will arrive at the condition where the increase in flow stress will not compensate for the decrease in area, while the other sections of the specimen will still be able to carry higher loads. → This phenomenon causes *necking*.

* Lardner, Thomas J. *An introduction to the mechanics of solids*. McGraw-Hill College, 1972.

Behavior Beyond Initial Yielding in the Tensile Test

A tensile test of a ductile material

Two different approaches in describing the strain in the tensile test:

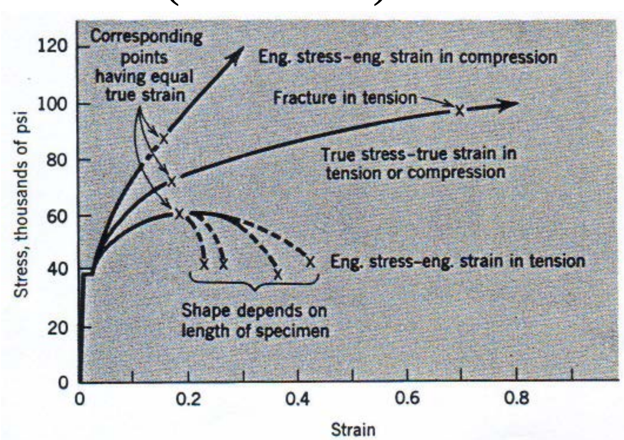
1. Defining the strain as the ratio of the change in length to the original length of the specimen. (engineering strain)

$$\varepsilon_x = \frac{\Delta L}{L_o} = \frac{L_f - L_o}{L_o} \qquad (9.4)$$

2. The total strain as being the sum of a number of increments of strain

$$\varepsilon_x = \sum \Delta\varepsilon_x = \sum \frac{\Delta L}{L} = \int_{L_o}^{L_f} \frac{dL}{L} = \ln\frac{L_f}{L_o} \qquad (9.5)$$

where L is the current length of the specimen when the increment of elongation occurs. (true strain)



The results obtained from uniaxial tensile and compression tests

(plotted on an engineering basis and on an true basis)

Criteria For a Continued Yielding

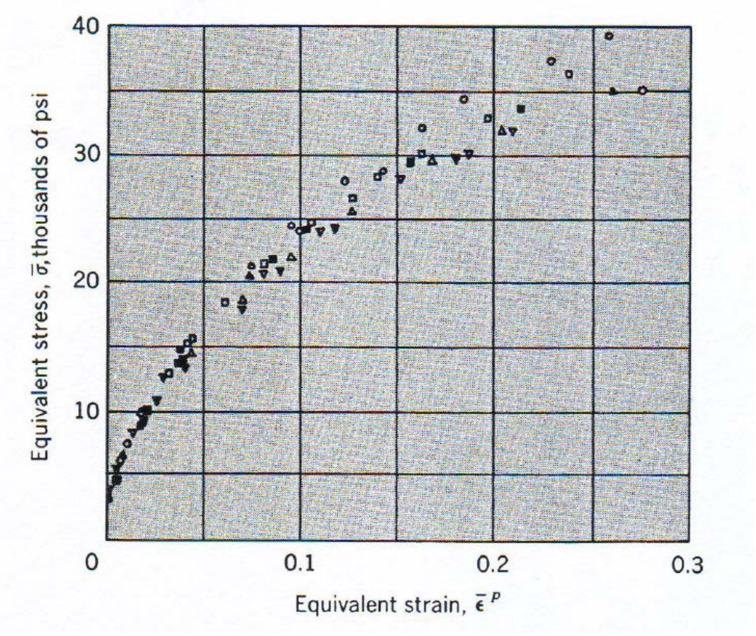
The tendency for further yielding can be measured by an equivalent stress:

$$\bar{\sigma} = \sqrt{1/2\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]} \qquad (9.6)$$

Initial yielding can occur when $\bar{\sigma} = Y$.

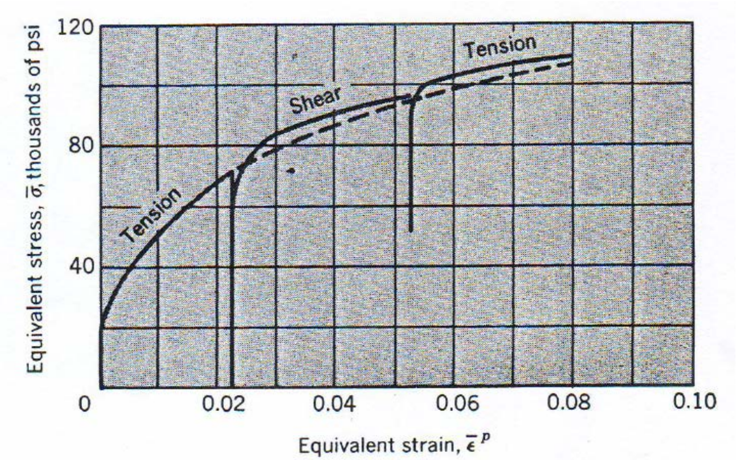
The equivalent stress depends upon the equivalent plastic strain:

$$\bar{\varepsilon}^p = \int \sqrt{2/9\left[\left(d\varepsilon_1^p - d\varepsilon_2^p\right)^2 + \left(d\varepsilon_2^p - d\varepsilon_3^p\right)^2 + \left(d\varepsilon_3^p - d\varepsilon_1^p\right)^2\right]} \qquad (9.7)$$

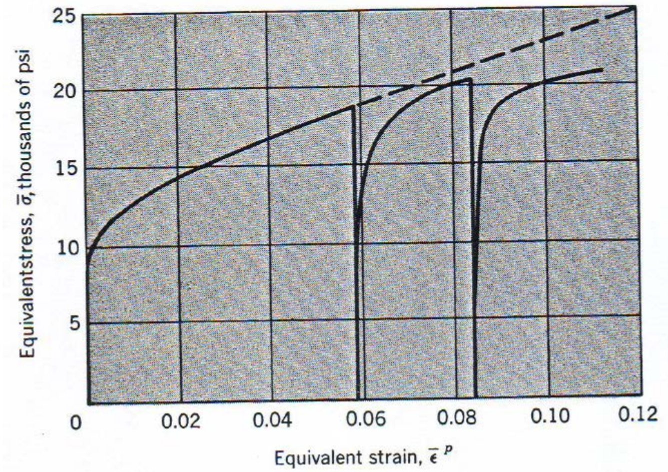


Criteria For a Continued Yielding

The correlation is put to a more severe test when the kind of stressing is changed during the test, as , for example, when first tensile, then shear, and then tensile stresses are applied (*a*). When the change in stress during the test is a complete reversal, the correlation is less satisfactory, as in (*b*).



