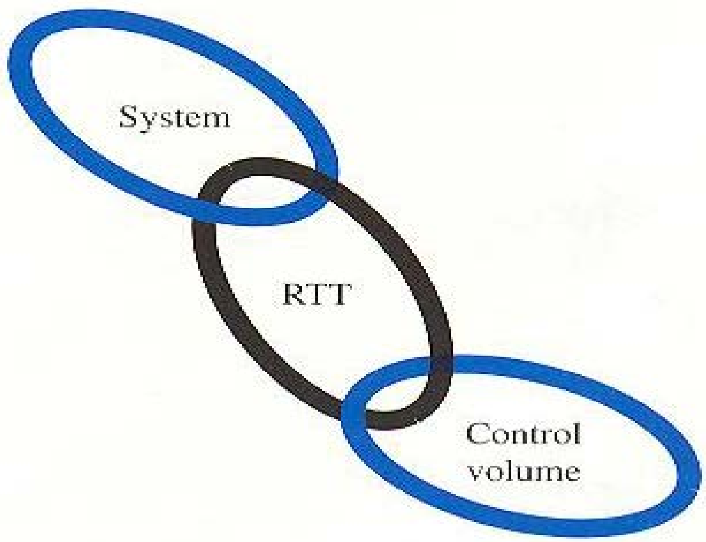
Chapter 4

Continuity Equation and Reynolds Transport Theorem



Contents

Objectives

- Apply the concept of the control volume to derive equations for the conservation of mass for steady one- and two-dimensional flows
- Derive the Reynolds transport theorem for three-dimensional flow
- Show that continuity equation can recovered by simplification of the Reynolds transport theorem

- 자연계 3대 법칙

1) 질량 보존의 법칙: 시스템내에 존재하는 질량은 새로이 생성되거나 소멸되지 않는다.

→ 연속방정식

2) 에너지 보존의 법칙: 시스템안에 존재하는 모든 에너지의 합은 그 형태가 변해도 보존된다.

→ 베르누이 방정식

3) 뉴턴의 제2법칙: 시스템(물체)에 가한 힘의 합은 운동량의 시간에 대한 변화율과 같다.

→ 운동량 방정식

4.1 Control Volume

- **System approach vs Control volume approach**

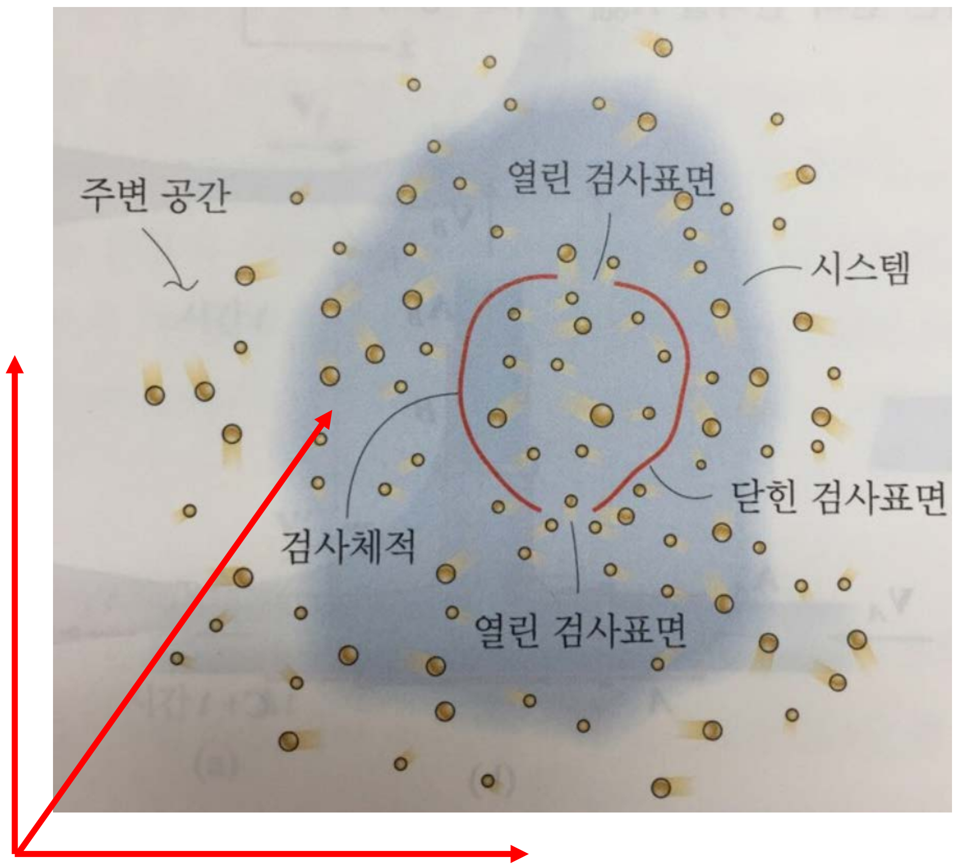
- 유체는 자유롭게 이동하고 주변과 상호 작용하는 특성을 가지고 있으므로 2개의 해석 방법을 사용함.
- System approach는 Lagrangian 해석법과 유사하며, Control volume approach는 Eulerian해석법과 유사함.
- System approach (Lagrangian 해석법)에서는 관찰자가 유체에 표시를 해서 유체가 움직일 때 이를 따라 가면서 계속해서 특성을 관찰함
- Control volume approach (Eulerian해석법)에서는 관찰자가 이동하지 않고 고정된 지점(영역)을 통과하는 유체의 운동을 관찰함

