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**457.646 Topics in Structural Reliability**

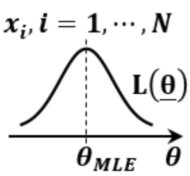
**In-Class Material: Class 21**

## ◎ Likelihood function $L(\boldsymbol{\theta})$ for distribution (statistical) parameters $\boldsymbol{\theta}_f$ (e.g. $\mu,\ \sigma,\ \lambda,\ \xi\ldots$)

① Measured value are available, $\mathbf{x}_i,\ i=1,\cdots,N$

Assuming the observations are s.i.

$$L(\boldsymbol{\theta}_f) \propto P\left(\bigcap_{i=1}^{N} \mathbf{X}=\mathbf{x}_i \,\middle|\, \boldsymbol{\Theta}_f=\boldsymbol{\theta}_f\right)$$
$$=\prod_{i=1}^{N} P(\mathbf{X}=\mathbf{x}_i \,|\, \boldsymbol{\Theta}_f=\boldsymbol{\theta}_f) \qquad (\because s.i.)$$
$$\propto \prod_{i=1}^{N} f_{\mathbf{x}}(\mathbf{x}_i \,|\, \boldsymbol{\theta}_f)$$




e.g. $\mathbf{x}=\{x\}$ uni-variate normal $N(\mu,\sigma^2)$

Two samples observed: 12.3($\leftarrow x_1$), 13.5($\leftarrow x_2$) $\quad f(\boldsymbol{\theta}) = cL(\boldsymbol{\theta})\cdot P(\boldsymbol{\theta})$

$$L(\boldsymbol{\theta}_f) \propto \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2}\left(\frac{12.3-\mu}{\sigma}\right)^2\right)\times\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2}\left(\frac{13.5-\mu}{\sigma}\right)^2\right)$$

※ $L(\boldsymbol{\theta})$
- MLE $\quad \boldsymbol{\theta}_{\text{MLE}} = \arg\max L(\boldsymbol{\theta})$ $\quad \to \ \dfrac{\partial L}{\partial \theta}=0$, $\quad$ prefer $\dfrac{\partial \ln L}{\partial \theta}=0$
- Bayesian Parameter Extimation

$$f(\boldsymbol{\theta}) = c\cdot L(\boldsymbol{\theta})\cdot p(\boldsymbol{\theta})$$

② No direct measurement $\mathbf{x}$ of available, but a set of events that involve $\mathbf{x}$ are available

e.g. no measurement for compressive strength of concrete $f_c'$ $(\leftarrow \mu,\ \sigma,\ \lambda\ldots)$

available but spalling observed under a certain condition

Inequality events : $h_i(\mathbf{x})\le 0,\ i=1,\cdots,N$

Equality events : $h_i(\mathbf{x})=0$

a) Inequality

e.g. $h_i(\mathbf{x}) = -C(\mathbf{x}) + D(\mathbf{x}) \le 0$ no failure observed

$h_i(\mathbf{x}) = C(\mathbf{x}) - D(\mathbf{x}) \le 0$ failure observed

$$L(\boldsymbol{\theta}_f) \propto \prod_{i=1}^{N} P(h_i(\mathbf{x}) \le 0 | \boldsymbol{\theta}_f)$$

$$= \prod_{i=1}^{N} \int_{h_i(\mathbf{x}) \le 0} f_{\mathbf{x}}(\mathbf{x};\boldsymbol{\theta}_f) d\mathbf{x} \Rightarrow \text{structural reliability analysis}$$

b) Equality

e.g. $h_i(\mathbf{x}) = a(\mathbf{x}) - a_o = 0$

$a(\mathbf{x})$ : fatigue crack growth model, e.g. Paris law

$a_o$ : measured crack size

$$L(\boldsymbol{\theta}_f) \propto \prod_{i=1}^{N} \lim_{\delta \to 0} P[0 < h_i(\mathbf{x}) \le \delta]$$

$$= \prod_{i=1}^{N} \frac{\partial}{\partial \delta} P[h_i(\mathbf{x}) - \delta \le 0]\Big|_{\delta=0}$$

**Proof**

$$\lim_{\Delta\delta \to 0} \frac{P[h_i(\underline{\mathbf{x}}) - \delta - \Delta\delta \le 0] - P[h_i(\underline{\mathbf{x}}) - \delta \le 0]}{\Delta\delta}\Bigg|_{\delta=0}$$

$$= \lim_{\Delta\delta \to 0} \frac{P[h_i(\underline{\mathbf{x}}) - \Delta\delta \le 0] - P[h_i(\underline{\mathbf{x}}) \le 0]}{\Delta\delta}$$

$$\propto \lim_{\Delta\delta \to 0} P[0 \le h_i(\underline{\mathbf{x}}) \le \Delta\delta]$$

$\nabla_{\boldsymbol{\delta}} P_f |_{\delta=0}$ : can be considered as parameter sensitivity of $P_f$ w.r.t $\delta$ (model parameter)

FORM-based (Madsen, 1987)

Good review & new development (Straub, 2011)

↘ a trick to transform equality constraint to ________ constraint

**◎ Likelihood function for limit-state model parameters,** $L(\boldsymbol{\theta}_g)$

e.g. $g(\mathbf{x};\boldsymbol{\theta}_g) = \underline{V_c(\mathbf{x};\boldsymbol{\theta}_g)} - V_d(\mathbf{x};\boldsymbol{\theta}_g) \le 0$

$$\frac{1}{6}\sqrt{f_c'}\, b_w d \quad \text{(ACI 11-3)}$$

① Statistical model (using original deterministic model)

