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457.646 Topics in Structural Reliability
In-Class Material: Class 15

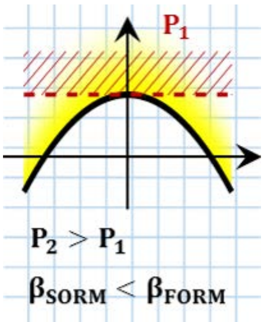
※ FERUM Example (SORM)

$$g(\mathbf{x}) = 1 - \frac{m_1}{s_1 y} - \frac{m_2}{s_2 y} - \left(\frac{P}{Ay}\right)^2 \le 0$$

$$\beta_{FORM} = 2.4661$$

(Curvature fitting)

$$\kappa_i \begin{cases} -1.548 \times 10^{-1} \\ -3.997 \times 10^{-2} \\ 8.903 \times 10^{-7} \end{cases}$$



$$\beta_{SORM} = 2.3506(T),\ 2.3596(B),\ 2.341(iB)$$

(Point fitting)

$$a_i \begin{array}{cc} + & - \\ \begin{cases} -6.2969 \times 10^{-2} \\ -1.1986 \times 10^{-2} \\ -1.3778 \times 10^{-1} \end{cases} & \begin{array}{c} -4.0358 \times 10^{-2} \\ -9.7461 \times 10^{-3} \\ -1.1050 \times 10^{-1} \end{array} \end{array}$$

$$\beta_{SORM} = 2.3599(T),\ 2.3693(B),\ 2.3537(iB)$$

See supplement, “Importance and Sensitivity Vectors” (by A. Der Kiureghian)

➔ Main reference: Bjerager & Krenk (1989)

◎ FORM importance vector $\hat{\boldsymbol{\alpha}}$

FORM approximation of the limit-state function

$$G(\mathbf{u}) \cong G(\mathbf{u}^*) + \nabla G(\mathbf{u}^*)(\mathbf{u} - \mathbf{u}^*)$$

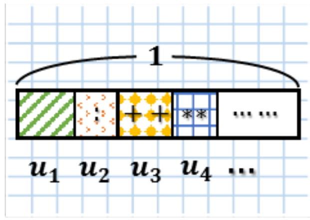
$$=$$

$$= \qquad\qquad (\beta - \hat{\boldsymbol{\alpha}}\mathbf{u})$$

$$G'(\mathbf{u}) = \frac{G(\mathbf{u})^{FO}}{\|\nabla G(\mathbf{u}^*)\|} =$$

Note $\sigma_{G'}^{2} = (\qquad)\boldsymbol{\Sigma}_{\mathbf{uu}}(\qquad)$

$= \quad \hat{\boldsymbol{\alpha}} \quad \hat{\boldsymbol{\alpha}}^{T} \quad = \hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\alpha}}^{T} = \boxed{\qquad\qquad} =$



Contribution (percentage) of u_i

to the total (variability)

of the limit-state function $G'(\mathbf{u})$

① Magnitude of α_i^{2} $\Rightarrow$ measure of relative importance (contribution to the uncertainty) of u_i's

② Sign of α_i $\Rightarrow$ nature of u_i's e.g., $g(\mathbf{X}) = R - S$

$$G'(\mathbf{u}) = \beta - \hat{\boldsymbol{\alpha}}\mathbf{u} = \beta -$$

α_i positive $\Rightarrow$ u_i capacity or demand

α_i negative $\Rightarrow$ u_i capacity or demand

Question) Importance of u_i $\overset{?}{=}$ Importance of X_i

i) Independent : $u_i = \Phi^{-1}\left[F_{X_i}(x_i)\right]$ OK

ii) Dependent: e.g., Nataf NOT OK

$$\mathbf{u} = \mathbf{L}_0^{-1}\mathbf{z} = \mathbf{L}_0^{-1}\begin{Bmatrix} \Phi^{-1}\left[F_{X_1}(x_1)\right] \\ \vdots \\ \Phi^{-1}\left[F_{X_n}(x_n)\right] \end{Bmatrix}$$

$\therefore$ $\hat{\alpha}_i$ does NOT $\begin{pmatrix} \text{Measure importance} \\ \text{Indicate the nature} \end{pmatrix}$ of x_i's

when X_i's are .