4.1 Control Volume

1) Physical system

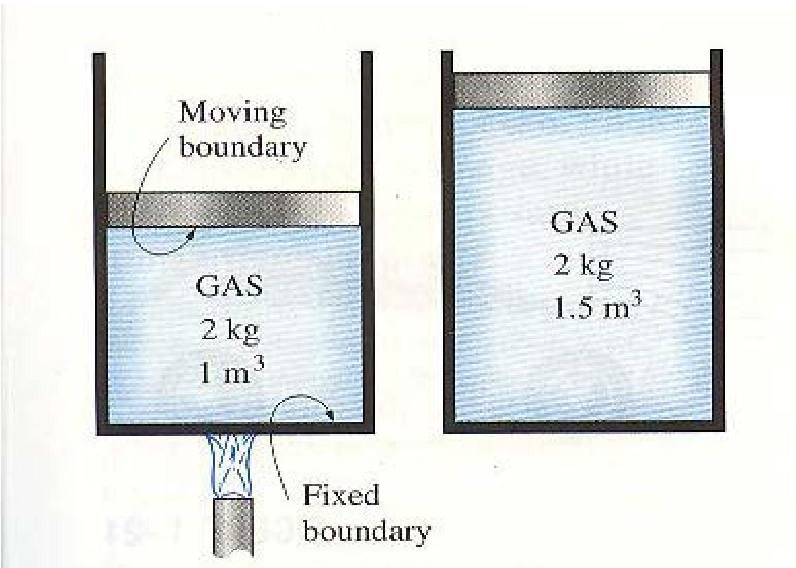
- is defined as a collection of matter of fixed identity (always the same atoms or fluid particles) → 물질의 집합체
- They can move, flow, and interact with its surroundings
- System may consist of a relatively large amount of mass (such as all of the air in the Earth's atmosphere)
- or it may be an infinitesimal size (such as a single fluid particle)
- The system may interact with its surroundings by various means (by transfer of heat or the exertion of a pressure force).
- It may continually <u>change size and shape</u>, but it always <u>contains the same mass.</u>

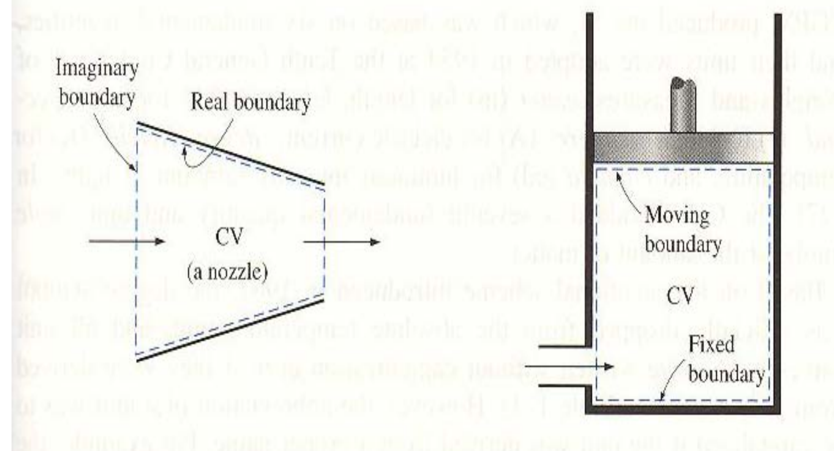
4.1 Control Volume



System vs Control volume

4.1 Control Volume





(*a*) A control volume (CV) with real and imaginary boundaries

(*b*) A control volume (CV) with fixed and moving boundaries

Closed system

Control volume

4.1 Control Volume

- Disadvantage of system approach

- In particle mechanics, the system is a convenient physical entity.

- A system-based analysis of fluid flow leads to the Lagrangian equations of motion in which particles of fluid are tracked.

- However, a fluid system is mobile and very deformable.

- Further, a large number of engineering problems involve mass flow in and out of a system.

→ This suggests the need to define a more convenient object for analysis.

→ Control volume (Eulerian view)

4.1 Control Volume

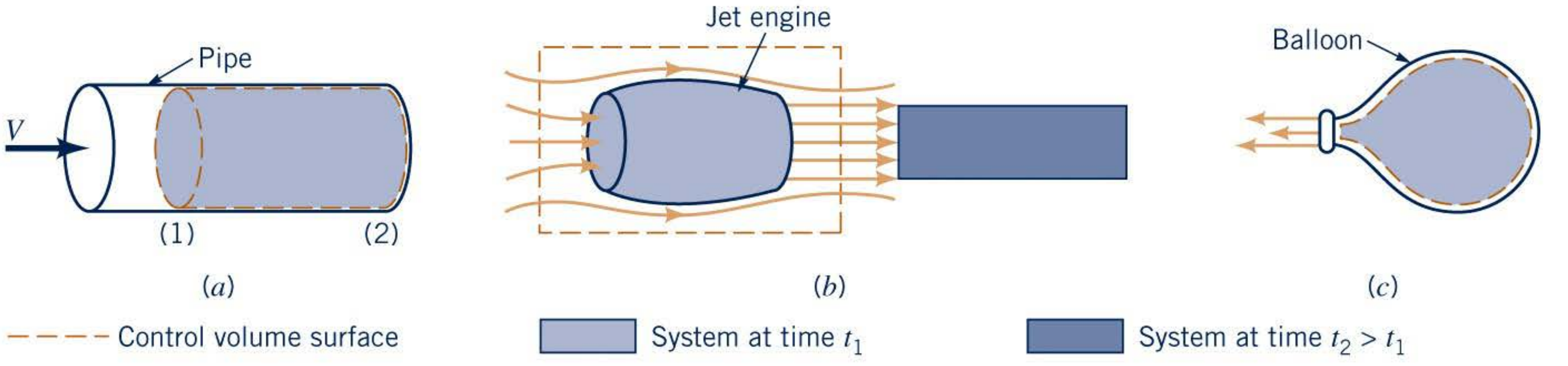
2) **Control volume (검사체적)**

- 유체 흐름장 내에서 취급대상으로 고려하는 한정된 체적
- a volume which is fixed in space and through whose boundary matter, mass, momentum, energy can flow
- Control volume can be any size (finite or infinitesimal), any space.
- In this course, we assume that the control volume can be fixed in size and shape.

→ This approach is consistent with the Eulerian view of fluid motion, in which attention is focused on particular points in the space filled by the fluid rather than on the fluid particles.

