Seoul National University
Dept. of Civil and Environmental Engineering

Instructor: Junho Song
junhosong@snu.ac.kr

457.646 Topics in Structural Reliability

In-Class Material: Class 10

◎ Second moment reliability index β_{MVFOSM}

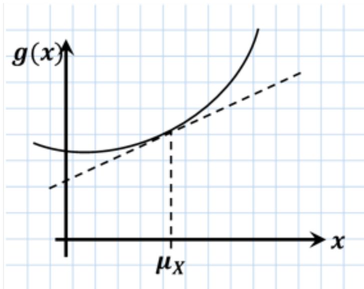
M V F O S M

- Failure : $g(\mathbf{x}) \le 0$ (NOT "elementary")
- Use ( ) & ( ) only. Therefore, can't compute P_f (index, not method)

- Ang & Cornell (1974) ASCE Journal of Structural Engineering

Use ( ) order approximation to estimate & of $g(\mathbf{x})$

$P_f = P(g \le 0) = P(u_g \le \quad)$



μ_g^{FO} & σ_g^{FO} ?

$$g(\mathbf{x}) \simeq g(\quad) + \sum_{i=1}^{n} \qquad (x_i - \mu_{X_i})$$

$$\begin{cases} \mu_g^{FO} = \\ \sigma_g^{2FO} = \sum \qquad\qquad + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \end{cases}$$

$$=$$

$$\boxed{\therefore \quad \beta_{MVFOSM} = \frac{\mu_g^{FO}}{\sigma_g^{FO}} = \underline{\qquad\qquad\qquad}}$$

If we assume $u_g \sim N(0,1)$

$P_f \cong \Phi(\quad)$

⇒ Popular for a while

⇒ But problem

i.e. equivalent limit-state functions could give different β_{MVFORM}

$$\left.\begin{aligned} g_1(x) &= X_1^2 + 3X_2 < 0 \\ g_2(x) &= g_1(X) / X_1^2 = 1 + 3\frac{X_2}{X_1^2} < 0 \end{aligned}\right\}$$ equivalent ⇒ the same β_{MVFORM} ?

Instructor: Junho Song
junhosong@snu.ac.kr

Example: lack of invariance of second order reliability methods

Consider a structural reliability problem with two random variables X_1 and X_2. The mean vector and the covariance matrix of X_1 and X_2 are

$$\mathbf{M}_{\mathrm{X}} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}, \quad \mathbf{\Sigma}_{\mathrm{XX}} = \begin{bmatrix} 4 & 5 \\ 5 & 25 \end{bmatrix}$$

<u>Case 1:</u> $g(X_1, X_2) = X_1^2 + 3X_2$

Gradient $\nabla g = [2X_1 \quad 3]$. At the mean point $\mathbf{X} = \mathbf{M}_{\mathrm{X}}$, $\nabla g = [10 \quad 3]$.

First order approximation on μ_g and σ_g^2:

$$\mu_g \cong 5^2 + 3 \times 10 = 55$$

$$\sigma_g^2 \cong \nabla g \mathbf{\Sigma}_{\mathrm{XX}} \nabla g^{\mathrm{T}} = 925$$

$$\beta_{MVFOSM} = \frac{\mu_g}{\sigma_g} = \frac{55}{\sqrt{925}} = 1.81$$

$$P_f = \Phi(-1.81) = 0.0351$$

<u>Case 2:</u> $g(X_1, X_2) = 1 + \dfrac{3X_2}{X_1^2}$

$\nabla g = [-6X_2 X_1^{-3} \quad 3X_1^{-2}]$.

At the mean point $\mathbf{X} = \mathbf{M}_{\mathrm{X}}$, $\nabla g = [-0.48 \quad 0.12]$.

$$\mu_g \cong 1 + 3 \times 10 / 25 = 2.20$$

$$\sigma_g^2 \cong \nabla g \mathbf{\Sigma}_{\mathrm{XX}} \nabla g^{\mathrm{T}} = 0.706$$

$$\beta_{MVFOSM} = \frac{\mu_g}{\sigma_g} = \frac{2.20}{\sqrt{0.706}} = 2.62$$

$$P_f = \Phi(-2.62) = 0.00440$$

Although the two limit-state functions are equivalent ones with the same failure domains, the second order reliability method yields different reliability indices and failure probability estimates.

///

Summary:

$$\beta_{SM} = \frac{\mu_M}{\sigma_M} = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2 - 2\sigma_R\sigma_S\rho_{RS}}}$$