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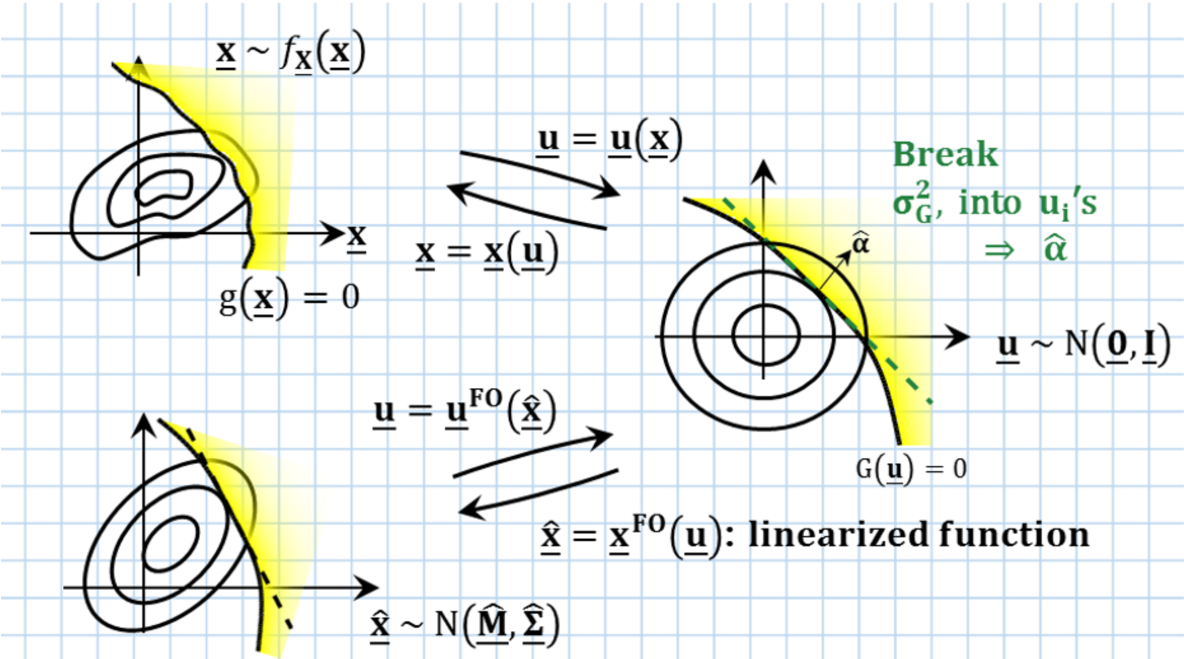
◎ Form importance vector $\hat{\gamma}$ (Question: contribution/nature of x_i? Not u_i's)



Transform to "normal equivalent" of $\mathbf{x}$

Why? Want to keep ( ) distribution

Want to recover ( )

$\mathbf{u}^{FO}(\mathbf{x})$?

$$\begin{cases} \mathbf{u} = \mathbf{u}(\mathbf{x}^*) + J_{\mathbf{u},\mathbf{x}}(\hat{\mathbf{x}} - \mathbf{x}^*) \\ \hat{\mathbf{x}} = \mathbf{x}^* + J_{\mathbf{u},\mathbf{x}}^{-1}(\mathbf{u} - \mathbf{u}^*) \end{cases} \quad \textbf{(*)}$$

Note: Jacobians evaluated at $\mathbf{x} =$

$\hat{\mathbf{X}} \sim N(\hat{\mathbf{M}}, \hat{\mathbf{\Sigma}})$

$$\begin{cases} \hat{\mathbf{M}} = \\ \hat{\mathbf{\Sigma}} = \end{cases}$$

Substituting (*) into $G'(\mathbf{u}) = \beta - \hat{\boldsymbol{\alpha}}\mathbf{u}$,

$$G'(\mathbf{u}) = G''(\hat{\mathbf{x}}) = \beta - \hat{\boldsymbol{\alpha}}[\mathbf{u}^* + J_{\mathbf{u},\mathbf{x}}(\hat{\mathbf{x}} - \mathbf{x}^*)]$$

$$= \beta - \hat{\boldsymbol{\alpha}}\mathbf{u}^* - \hat{\boldsymbol{\alpha}}J_{\mathbf{u},\mathbf{x}}(\hat{\mathbf{x}} - \mathbf{x}^*)$$

$$= -\hat{\boldsymbol{\alpha}}J_{\mathbf{u},\mathbf{x}}(\hat{\mathbf{x}} - \mathbf{x}^*)$$