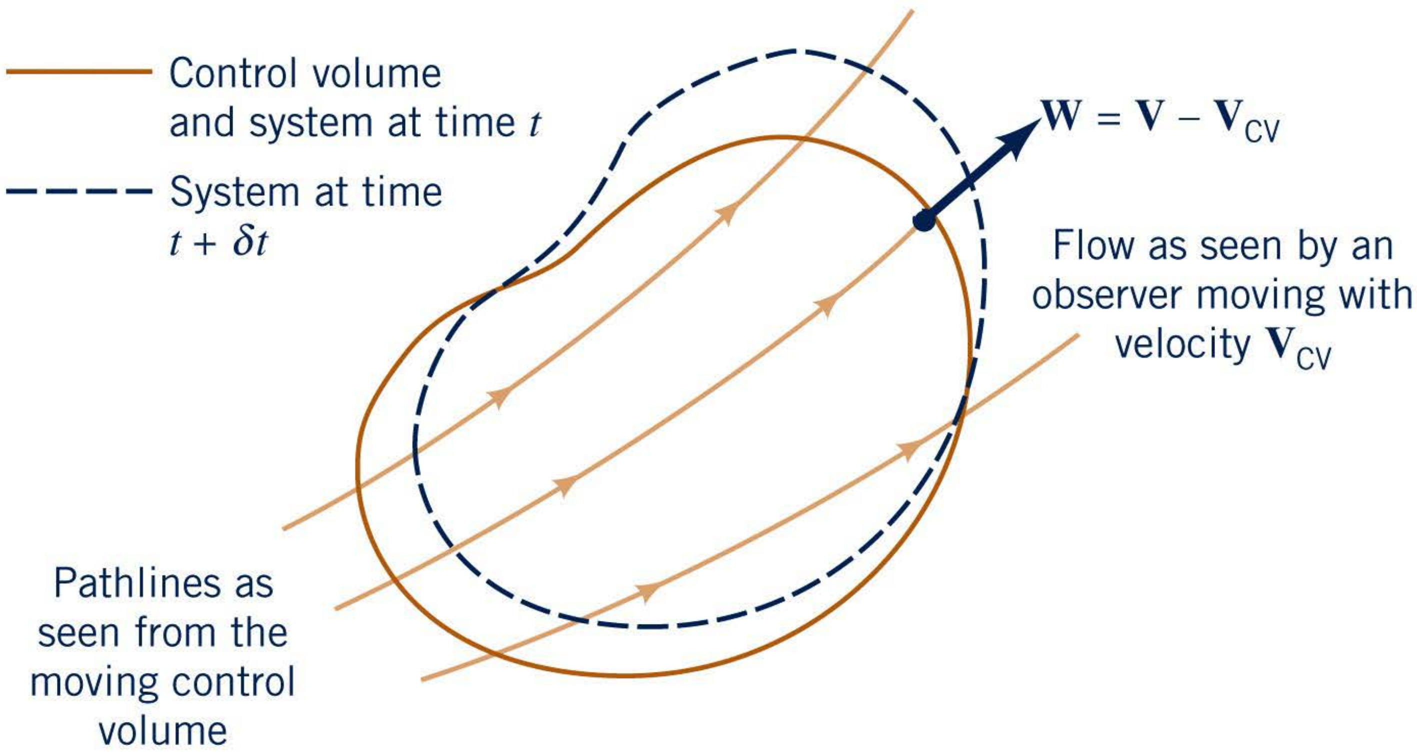
- Control surface (검사표면): boundary of control volume

4.1 Control Volume



a) Fixed control volume   b) Moving control volume   c) Deforming control volume

4.1 Control Volume



System vs Control volume

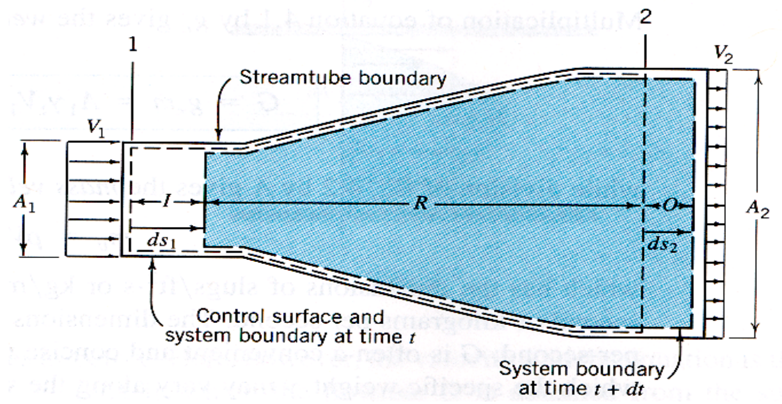
4.2 Continuity Equation for One-Dimensional Steady Flow

- Principle of conservation of mass

The application of principle of conservation of mass to a steady flow in a stream tube results in the **continuity equation**.

[Re] Continuity equation

~ describes the continuity of flow from section to section of the stream tube (control volume)



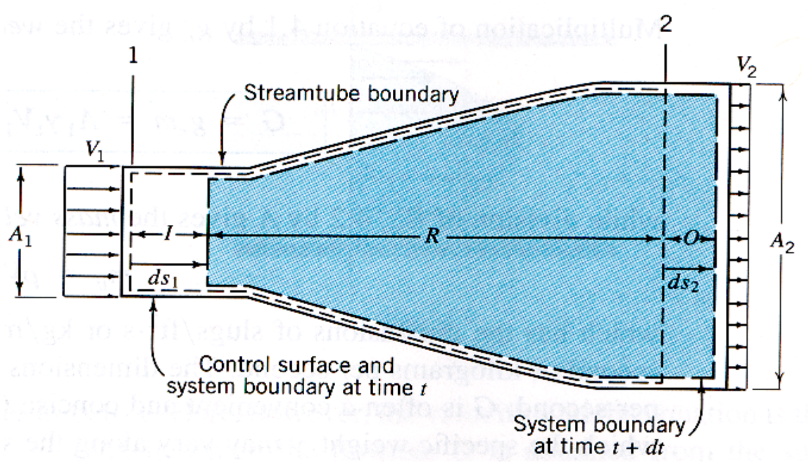
4.2 Continuity Equation for One-Dimensional Steady Flow