(*a*)



(*b*)

The lowered elastic limit observed on the reversals of load in (*b*) is called the "*Bauschinger* effect".

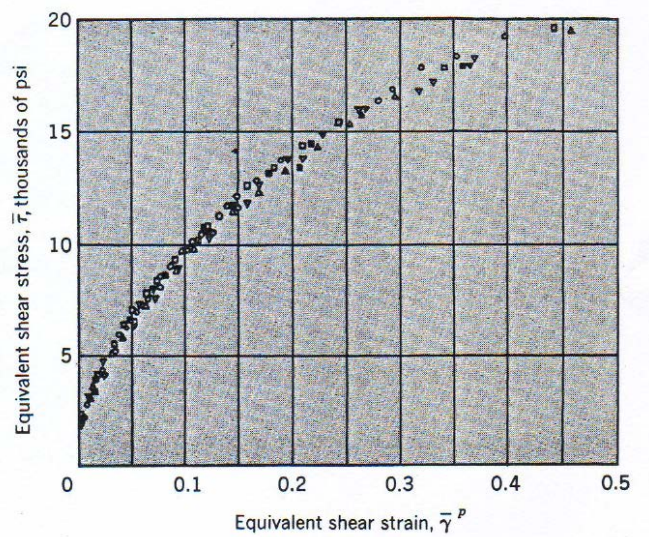
Criteria For a Continued Yielding

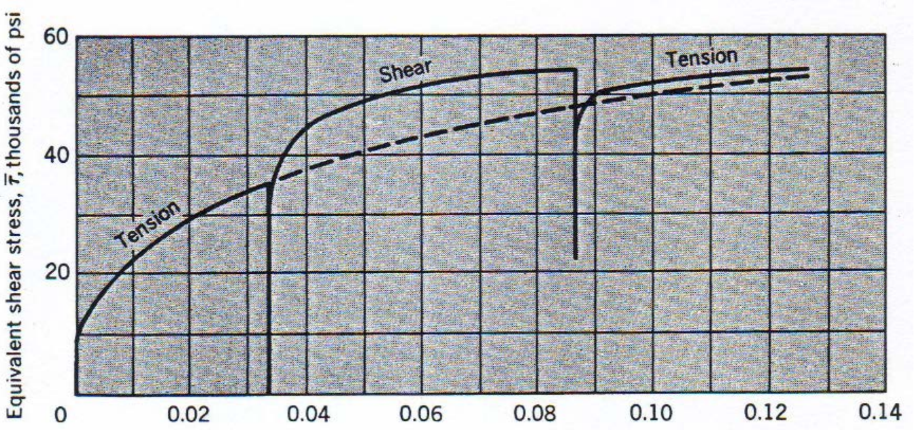
For materials which yield initially according to the maximum shear-stress criterion, it has been found that the tendency for further yielding can be measured by an equivalent shear stress:

$$\bar{\tau} = \frac{\sigma_{\max} - \sigma_{\min}}{2} \tag{9.8}$$

The equivalent plastic shear strain:

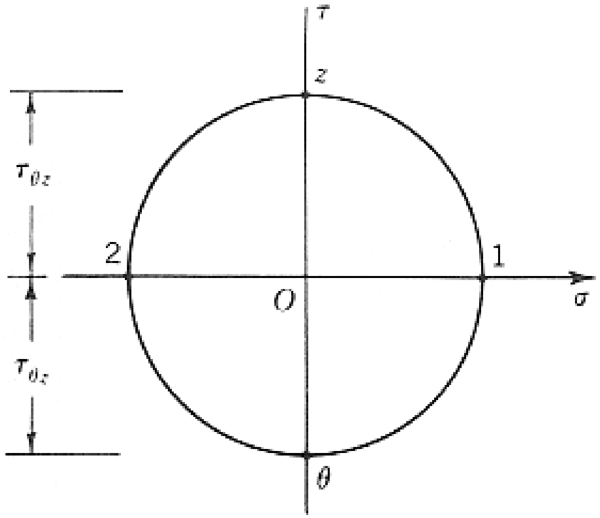
$$\bar{\gamma}^p = \int [(d\varepsilon^p)_{\max} - (d\varepsilon^p)_{\min}] \tag{9.9}$$





Alternating tension and shear correlated on the basis of equivalent shear stress and equivalent plastic shear strain.

The Onset of Yielding in Torsion



Using the Mises criterion,

the effective stress: $\bar{\sigma}$

$$\sigma_1 = \tau_{\theta z} \quad \sigma_2 = -\tau_{\theta z} \quad \sigma_3 = 0 \qquad (9.10)$$

$$\bar{\sigma} = \sqrt{\frac{1}{2}[(2\tau_{\theta z})^2 + (-\tau_{\theta z})^2 + (-\tau_{\theta z})^2]}$$

