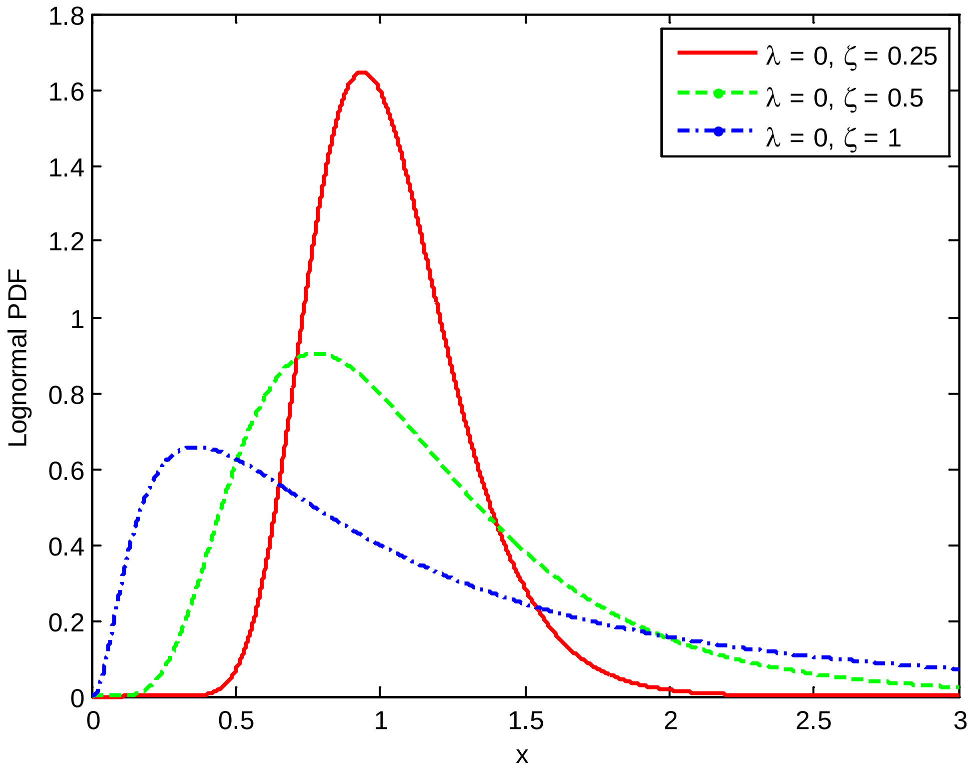
Seoul National University
Dept. of Civil and Environmental Engineering

Instructor: Junho Song
junhosong@snu.ac.kr

457.646 Topics in Structural Reliability
Lognormal Distribution

1. **Lognormal** distribution

- Closely related to the normal distribution (____________ function of a normal r.v.)
- Defined for _________ values only.



(a) PDF: $X \sim LN(\lambda, \zeta)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\zeta x}\exp\left[-\frac{1}{2}\left(\frac{\ln x - \lambda}{\zeta}\right)^2\right], \quad 0 < x < \infty$$

(b) CDF:

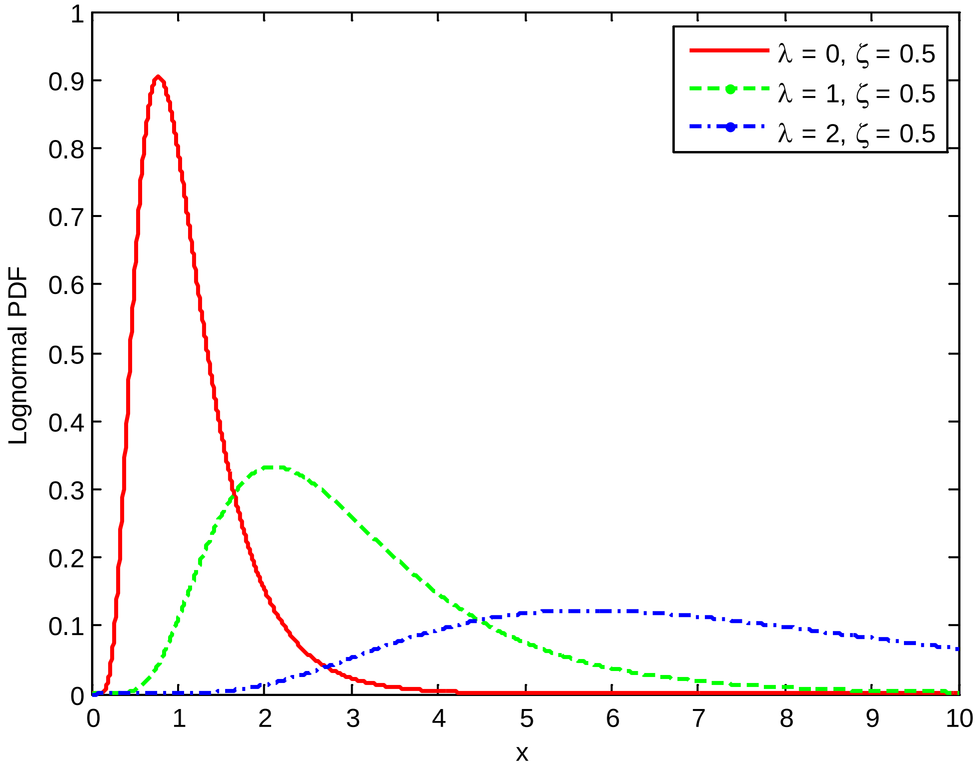
$$F_X(x) = \int_{-\infty}^{x} f_X(x)dx, \quad 0 < x < \infty$$