- One-dimensional steady flow

- Consider the element of a finite stream tube through which passes a steady, 1D flow of a compressible fluid

- no net velocity normal to a streamline

- no fluid can leave or enter the stream tube except at two ends



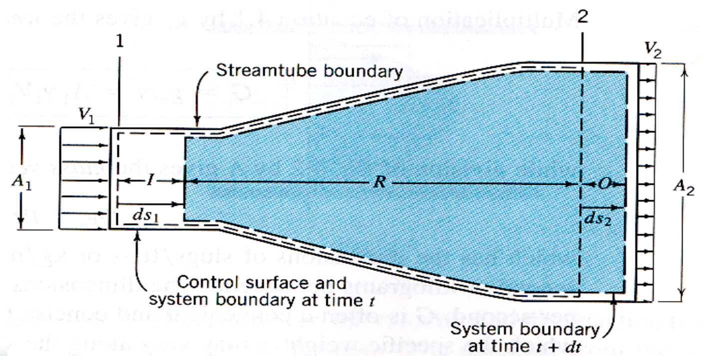
4.2 Continuity Equation for One-Dimensional Steady Flow

- Now, define the control volume as marked by the control surface that bounds the region between sections 1 and 2 and <u>lies along the inner wall of the streamtube</u>

→ To be consistent with the assumption of one-dimensional steady flow, the velocities at sections 1 and 2 are <u>assumed to be uniform</u>.

→ The control volume comprises volumes *I* and *R* at time *t*.

→ The control volume is fixed in space, but in *dt* <u>the system moves downstream</u>.



4.2 Continuity Equation for One-Dimensional Steady Flow

From the conservation of system mass

$$(m_I + m_R)_t = (m_R + m_O)_{t+\Delta t} \qquad (1)$$

For steady flow, the fluid properties at points in space are not functions of time, $\dfrac{\partial m}{\partial t} = 0$

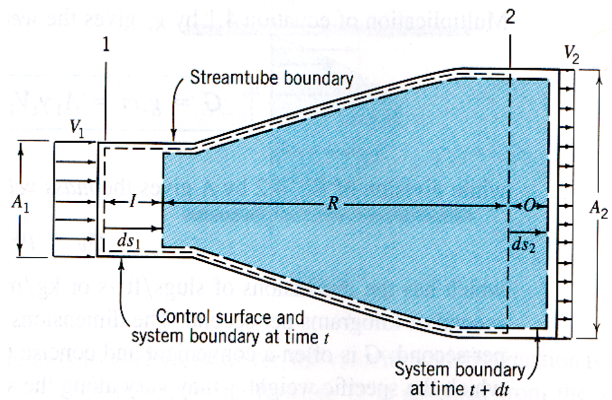
$$\rightarrow (m_R)_t = (m_R)_{t+\Delta t} \qquad (2)$$

Substituting (2) into (1) yields

$$(m_I)_t = (m_O)_{t+\Delta t} \qquad (3)$$

Inflow

Outflow



4.2 Continuity Equation for One-Dimensional Steady Flow

Express inflow and outflow in terms of the mass of fluid moving across the control surface in time dt

$$(m_I)_t = \rho_1 A_1 ds_1$$

$$(m_O)_{t+\Delta t} = \rho_2 A_2 ds_2 \qquad (4)$$

Substituting (4) into (3) yields

$$\rho_1 A_1 ds_1 = \rho_2 A_2 ds_2$$

Dividing by dt gives

$$\boxed{\frac{ds_1}{dt} = V_1}$$

$$\boxed{\dot{m} = \rho_1 A_1 V_1 = \rho_2 A_2 V_2} \qquad (4.1)$$

→ Continuity equation

4.2 Continuity Equation for One-Dimensional Steady Flow

In steady flow, the mass flow rate (질량유량), $\dot{m}$ passing all sections of a stream tube is constant.

$$\dot{m} = \rho A V \text{ = constant (kg/sec)}$$

$$d(\rho A V) = 0 \qquad \text{(4.2a)}$$

$$\rightarrow \; d\rho(AV) + dA(\rho V) + dV(\rho A) = 0 \qquad \text{(5)}$$

Dividing (a) by $\rho A V$ results in

$$\boxed{\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0} \qquad \text{(4.2b)}$$

→ 1-D steady compressible fluid flow

4.2 Continuity Equation for One-Dimensional Steady Flow

For <u>incompressible</u> fluid flow; constant density

$$\rightarrow \quad d\rho = 0, \; \frac{\partial \rho}{\partial t} = 0 \qquad (4.3)$$

From Eq. (4.2a)

$$\rho d(AV) = 0 \qquad (4.4)$$

$$d(AV) = 0 \qquad (6)$$

Set Q = <u>volume flowrate (유량)</u> (m^3/s, cms)

Then (6) becomes

$$\boxed{Q = AV = \text{const.} = A_1V_1 = A_2V_2} \qquad (4.5)$$

4.2 Continuity Equation for One-Dimensional Steady Flow

- For 2-D flow, flowrate is usually quoted per unit distance normal to the plane of the flow.