$$= \sqrt{3}\tau_{\theta z} \qquad (9.11)$$

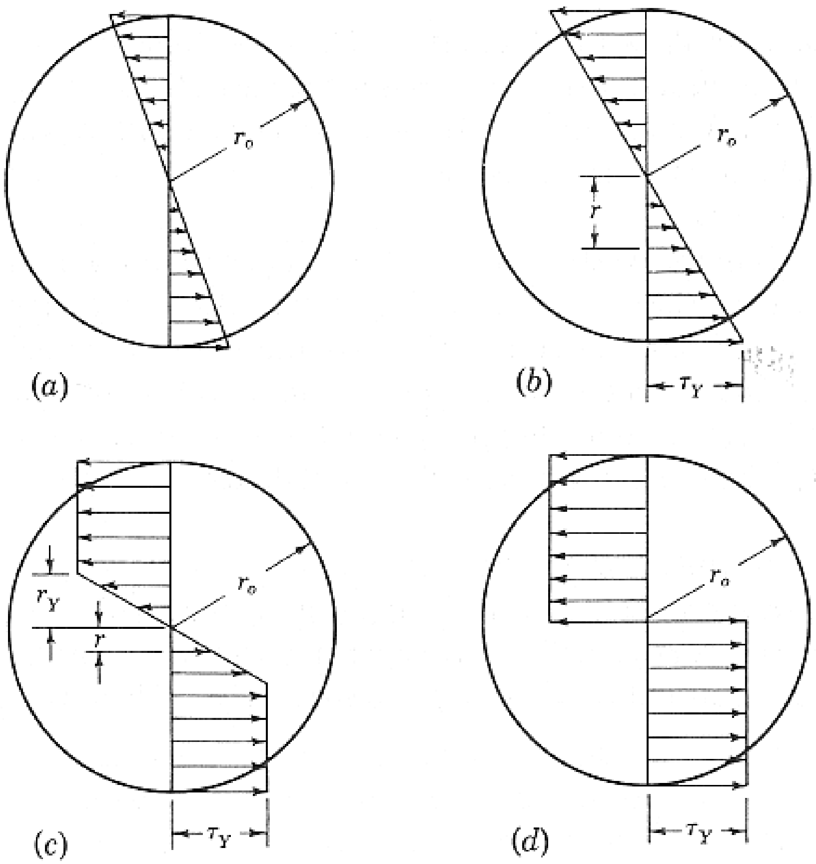
$$\tau_{\theta z} = \frac{1}{\sqrt{3}}Y = 0.577Y \qquad (9.12)$$

Using the maximum shear-stress criterion,
the equivalent shear stress: $\bar{\tau}$

$$\bar{\tau} = \tau_{\theta z} \qquad (9.13)$$

$$\tau_{\theta z} = \frac{1}{2}Y = 0.500Y \qquad (9.14)$$

Plastic Deformation in Torsion



Shear-stress distributions in a twisted shaft of material having the stress-strain curve of perfectly elastic material. (*a*) Entirely elastic; (*b*) onset of yield; (*c*) partially plastic; (*d*) fully plastic.

Twisting moment and twisting angle

$$T_Y = \frac{\tau_Y I_z}{r_o} = \frac{\pi}{2}\tau_Y r_o^3 \qquad \phi_Y = \frac{\tau_Y L}{G r_o} \tag{9.15}$$

$$\gamma_{\theta z} = r\frac{d\phi}{dz} = r\frac{\phi}{L} \tag{9.16}$$

$$r_Y = \frac{L\gamma_Y}{\phi} \tag{9.17}$$

$$r_Y = \frac{L\tau_Y}{G\phi} = r_o\frac{\phi_Y}{\phi} \tag{9.18}$$

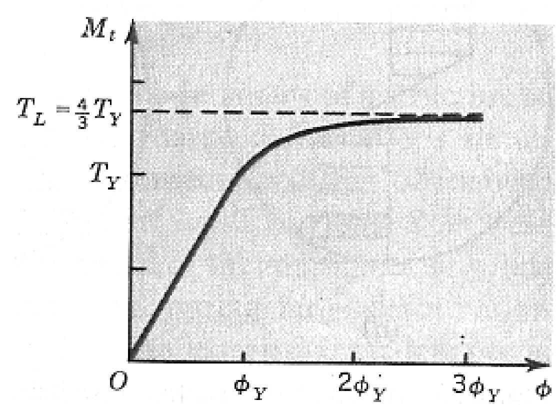
Plastic Deformation in Torsion

$$\tau_{\theta Z} = G\gamma_{\theta Z} = G\frac{\phi}{L}r = \tau_Y\frac{r}{r_Y} \quad (0 < r < r_Y) \qquad (9.19)$$

$$\tau_{\theta Z} = \tau_Y \qquad (r_Y < r < r_o) \qquad (9.20)$$

$$\begin{aligned} M_t &= \int_A r\tau_{\theta z}\, dA \\ &= \int_0^{r_Y} r\left(\frac{r}{r_Y}\tau_Y\right)2\pi r dr + \int_{r_Y}^{r_o} r\tau_Y 2\pi r dr \\ &= \frac{\pi}{2}{r_Y}^3\tau_Y + \frac{2\pi}{3}({r_o}^3 - {r_Y}^3)\tau_Y \\ &= \frac{2\pi}{3}\tau_Y {r_o}^3\left(1 - \frac{1}{4}\frac{{r_Y}^3}{{r_o}^3}\right) \end{aligned} \qquad (9.21)$$

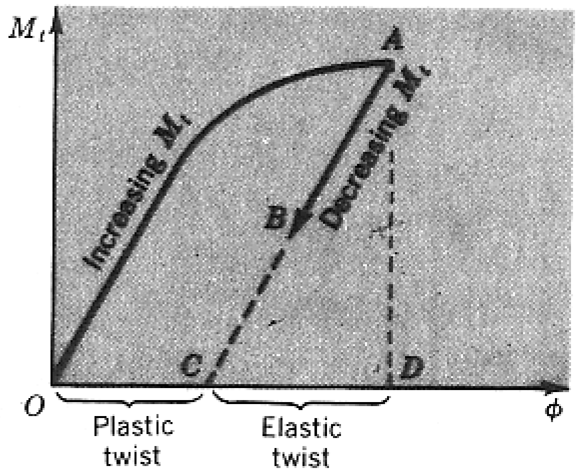
$$M_t = \frac{4}{3}T_Y\left(1 - \frac{1}{4}\frac{{\phi_Y}^3}{\phi^3}\right) \qquad (9.22)$$