$y = \underline{\hat{g}(\mathbf{x};\boldsymbol{\theta}_g)} + \sigma\varepsilon$ ~ submodel or limit state function

e.g. $\theta_1 f_c'^{\theta_2} b_w d$ (ACI 11-3) $\boldsymbol{\theta}_g = \{\theta_1, \cdots, \theta_n, \sigma\}$

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$\mathbf{x}$ : observable input parameters ( $f_c^{'}$, $b_w$, $d, \cdots$)

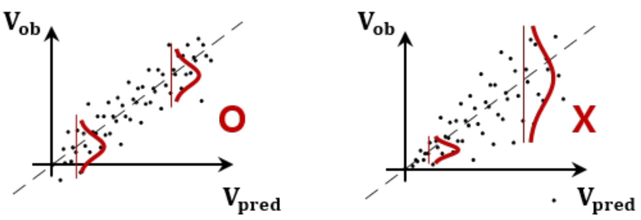
$\mathbf{y}$ : observable output parameters ($V_c$)

$\boldsymbol{\theta}_g$ : uncertain model parameters ($\theta_1$, $\theta_2 \cdots$)

$\sigma\varepsilon$ : uncertainty due to missing variables and/or inexact mathematical form

- $\varepsilon$ : std. normal r.v “ ” assumption
- $\sigma$ : magnitude of model error (uncertain parameter)

  → constant over $\mathbf{x}$ “ ” assumption
- $\mu_\varepsilon = 0$ : unbiased model




May achieve H_______________ by a proper nonlinear transformation

e.g. $\ln y = \ln \hat{g}(\mathbf{x}, \boldsymbol{\theta}_g) + \sigma\varepsilon$

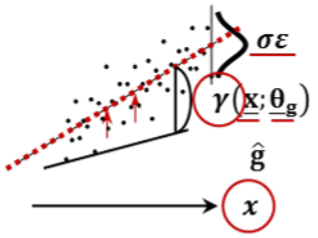
①' Statistical model (based on deterministic model, Gardoni et al. 2002)

$$y = \hat{g}(\mathbf{x}) + \gamma(\mathbf{x}; \boldsymbol{\theta}_g) + \sigma\varepsilon$$

$\hat{g}(\mathbf{x})$ : original deterministic model (e.g. $\frac{1}{6}\sqrt{f_c^{'}} b_w d$ )

$\gamma(\mathbf{x}; \boldsymbol{\theta}_g)$ : corrects the bias

$\sigma\varepsilon$ : remaining scatter




e.g. RC beam w/o stirrups shear capacity
(Song et al. 2010, Structural Eng & Mechanics)

$$\ln V = \ln \hat{v}(\mathbf{x}) + \Sigma\theta_g \ln h_i(\mathbf{x}) + \sigma\varepsilon$$

$\hat{v}(\mathbf{x})$ : 8 models from codes & papers

$h_i(\mathbf{x})$ : explanatory terms from the shear transfer mechanism

Find $\mathbf{M}_{\boldsymbol{\theta}}$, $\boldsymbol{\Sigma}_{\boldsymbol{\theta\theta}}$ ($\frac{\sigma_\theta}{\mu_\theta} = \delta_\theta$, $\rho_{\theta_i \theta_j}$)
using Bayesian Parameter Estimation

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② Likelihood function $L(\boldsymbol{\theta}_g)$?

Observed event Equality: $y = y_i$, $i = 1, \cdots, m$ know $v_c$ when failed

Inequality: $\begin{cases} y > a_i & i = m+1, \cdots, m+n \\ y > b_i & i = m+1, \cdots, m+n+N \end{cases}$ No failure up to Vc

Failed but do not know when

Model $Y = \hat{g} + \gamma + \sigma\varepsilon$

a) $P(Y = y_i) = P(\sigma\varepsilon = y_i - \hat{g}(\mathbf{x}) - \gamma(\mathbf{x}, \theta_g))$

$$P(Y = y_i) \propto f_Y(y_i)$$

$$= f_Q(q_i) \cdot \frac{dq}{dy}$$

$$= f_\varepsilon(\varepsilon_i) \cdot \frac{d\varepsilon}{dq}$$

$$= \frac{1}{\sigma} \varphi\left(\frac{y_i - \hat{g} - \gamma}{\sigma}\right)$$

$$f_Y(y_i) = f_Q(q) \cdot \frac{dq}{dy_i}$$

$$f_Q(q) = f_\varepsilon(\varepsilon) \cdot \frac{d\varepsilon}{dq}$$

$$q = \sigma \cdot \varepsilon$$

b) $P(Y > a_i) = P(\hat{g} + \gamma + \sigma\varepsilon > a_i)$

$$= P(\sigma\varepsilon > a_i - \hat{g} - \gamma)$$

$$= \Phi\left(-\frac{a_i - \hat{g} - \gamma}{\sigma}\right)$$

c) $P(Y < b_i) = P(\hat{g} + \gamma + \sigma\varepsilon < b_i)$

$$= P(\sigma\varepsilon < b_i - g - \gamma)$$

$$= \Phi\left(\frac{b_i - g - \gamma}{\sigma}\right)$$

$$\therefore L(\theta_g) = \prod_{i=1}^{m} \frac{1}{\sigma} \varphi\left(\frac{y_i - \hat{g} - \gamma}{\sigma}\right) \times \prod_{i=m+1}^{m+n} \Phi\left(-\frac{a_i - \hat{g} - \gamma}{\sigma}\right) \times \prod_{i=m+n+1}^{m+n+N} \Phi\left(\frac{b_i - \hat{g} - \gamma}{\sigma}\right)$$

※ Matlab codes for "Model Development by Bayesian method"

→ MDB (by Prof. S.Y. Ok at Hankyoung univ. for educational purpose)