$\rightarrow$ $q =$ flowrate per unit distance normal to the plane of flow $\left(\mathrm{m}^3/\mathrm{s}\cdot\mathrm{m}\right)$

$$q = \frac{Q}{b} = \frac{AV}{b} = hV \tag{4.6}$$

$$\boxed{h_1V_1 = h_2V_2} \tag{4.7}$$

[Re] For unsteady flow

$$mass_{t+\Delta t} = mass_t + \text{inflow} - \text{outflow}$$

$$(m_R)_{t+\Delta t} - (m_R)_t = (m_I)_t - (m_O)_{t+\Delta t}$$

4.2 Continuity Equation for One-Dimensional Steady Flow

Divide by dt

$$\frac{(m_R)_{t+\Delta t} - (m_R)_t}{dt} = (m_I)_t - (m_O)_{t+\Delta t}$$

Define

$$\frac{\partial m}{\partial t} = \frac{(m_R)_{t+\Delta t} - (m_R)_t}{dt} = \frac{\partial(\rho vol)}{\partial t}$$

Then

$$\frac{\partial(\rho vol)}{\partial t} = (m_I)_t - (m_O)_{t+\Delta t}$$

4.2 Continuity Equation for One-Dimensional Steady Flow

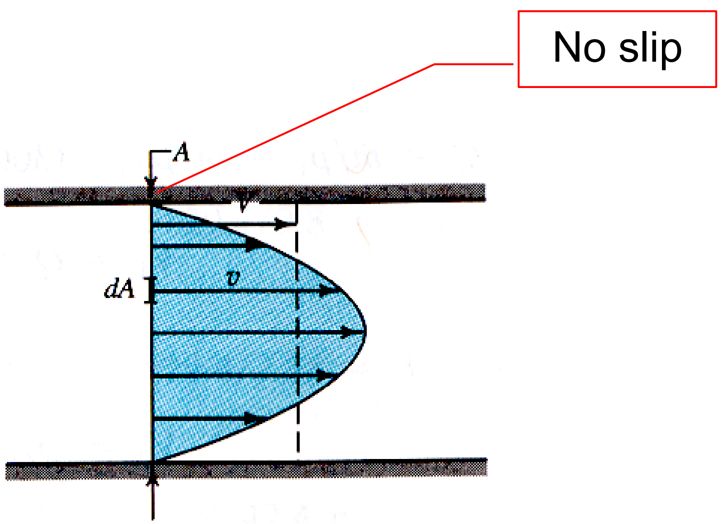
• Non-uniform velocity distribution through flow cross section

Eq. (4.5) can be applied. However, velocity in Eq. (4.5) should be the mean velocity.

$$V = \frac{Q}{A}$$

$$Q = \int_A dQ = \int_A v dA$$

$$\therefore \quad V = \frac{1}{A}\int_A v dA$$



4.2 Continuity Equation for One-Dimensional Steady Flow

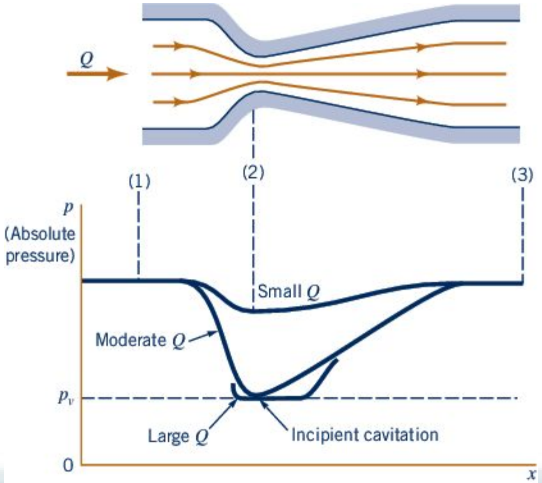
•The fact that the product AV remains constant along a streamline in a fluid of constant density allows a physical interpretation of streamline pictures.

→ As the cross-sectional area of stream tube increases, the velocity must decrease.

→ Streamlines widely spaced indicate regions of low velocity, streamlines closely spaced indicate regions of high velocity.

$$A_1V_1 = A_2V_2$$

$$A_1 > A_2 \rightarrow V_1 < V_2$$



4.2 Continuity Equation for One-Dimensional Steady Flow

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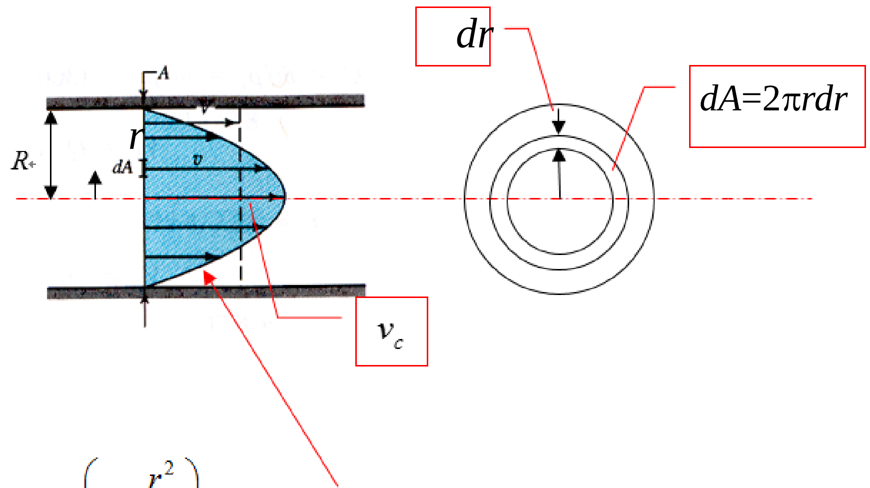
The velocity in a cylindrical pipe of radius *R* is represented by an <u>axisymmetric parabolic distribution (laminar flow)</u>.