The limit or fully plastic twisting moment

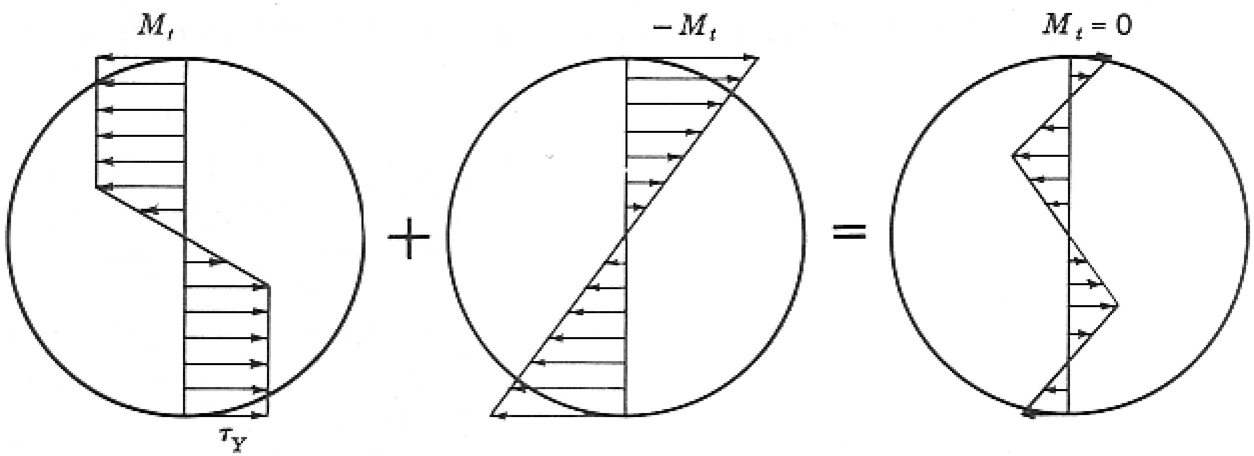
$$T_L = \frac{4}{3}T_Y \qquad (\phi \to \infty)$$

Residual Stresses in Torsion



Residual stresses:

Internal stresses are "locked in" the material by the plastic deformation.



(a) (b) (c)

The Onset of Yielding in Bending

Pure bending : $\sigma_1 = \sigma_x; \qquad \sigma_2 = \sigma_3 = 0$ (9.23)

Yielding will occur when $\sigma_x = Y$ (9.24)

Review two criteria available to signal the onset of yielding

The Mises criterion : $\sqrt{\frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]} = Y$ (9.25)

The maximum shear-stress criterion : $\tau_{\max} = \dfrac{\sigma_{\max} - \sigma_{\min}}{2} = \dfrac{Y}{2}$ (9.26)

The Onset of Yielding in Bending

Example 9.3*

A circular rod of radius *r* is bent into the shape of a U to form the structure of Fig. 7.25*a*. The material in the rod has a yield stress *Y* in simple tension. We wish to determine the load *P* that will cause yielding to begin at some point in the structure.

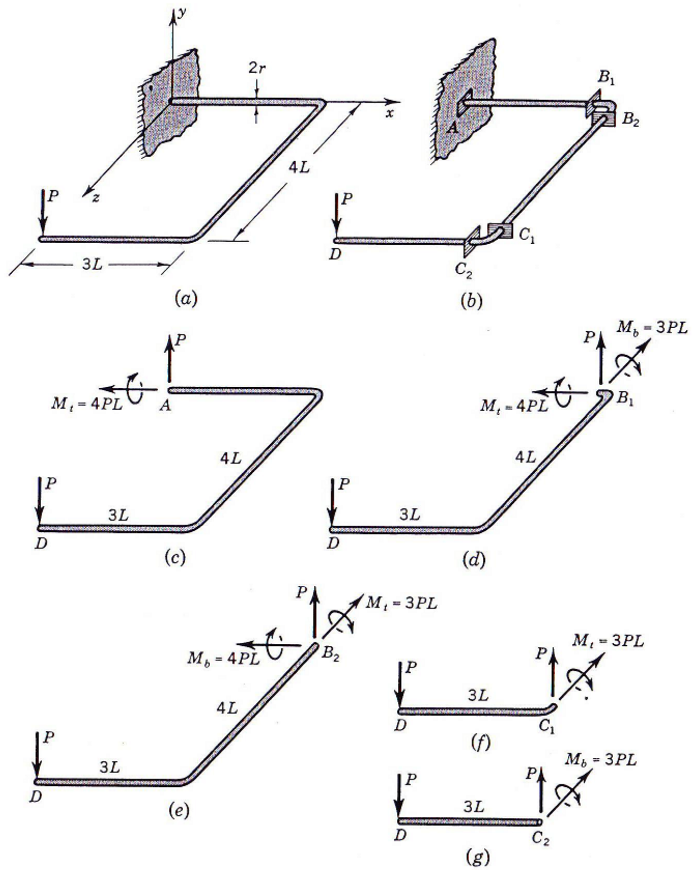


Fig. 9.9 Example 7.7. Bending and twisting moments at five critical locations in a structure.

* Lardner, Thomas J. *An introduction to the mechanics of solids*. McGraw-Hill College, 1972.