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$$\sigma_{G''}^{2} = (-\hat{\boldsymbol{\alpha}} J_{\mathbf{u,x}})\hat{\Sigma}(-J_{\mathbf{u,x}}{}^{\mathrm{T}}\hat{\boldsymbol{\alpha}}^{\mathrm{T}})$$

$$= \hat{\boldsymbol{\alpha}} J_{\mathbf{u,x}} J_{\mathbf{u,x}}{}^{-1} (J_{\mathbf{u,x}}{}^{-1})^{\mathrm{T}} J_{\mathbf{u,x}}{}^{\mathrm{T}} \hat{\boldsymbol{\alpha}}^{\mathrm{T}}$$

$$= \qquad\qquad = \left\| \qquad\qquad \right\|^{2} \quad =$$

Contribution of each $\hat{x}_i$?

$$\hat{\boldsymbol{\Sigma}} = \hat{\mathbf{D}}\hat{\mathbf{D}} + (\hat{\boldsymbol{\Sigma}} - \hat{\mathbf{D}}\hat{\mathbf{D}})$$

diagonal off-diagonal

$$\sigma_{G''}^{2} = \quad \hat{\boldsymbol{\alpha}} J_{\mathbf{u,x}} (\hat{\mathbf{D}}\hat{\mathbf{D}}) J_{\mathbf{u,x}}{}^{T} \hat{\boldsymbol{\alpha}}^{T} + \qquad \hat{\boldsymbol{\alpha}} J_{\mathbf{u,x}} (\hat{\boldsymbol{\Sigma}} - \hat{\mathbf{D}}\hat{\mathbf{D}}) J_{\mathbf{u,x}}{}^{T} \hat{\boldsymbol{\alpha}}^{T} = 1$$

Contribution from variances $\sigma_{\hat{x}_i}^2$ Contribution from covariances $COV[\hat{x}_i, \hat{x}_j]$

Then, how about using $\hat{\boldsymbol{\alpha}} J_{\mathbf{u,x}} \hat{\mathbf{D}}$ instead of $\hat{\boldsymbol{\alpha}}$?

But not normalized yet.

$$\therefore \quad \hat{\boldsymbol{\gamma}} = \frac{\qquad\qquad}{\qquad\qquad}$$

i) Magnitude of $\hat{\gamma}_i^2$ $\rightarrow$ contribution (importance) of $\hat{x}_i$ or x_i

ii) Sign of $\hat{\gamma}_i$ $\rightarrow$nature of $\hat{x}_i$ or x_i

Note : $G''(\hat{\mathbf{x}}) = -\hat{\boldsymbol{\alpha}} J_{\mathbf{u,x}} (\hat{\mathbf{x}} - \mathbf{x}^*)$

$\hat{\gamma}_i$ positive $\rightarrow$ ______________ type r.v x_i

$\hat{\gamma}_i$ negative $\rightarrow$ ______________ type r.v x_i

Note : when $\mathbf{x}$ are independent, $\hat{\boldsymbol{\alpha}} = \hat{\boldsymbol{\gamma}}$?

$$\hat{\boldsymbol{\Sigma}} = (J_{\mathbf{u,x}}{}^{-1})(J_{\mathbf{u,x}}{}^{-1})^{T} = \hat{\mathbf{D}}\hat{\mathbf{D}} + (\hat{\boldsymbol{\Sigma}} - \hat{\mathbf{D}}\hat{\mathbf{D}})$$

$$\hat{\mathbf{D}} =$$

$$\hat{\boldsymbol{\gamma}} = \frac{\hat{\boldsymbol{\alpha}} J_{\mathbf{u,x}} \hat{\mathbf{D}}}{\left\| \hat{\boldsymbol{\alpha}} J_{\mathbf{u,x}} \hat{\mathbf{D}} \right\|} =$$

※ **FERUM Example** ($\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\gamma}}$)