What is *V* in terms of maximum velocity, v_c ?



$$v = v_c\left(1 - \frac{r^2}{R^2}\right)$$

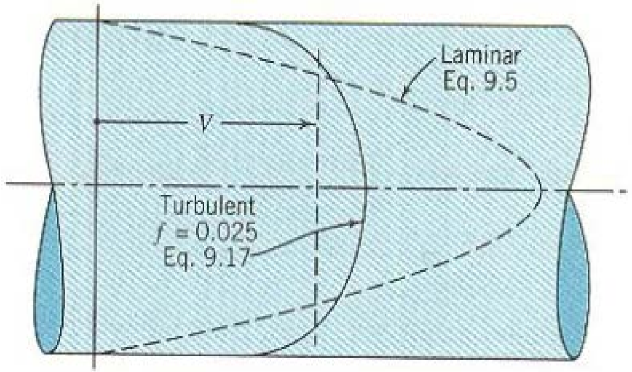
4.2 Continuity Equation for One-Dimensional Steady Flow

$$V = \frac{Q}{A} = \frac{1}{A}\int_A v\,dA = \frac{1}{\pi R^2}\int_0^R v_c\left(1 - \frac{r^2}{R^2}\right)2\pi r\,dr$$

$$= \frac{2v_c}{R^2}\int_0^R\left(r - \frac{r^3}{R^2}\right)dr = \frac{2v_c}{R^2}\left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R = \frac{2v_c}{R^2}\left[\frac{R^2}{2} - \frac{R^2}{4}\right] = \frac{v_c}{2}$$ → Laminar flow

[Cf] Turbulent flow

→ logarithmic velocity distribution



4.3 Continuity Equation for Two-Dimensional Steady Flow

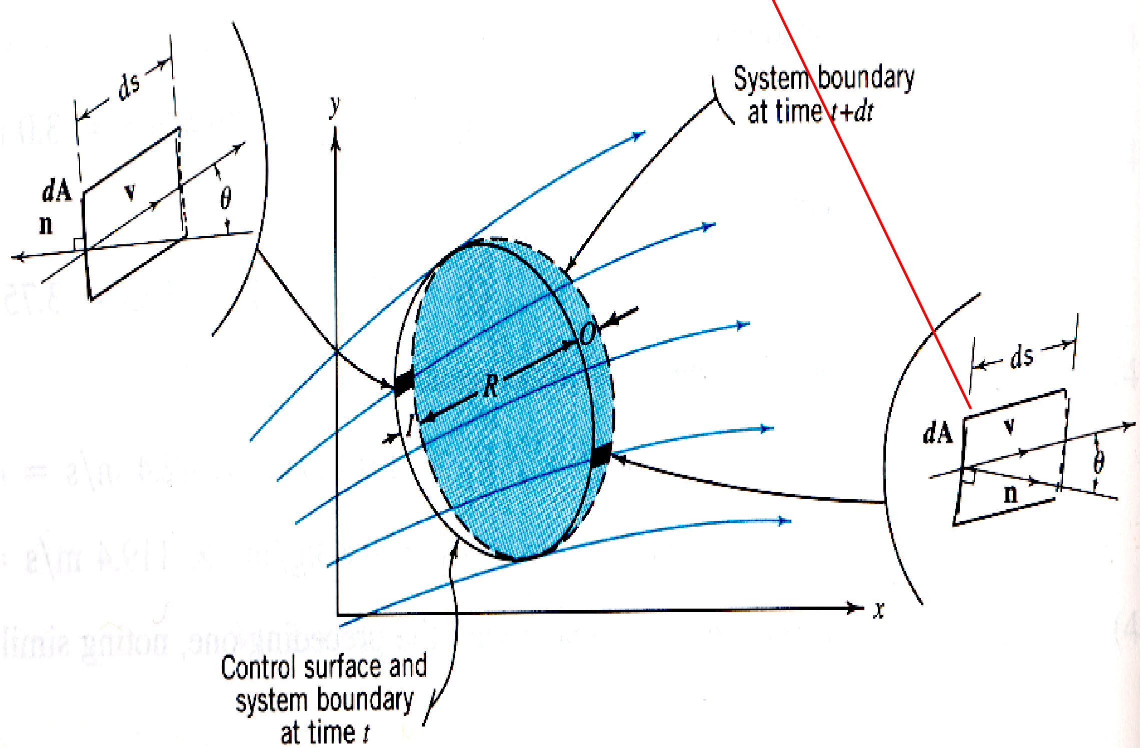
(1) Finite control volume

- Consider a general control volume and apply

conservation of mass

- 유한검사체적법

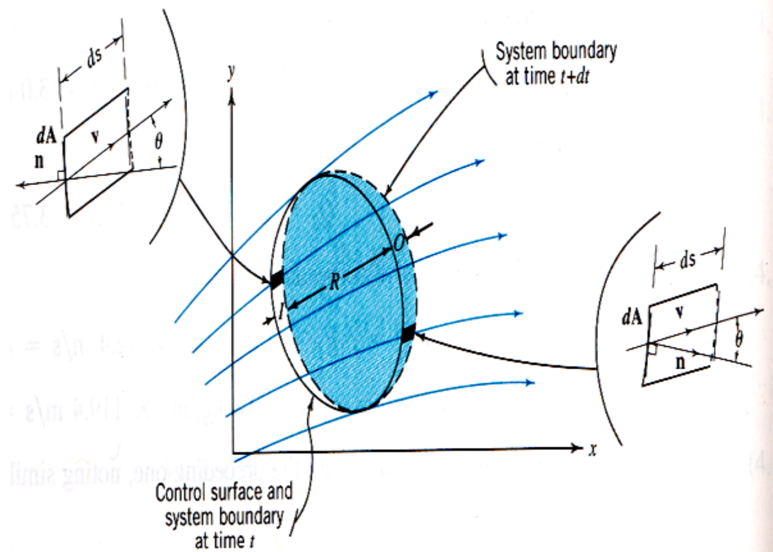
Area of parallelogram
= $(ds\cos\theta)dA$



4.3 Continuity Equation for Two-Dimensional Steady Flow

- Apply conservation of mass

$$(m_I + m_R)_t = (m_R + m_O)_{t+\Delta t} \qquad \text{(a)}$$



For steady flow: $(m_R)_t + (m_R)_{t+\Delta t}$

Then (a) becomes

$$(m_I)_t = (m_O)_{t+\Delta t} \qquad \text{(b)}$$

i) Mass in O moving out through control surface

$$(m_O)_{t+\Delta t} = \int_{C.S.out} \rho (ds\cos\theta) dA$$

$$\boxed{\text{mass} = \rho\ vol = \rho \times \text{area} \times 1 = \rho\, ds\, dA \cos\theta}$$