The Onset of Yielding in Bending

Example 9.3*

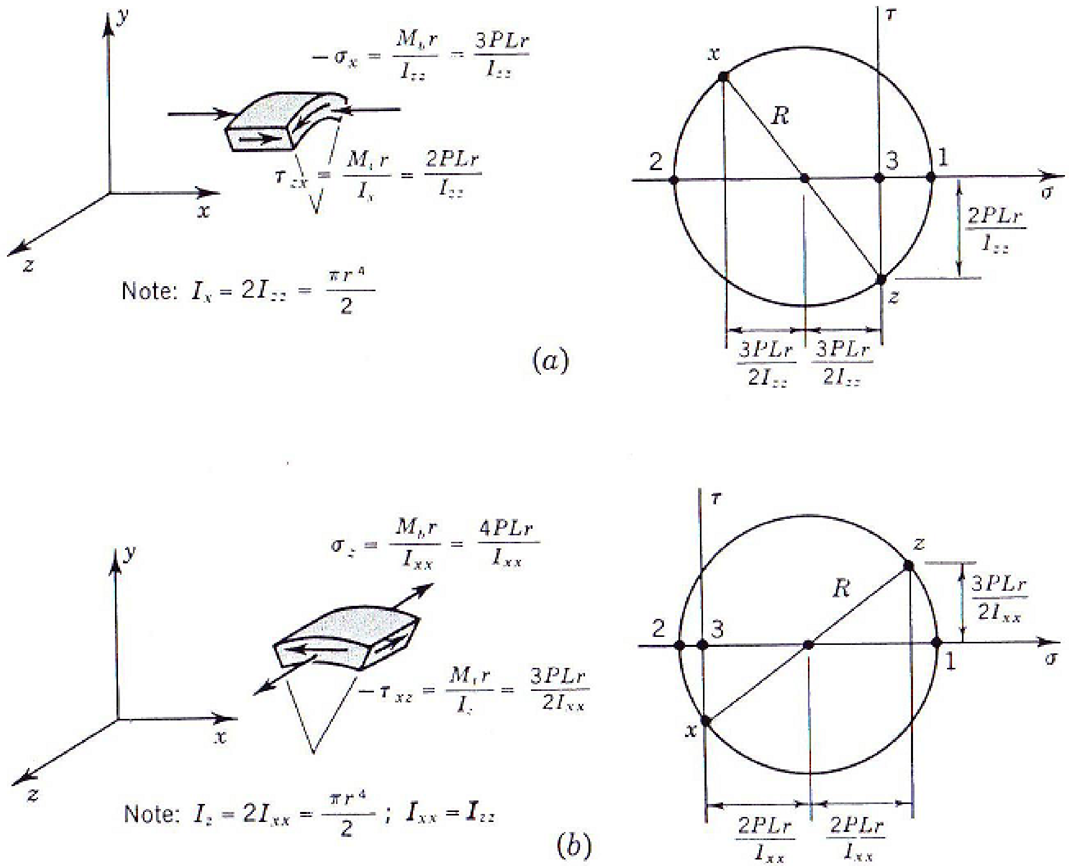


Fig. 9.10 Example 7.7. (a) Maximum stress condition at location B_1;

(b) maximum stress condition at location B_2.

* Lardner, Thomas J. *An introduction to the mechanics of solids*. McGraw-Hill College, 1972.

The Onset of Yielding in Bending

Example 9.3*

The radius of Mohr's circle for the element at B_1 , shown in Fig.9.10 (a), is

$$R = \sqrt{\left(\frac{3}{2}\frac{PLr}{I_{zz}}\right)^2 + \left(\frac{2PLr}{I_{zz}}\right)^2} = \frac{5}{2}\frac{PLr}{I_{zz}} \qquad (a)$$

The principal stresses at the point are

$$\sigma_2 = -4\frac{PLr}{I_{zz}} \qquad (b)$$

The Mises yield criterion is

$$\sqrt{\frac{1}{2}\left(\frac{PLr}{I_{zz}} + 4\frac{PLr}{I_{zz}}\right)^2 + \left(-4\frac{PLr}{I_{zz}} - 0\right)^2 + \left(0 - \frac{PLr}{I_{zz}}\right)^2} = Y \qquad (c)$$

Yielding begins when

$$P = 0.218\frac{I_{zz}Y}{Lr} \qquad (d)$$

The maximum shear-stress criterion is

$$\tau_{\max} = \frac{1}{2}\left(\frac{PLr}{I_{zz}} + 4\frac{PLr}{I_{zz}}\right) = \frac{Y}{2} \qquad (e)$$

* Lardner, Thomas J. *An introduction to the mechanics of solids*. McGraw-Hill College, 1972.

The Onset of Yielding in Bending

Example 9.3*

Yielding is predicted when

$$P = 0.200\frac{I_{zz}Y}{Lr} \qquad (f)$$

Repeating for the element on top of the beam at location B_2 (Fig. 9.10 (*b*)):

The principle stresses to be

$$\sigma_1 = +\frac{9}{2}\frac{PLr}{I_{zz}} \qquad \sigma_2 = -\frac{1}{2}\frac{PLr}{I_{zz}} \qquad \sigma_3 = 0 \qquad (g)$$

The Mises yield criterion is

$$P = 0.210\frac{I_{xx}Y}{Lr} \qquad (h)$$

The maximum shear-stress criterion is

$$P = 0.200\frac{I_{xx}Y}{Lr} \qquad (i)$$

* Lardner, Thomas J. *An introduction to the mechanics of solids*. McGraw-Hill College, 1972.

Plastic Deformation in Bending

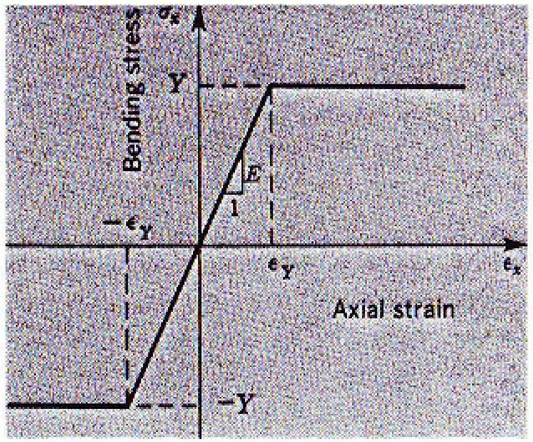


Fig. 9.11 Elastic-perfectly plastic material.