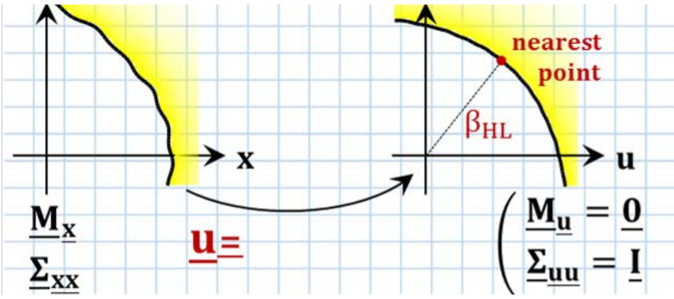
$$\beta_{SF} = \frac{\mu_F}{\sigma_F}, \text{ for LN } \beta_{SF} = \frac{\lambda_R - \lambda_S}{\sqrt{\zeta_R^2 + \zeta_S^2 - 2\zeta_R\zeta_S\rho_{\ln R \ln S}}}$$

$$\beta_{MVFOSM} = \frac{\mu_g^{FO}}{\sigma_g^{FO}} = \frac{g(\mathbf{M}_{\mathrm{X}})}{\nabla g(\mathbf{M}_{\mathrm{X}})\mathbf{\Sigma}_{\mathrm{XX}}\nabla g(\mathbf{M}_{\mathrm{X}})^{\mathrm{T}}} \quad \text{(Oct1974)}$$

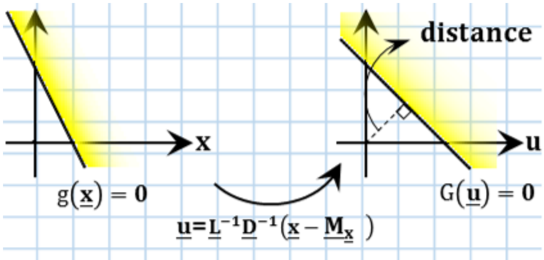
Instructor: Junho Song
junhosong@snu.ac.kr

◎ Hasofer-Lind Reliability Index, β_{HL} (JEM, May 1974)

(or β_{AFOSM} "Advanced" FOSM)



Linear Limit-State Function

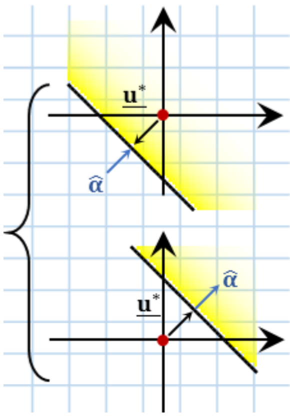


$$g(\mathbf{x}) = a_0 + \mathbf{a}^{\mathrm{T}}\mathbf{x}$$
$$= a_0 + \mathbf{a}^{\mathrm{T}}(\qquad\qquad)$$
$$= a_0 + \mathbf{a}^{\mathrm{T}}\mathbf{M} + \mathbf{a}^{\mathrm{T}}\mathbf{DLu}$$
$$= b_0 + \mathbf{b}^{\mathrm{T}}\mathbf{u} = G(\mathbf{u})$$

$$\boxed{\beta = \frac{\mu_G}{\sigma_G} = \frac{b_0}{\|\mathbf{b}\|}} \quad \text{VS} \quad \text{distance} = \frac{|b_0|}{\|\mathbf{b}\|}$$

Can have $+\!/\!-$ sign   always positive

For $G(\mathbf{u}) = b_0 + \mathbf{b}^{\mathrm{T}}\mathbf{u}$



$b_o = G(\mathbf{0}) < 0$

(in failure domain)

$\beta < 0$

$b_o = G(\mathbf{0}) > 0$

(in safe domain)

$\beta > 0$

Instructor: Junho Song
junhosong@snu.ac.kr

i. $\hat{\boldsymbol{\alpha}} = -\dfrac{\nabla G}{\|\nabla G\|}$: "Negative normalized gradient vector"

: Unit row vector pointing toward the _______ domain

e.g. linear function : $\hat{\boldsymbol{\alpha}} = -\dfrac{\mathbf{b}^{\mathrm{T}}}{\|\mathbf{b}\|}$

ii. $\mathbf{u}^*$: "Design point"

"Most probable failure point (MPP)"

"Beta point"

e.g. linear function : $\mathbf{u}^* \equiv -b_0 \dfrac{\mathbf{b}}{\|\mathbf{b}\|^2}$

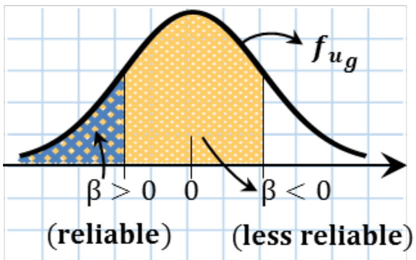
iii.

$$\boxed{\beta_{HL} \equiv \hat{\boldsymbol{\alpha}}\mathbf{u}^*}$$

Hasofer-Lind Reliability Index

$$\begin{cases} |\beta_{HL}| : \text{distance between origin and } \mathbf{u}^* \\ \text{sign : directions of } \hat{\boldsymbol{\alpha}} \text{ and } \mathbf{u}^* \end{cases}$$

e.g. linear function : $\beta_{HL} = \dfrac{b_0}{\|\mathbf{b}\|}\left(= \dfrac{\mu_G}{\sigma_G}\right)$



$P_f = F_{u_g}(-\beta_{HL})$

What if $\mathbf{X} \sim N(\mathbf{M}_{\mathbf{X}}, \Sigma_{\mathbf{XX}})$ and $g(\mathbf{x})$ linear?

$\Rightarrow G, g \sim N$

$P_f = \Phi(-\beta_{HL})$