4.3 Continuity Equation for Two-Dimensional Steady Flow

- Displacement along a streamline is

$$ds = vdt \qquad \text{(c)}$$

Substituting (c) into (b) gives

$$(m_O)_{t+\Delta t} = \int_{C.S.out} \rho(v\cos\theta)\, dA\, dt \qquad \text{(d)}$$

By the way, $v\cos\theta$ = normal velocity component normal to C.S. at dA

Set $\vec{n}$ = outward unit normal vector at dA $\left(|\vec{n}| = 1\right)$

$$\therefore v_n = \vec{v} \cdot \vec{n} = v\cos\theta \leftarrow \text{scalar or dot product} \qquad \text{(e)}$$

Substitute (e) into (d)

$$(m_O)_{t+\Delta t} = dt \int_{C.S.out} \rho \vec{v} \cdot \vec{n}\, dA = dt \int_{C.S.out} \rho \vec{v} \cdot \overrightarrow{dA} \qquad \text{(4.8)}$$

where $\overrightarrow{dA} = \vec{n}\, dA$ =directed area element

4.3 Continuity Equation for Two-Dimensional Steady Flow

[Cf] Tangential component of velocity does not contribute to flow through the C.S.

→ It contributes to circulation

ii) Mass flow into CV

$$(m_I)_t = \int_{C.S.in} \rho (ds \cos\theta) dA$$

$$\boxed{\theta > 90^\circ \rightarrow \cos\theta < 0}$$

$$\int_{C.S.in} \rho (v\cos\theta) dA dt = dt \int_{C.S.in} \rho \vec{v} \cdot (-\vec{n}) dA$$

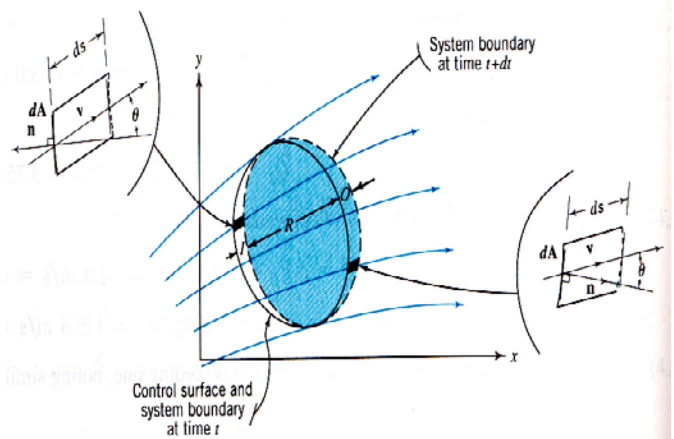
$$= dt \left\{ -\int_{C.S.in} \rho \vec{v} \cdot \vec{n} dA \right\} = dt \left\{ -\int_{C.S.in} \rho \vec{v} \cdot \overrightarrow{dA} \right\}$$

For steady flow, *mass in = mass out*

$$dt \int_{C.S.out} \rho \vec{v} \cdot \overrightarrow{dA} = dt \left\{ -\int_{C.S.in} \rho \vec{v} \cdot \overrightarrow{dA} \right\}$$

Divide by dt

$$-\int_{C.S.in} \rho \vec{v} \cdot \overrightarrow{dA} = \int_{C.S.out} \rho \vec{v} \cdot \overrightarrow{dA}$$



4.3 Continuity Equation for Two-Dimensional Steady Flow

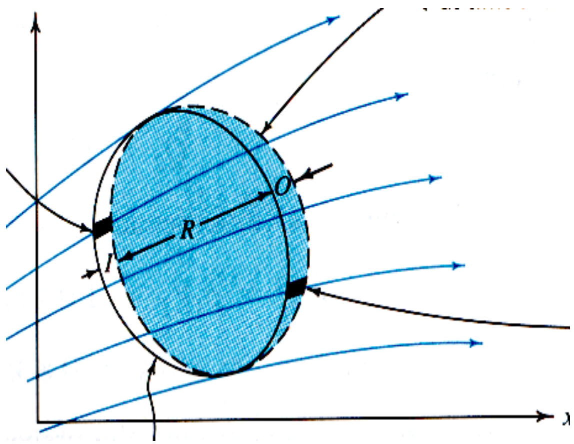
$$\int_{C.S.out} \rho \vec{v} \cdot \overrightarrow{dA} + \int_{C.S.in} \rho \vec{v} \cdot \overrightarrow{dA} = 0 \qquad \text{(f)}$$

Combine C.S. in and C.S. out

Integral form

$$\boxed{\oint_{C.S.} \rho \vec{v} \cdot \overrightarrow{dA} = \oint_{C.S.} \rho \vec{v} \cdot \vec{n} dA = 0} \qquad (4.9)$$

where $\oint_{C.S.}$ = integral around the control surface in the counterclockwise direction