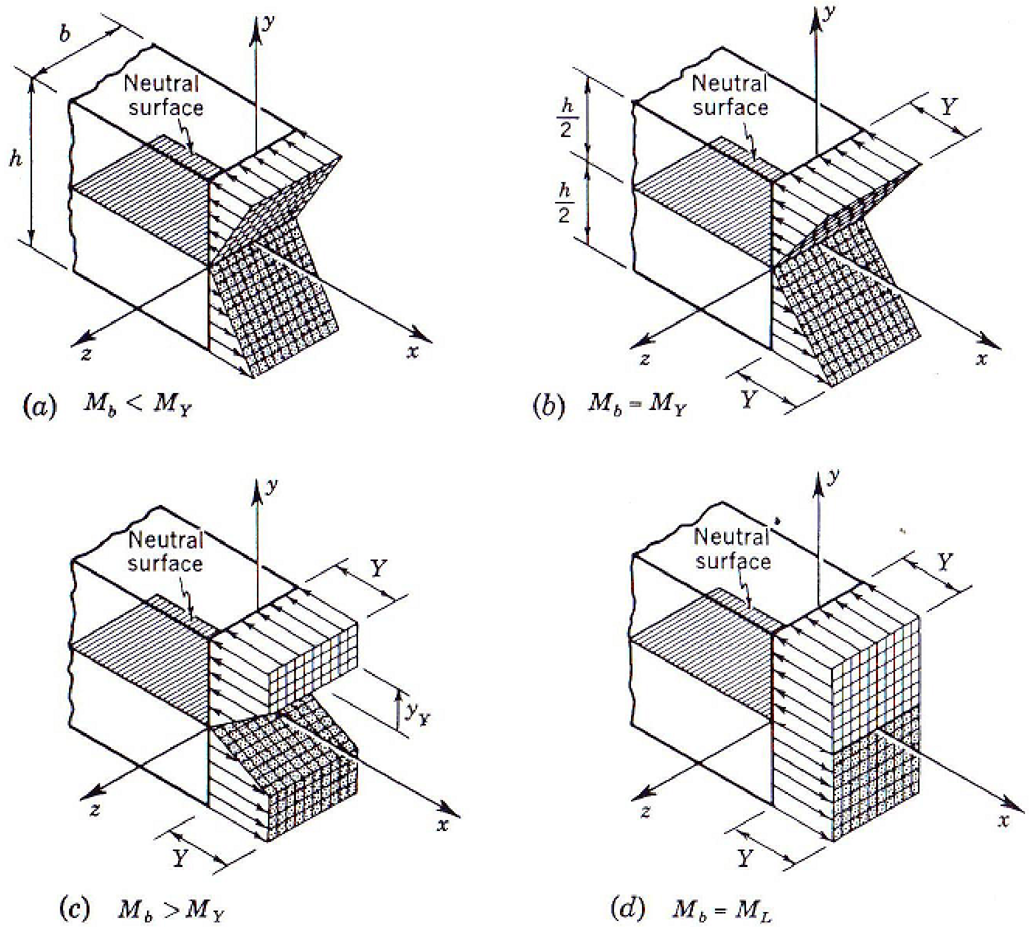


Fig. 9.12 Bending-stress distribution in a rectangular beam of elastic-perfectly plastic material as the curvature is increased until the fully plastic moment M_L is reached at infinite curvature.

System Health & Risk Management

Plastic Deformation in Bending

The bending deformation of beam is

$$\varepsilon_x = -\frac{y}{\rho} = -\frac{d\phi}{ds} y \qquad (9.27)$$

In the elastic region $(0 < (\sigma_x)_{\max} < Y)$, the moment-curvature relation is

$$\frac{d\phi}{ds} = \frac{1}{\rho} = \frac{M_b}{EI_{zz}} \qquad (9.28)$$

and the stress distribution is

$$\sigma_x = -\frac{M_b y}{I_{zz}} \qquad (9.29)$$

In Fig. 9.12 *(b)*

(M_Y : the bending moment which corresponds to the onset of yielding in the beam)

$$M_Y = \frac{Y(b\,h^3/12)}{h/2} = \frac{bh^2}{6} Y \qquad (9.30)$$

$$\left(\frac{1}{\rho}\right)_Y = \frac{\varepsilon_Y}{h/2} \text{: the curvature corresponding to } M_Y \qquad (9.31)$$

Plastic Deformation in Bending

In Fig. 9.12 (c)

(y_Y : the coordinate which defines the extent of the inner elastic region of behavior)

$$\sigma_x = -\frac{y}{y_Y}Y \qquad \text{when} \quad 0 < y < y_Y$$

$$\sigma_x = -Y \qquad \text{when} \quad y_Y < y < \frac{h}{2} \tag{9.32}$$

The bending moment is

$$M_b = -\int_A \sigma_x y \, dA$$

$$= 2\left(-\int_0^{y_Y} \sigma_x yb \, dy - \int_{y_Y}^{h/2} \sigma_x yb \, dy\right) \tag{9.33}$$

$$= \frac{bh^2}{4}Y\left[1 - \frac{1}{3}\left(\frac{y_Y}{h/2}\right)^2\right] \tag{9.34}$$

System Health & Risk Management

Plastic Deformation in Bending

The strain at y_Y has the value $-\varepsilon_Y$, and using this, we obtain from (9.27) the curvature corresponding to the moment given by (9.37).

$$\frac{1}{\rho} = \frac{\varepsilon_Y}{y_Y} \tag{9.35}$$

Combining (9.31) and (9.35), we get

$$\frac{y_Y}{h/2} = \frac{(1/\rho)_Y}{1/\rho} \tag{9.36}$$

Finally, substituting (9.30) and (9.36) in (9.34), we find the bending moment to be given by

$$M_b = \frac{3}{2} M_Y \left\{ 1 - \frac{1}{3} \left[\frac{(1/\rho)_Y}{1/\rho} \right]^2 \right\} \tag{9.37}$$

Plastic Deformation in Bending

Fully plastic moment

(= limit moment)

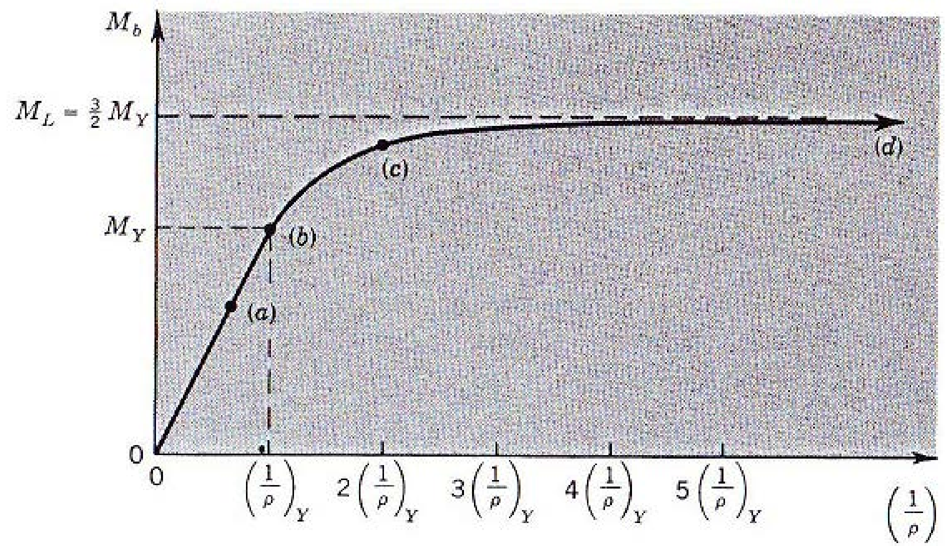
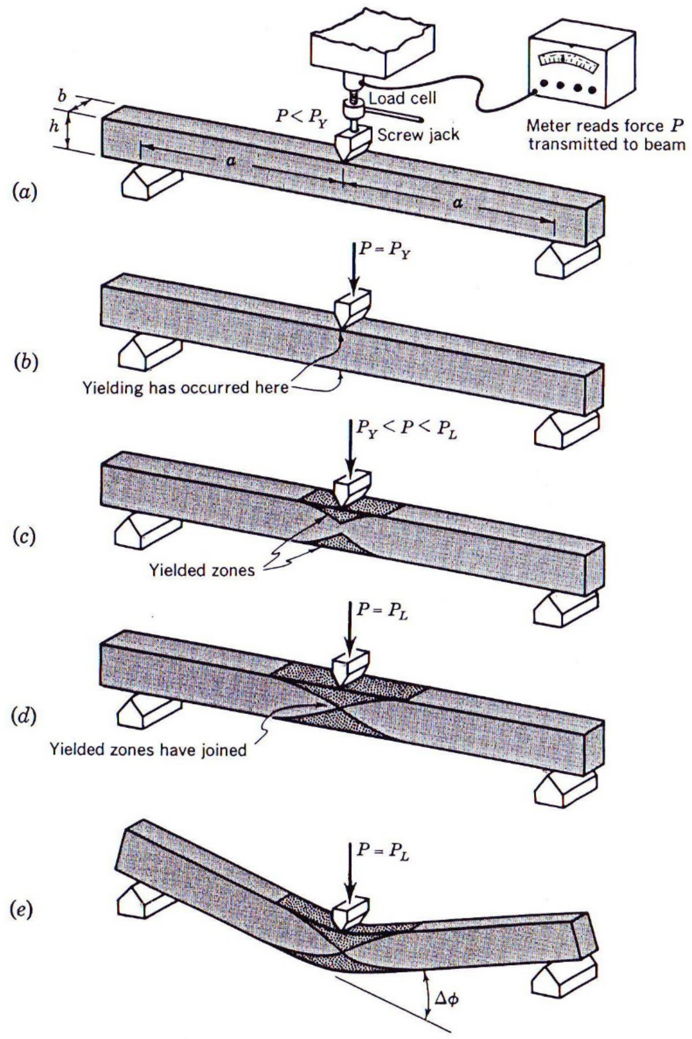