→ no closed-form expression available, but can be computed by use of the table of the standard normal CDF $\Phi(\cdot)$ (as shown below)

(c) Parameters: λ, ζ

- λ: mean of _____________, i.e. $\lambda = \lambda_X \equiv \mathrm{E}[\ln X]$
- σ: standard deviation of______________, i.e. $\zeta^2 = \zeta_X^2 = \sigma_{\ln X}^2$

(d) Shape of the PDF plots (see the plot above and below)



(e) Relationship between normal and lognormal distribution:

"The logarithm of a ____________ random variable is a ___________ random variable."

$$X \sim LN(\lambda, \zeta) \Rightarrow \ln X \sim N(\lambda, \zeta)$$

(f) "The exponential function of a ______________ random variable is a _______________ random variable."

$$\boxed{X \sim LN(\lambda, \zeta)} \; \underset{X = e^{Y}}{\overset{Y = \ln X}{\rightleftarrows}} \; \boxed{Y \sim N(\lambda, \zeta)}$$

(g) Can obtain the CDF of lognormal $X \sim LN(\lambda, \zeta)$ from the CDF of standard normal:

$$\begin{aligned} F_X(a) &= P(X \le a) \\ &= P(\ln X \le \ln a) \qquad \text{Since } \ln X \sim N(\lambda, \zeta), \\ &= \Phi\left(\frac{\ln a - \lambda}{\zeta}\right) \end{aligned}$$

(h) $(\lambda,\zeta) \rightarrow (\mu,\delta)$: Find the mean and c.o.v. from the distribution parameters

$$\mu = \mathrm{E}[X] = \exp(\lambda + 0.5\zeta^2)$$

$$\delta = \sigma/\mu = \sqrt{\exp(\zeta^2) - 1} \quad (\cong \zeta \text{ for } \zeta << 1)$$

(i) $(\mu,\delta) \rightarrow (\lambda,\zeta)$: Find the distribution parameters from the mean and c.o.v.

$$\zeta = \sqrt{\ln(1+\delta^2)} \quad (\cong \delta \text{ for } \delta << 1)$$

$$\lambda = \ln\mu - 0.5\ln(1+\delta^2)$$

(j) $(x_{0.5}) \leftrightarrow (\lambda)$: Relationship between the median and λ

$$\lambda = \ln x_{0.5}, \; x_{0.5} = e^{\lambda}$$

(k) $(\mu,\delta) \rightarrow (x_{0.5})$: Find the median from the mean and c.o.v.

$$x_{0.5} = \frac{\mu}{\sqrt{1+\delta^2}}$$

Note: $x_{0.5} < \mu$ for the lognormal distribution.

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Example 1: The drainage demand during a storm (in mgd: million gallons/day) is assumed to follow the lognormal distribution with the same mean and standard deviation as Example 2 of the previous lecture (mean 1.2, standard deviation 0.4). The maximum drain capacity is 1.5 mgd.

(a) Distribution parameters, i.e. λ and ζ?

(b) Probability of the flooding?

(c) Probability that the drainage demand during a storm will be between 1.0 and 1.6 mgd?

(d) The 90-percentile drainage demand?

Example 2: Consider a bridge whose uncertain capacity against "complete damage" limit-state caused by earthquake events is defined in terms of peak ground acceleration (PGA; unit: g) that the bridge can sustain. Suppose the median of the capacity is 1.03g and the coefficient of variation is 0.50. It is assumed that the capacity follows a lognormal distribution.

(a) Distribution parameters of the lognormal distribution, i.e. λ and ζ?

(b) The mean and standard deviation of the uncertain capacity, i.e. μ and σ?

(c) Suppose the peak ground acceleration from an earthquake event is 0.5g. What is the probability that the structure will exceed "complete damage" limit state?