Fig. 9.13 Moment-curvature relation for the rectangular beam of Fig.9.12. The positions (*a*), (*b*), (*c*), and (*d*) correspond to the stress distributions shown in Fig.9.12.

Table. Ratio of limit bending moment to bending moment at onset of yielding

Cross section	$K = M_L/M_Y$
Solid rectangle	1.5
Solid circle	1.7
Thin-walled circular tube	1.3
Typical I beam	1.1–1.2

Plastic Deformation in Bending



(*a*) A rectangular beam of elastic-perfectly plastic material.

(*b*) The central bending moment is M_Y, and

$$P_Y = \frac{2}{a} M_Y = \frac{bh^2}{3a} Y. \qquad (9.38)$$

(*d*) The central bending moment is M_L, and

$$P_L = \frac{2}{a} M_L = \frac{bh^2}{2a} Y. \qquad (9.39)$$

(*e*) A plastic hinge (a finite discontinuity)

Fig. 9.14 Creation of a plastic hinge as the center of the beam is forced downward by a screw jack. The load cell measures the force *P* developed by the screw jack.

Plastic Deformation in Bending

Example 9.4*

An originally straight rectangular bar is bent around a circular mandrel of radius

$R_0 - h/2$, as shown in Fig. 9.15 (a). As the bar is released from the mandrel, its radius of curvature increases to R_1, as indicated in Fig. 9.15 (b). This change of curvature is called ***elastic springback***; it becomes a factor of great importance when metals must be formed to close dimensional tolerances. Our interest here is in the moment of this springback and in the residual stresses which remain after the bar

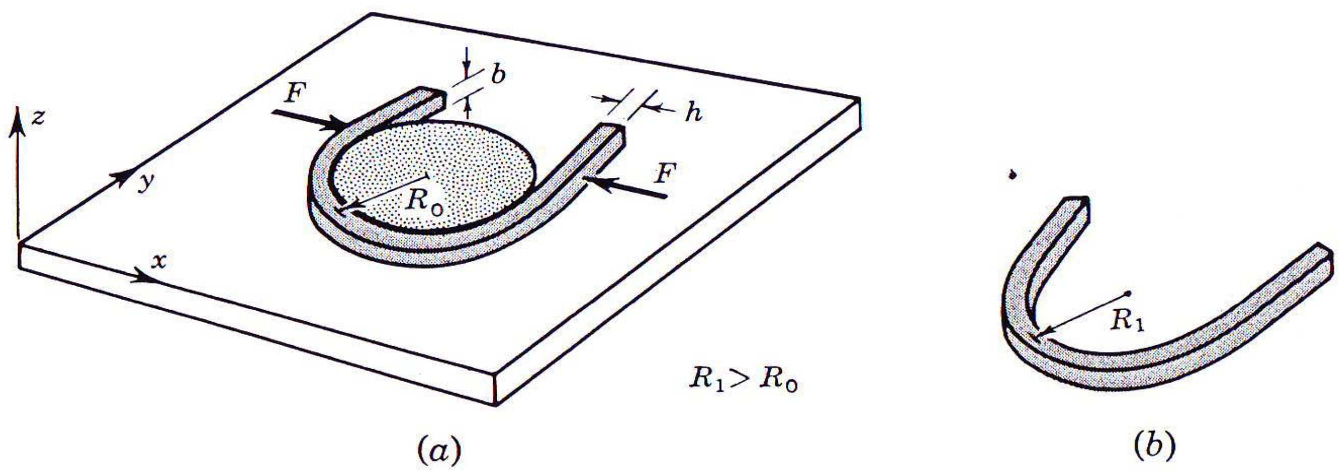


Fig. 9.15 Example 7.8. Illustration of elastic springback which occurs when an originally straight rectangular bar is released after undergoing large plastic bending deformation.

* Lardner, Thomas J. *An introduction to the mechanics of solids*. McGraw-Hill College, 1972.

Plastic Deformation in Bending

Example 9.4*

Fig. 9.16 Example 7.8. Moment-curvature relation for the complete cycle of loading and unloading the rectangular bar in Fig. 9.15.

The decrease in curvature due to the springback is

$$\frac{1}{R_0} - \frac{1}{R_1} = \frac{3}{2}\left(\frac{1}{\rho}\right)_Y \qquad (a)$$

Using (7.39),

$$\left(\frac{1}{\rho}\right)_Y = \frac{\varepsilon_Y}{h/2} = \frac{Y}{E}\frac{2}{h} \qquad (b)$$

Combining (a) and (b),

$$\frac{1}{R_0} - \frac{1}{R_1} = \frac{Y}{E}\frac{3}{h} \qquad (c)$$

* Lardner, Thomas J. *An introduction to the mechanics of solids*. McGraw-Hill College, 1972.

Plastic Deformation in Bending

Example 9.4*

Fig. 9.17 Example 7.8. Illustrating calculation of the residual-stress distribution in the bar of Fig. 9.15 (*b*).

* Lardner, Thomas J. *An introduction to the mechanics of solids*. McGraw-Hill College, 1972.

Limit Analysis in Bending

Two simple examples of ***collapse mechanisms***:

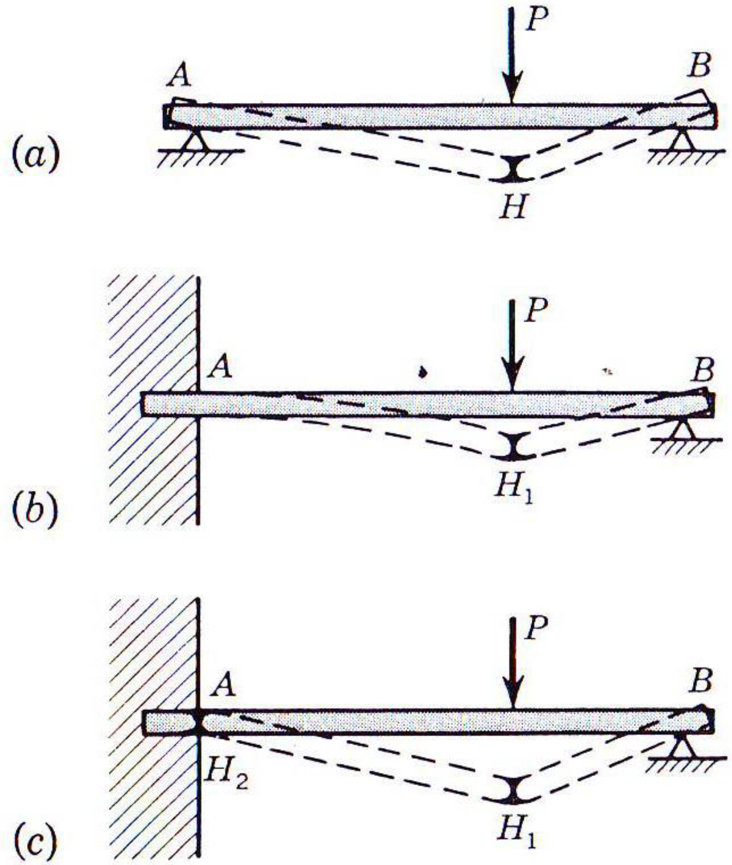


Fig. 9.18 One plastic hinge causes collapse in (*a*). Two plastic hinges are required for collapse of the beam shown in (*b*) and (*c*).

Limit Analysis in Bending

Example 9.4*

Figure 9.19 shows a beam built in at *C*, simply supported at *A*, and subjected to a concentrated load *P* at *B*. It is desired to find the magnitude of the limit load P_L which corresponds to the condition of plastic collapse. Let the bending moment corresponding to the onset of yielding for the beam section be M_Y, and let the limiting or fully plastic bending moment be M_L.

Fig. 9.19 Example 8.13. Equilibrium analysis of statically indeterminate beam, (a) and (b). Geometry of collapse, (c).

* Lardner, Thomas J. *An introduction to the mechanics of solids*. McGraw-Hill College, 1972.

Limit Analysis in Bending

Example 9.4*

In Fig. 9.19 (*b*),

$$\frac{2Pa}{3} - \frac{2M_C}{3} = M_L \qquad (a)$$

$$M_C = M_L$$

Eliminating M_C gives

$$P_L = 2.5\frac{M_L}{a} \qquad (b)$$

In the purely elastic case,

$$P_Y = 1.8\frac{M_Y}{a} \qquad (c)$$

Thus in terms of $K = M_L/M_Y$ we can write

$$P_L = 2.5\frac{KM_Y}{a} = \frac{2.5}{1.8}KP_Y = 1.39KP_Y \qquad (d)$$

* Lardner, Thomas J. *An introduction to the mechanics of solids*. McGraw-Hill College, 1972.

Limit Analysis in Bending

Example 9.5*

The structure shown in Fig. 8.28 consists of two equal cantilever beams *AC* and *CD* with roller contact at *C*. Given the limiting bending moment M_L for the beams, it is desired to find the limiting value of the load *P* which corresponds to plastic collapse of the structure.

Fig. 9.20 Example 8.14. Structure with two possible modes of collapse.

* Lardner, Thomas J. *An introduction to the mechanics of solids*. McGraw-Hill College, 1972.

Limit Analysis in Bending

Example 9.5*

For the mechanism of Fig. 9.21 (*c*),

$$\left.\begin{aligned} \frac{PL}{2} - FL &= M_L \\ \frac{FL}{2} &= M_L \end{aligned}\right\} \qquad (a)$$

Eliminating *F*, we obtain

$$P = 6\frac{M_L}{L} \qquad (b)$$

For the mechanism of Fig.9.21 (*d*),

$$\left.\begin{aligned} \frac{PL}{2} - FL &= M_L \\ FL &= M_L \end{aligned}\right\} \qquad (c)$$

Eliminating *F*, we obtain

$$P = 4\frac{M_L}{L} \qquad (d)$$

Fig. 9.21 Example 8.14. Structure with two possible modes of collapse.

* Lardner, Thomas J. *An introduction to the mechanics of solids*. McGraw-Hill College, 1972.

Limit Analysis in Bending

Example 9.5*

Since (*d*) is smaller than (*b*), the structure collapses in the mechanism of Fig. 9.21 (*d*) under the limit load

$$P = 4\frac{M_L}{L} \qquad (e)$$

An alternative procedure for deciding against the result (*b*) is to continue the force analysis in Fig. 9.21 (*c*) , obtaining the bending moment at *D* which corresponds to (*b*). If we do this we find that the magnitude of the bending moment at *D* must be 2*ML*, which is incompatible with the fact that the maximum bending moment can be developed in these beams is *ML*. This indicates that a hinge will form at *D* before the mode of Fig. 9.21 (*c*) can ever develop.

* Lardner, Thomas J. *An introduction to the mechanics of solids*. McGraw-Hill College, 1972.

THANK YOU
FOR LISTENING