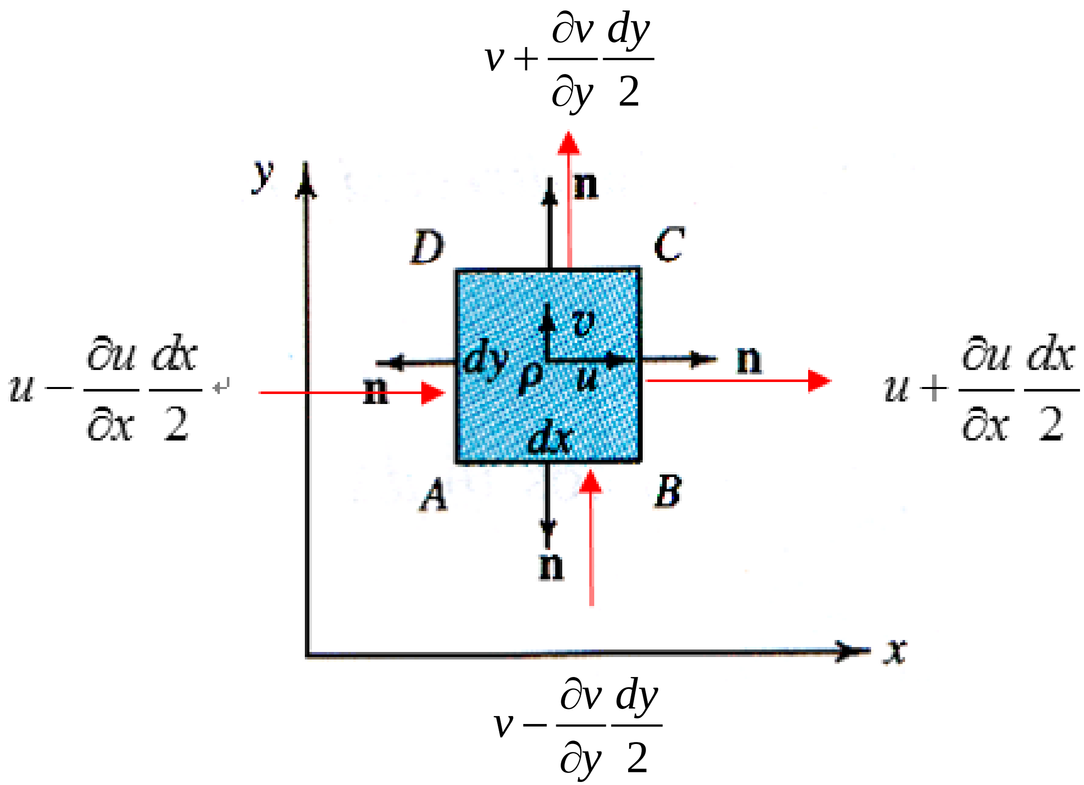


→ Continuity equation for 2-D steady flow of compressible fluid

4.3 Continuity Equation for Two-Dimensional Steady Flow

(2) Infinitesimal control volume

- 미소검사체적법



4.3 Continuity Equation for Two-Dimensional Steady Flow

Apply (4.9) to control volume ABCD

$$\int_{AB}\rho\vec{v}\cdot\vec{n}dA+\int_{BC}\rho\vec{v}\cdot\vec{n}dA+\int_{CD}\rho\vec{v}\cdot\vec{n}dA+\int_{DA}\rho\vec{v}\cdot\vec{n}dA=0 \qquad \text{(f)}$$

Expand to first-order accuracy

$$\boxed{\rho_{AB}\approx\rho-\frac{\partial\rho}{\partial x}\frac{dx}{2}}$$

$$\int_{AB}\rho\vec{v}\cdot\vec{n}dA\cong-\left(\rho-\frac{\partial\rho}{\partial y}\frac{dy}{2}\right)\left(v-\frac{\partial v}{\partial y}\frac{dy}{2}\right)dx$$

$$\boxed{\vec{v}\cdot\vec{n}=-\left(v-\frac{\partial v}{\partial y}\frac{dy}{2}\right)}$$

$$\int_{BC}\rho\vec{v}\cdot\vec{n}dA\cong\left(\rho+\frac{\partial\rho}{\partial x}\frac{dx}{2}\right)\left(u+\frac{\partial u}{\partial x}\frac{dx}{2}\right)dy$$

$$\int_{CD}\rho\vec{v}\cdot\vec{n}dA\cong\left(\rho+\frac{\partial\rho}{\partial y}\frac{dy}{2}\right)\left(v+\frac{\partial v}{\partial y}\frac{dy}{2}\right)dx \qquad \text{(g)}$$

$$\int_{DA}\rho\vec{v}\cdot\vec{n}dA\cong-\left(\rho-\frac{\partial\rho}{\partial x}\frac{dx}{2}\right)\left(u-\frac{\partial u}{\partial x}\frac{dx}{2}\right)dy$$

4.3 Continuity Equation for Two-Dimensional Steady Flow

Substitute (g) to (f), and expand products, and then retain only terms of lowest order (largest order of magnitude)

$$-\rho v dx + \rho \frac{\partial v}{\partial y}\frac{dy}{2}dx + v\frac{\partial \rho}{\partial y}\frac{dy}{2}dx - \cancel{\frac{\partial \rho}{\partial y}\frac{\partial v}{\partial y}\frac{(dy)^2}{4}dx}$$

$$+\rho v dx + \rho \frac{\partial v}{\partial y}\frac{dy}{2}dx + v\frac{\partial \rho}{\partial y}\frac{dy}{2}dx + \cancel{\frac{\partial \rho}{\partial y}\frac{\partial v}{\partial y}\frac{(dy)^2}{4}dx}$$

$$+\rho u dy + \rho \frac{\partial u}{\partial x}\frac{dx}{2}dy + u\frac{\partial \rho}{\partial x}\frac{dx}{2}dy + \cancel{\frac{\partial \rho}{\partial x}\frac{\partial u}{\partial x}\frac{(dx)^2}{4}dy}$$

$$-\rho u dy + \rho \frac{\partial u}{\partial x}\frac{dx}{2}dy + u\frac{\partial \rho}{\partial x}\frac{dx}{2}dy - \cancel{\frac{\partial \rho}{\partial x}\frac{\partial v}{\partial x}\frac{(dx)^2}{4}dy} = 0$$

$$\therefore \quad \rho\frac{\partial v}{\partial y}dxdy + v\frac{\partial \rho}{\partial y}dxdy + \rho\frac{\partial u}{\partial x}dxdy + u\frac{\partial \rho}{\partial x}dxdy = 0$$

4.3 Continuity Equation for Two-Dimensional Steady Flow

$$\rho\frac{\partial v}{\partial y}+v\frac{\partial \rho}{\partial y}+\rho\frac{\partial u}{\partial x}+u\frac{\partial \rho}{\partial x}=0$$

Point form

$$\boxed{\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0} \qquad (4.10)$$

→ Continuity equation for 2-D steady flow of compressible fluid

• Continuity equation of incompressible flow for both steady and unsteady flow ($\rho=$ const.)

$$\boxed{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0} \qquad (4.11)$$

4.3 Continuity Equation for Two-Dimensional Steady Flow

Continuity equation for unsteady 3-D flow of compressible fluid

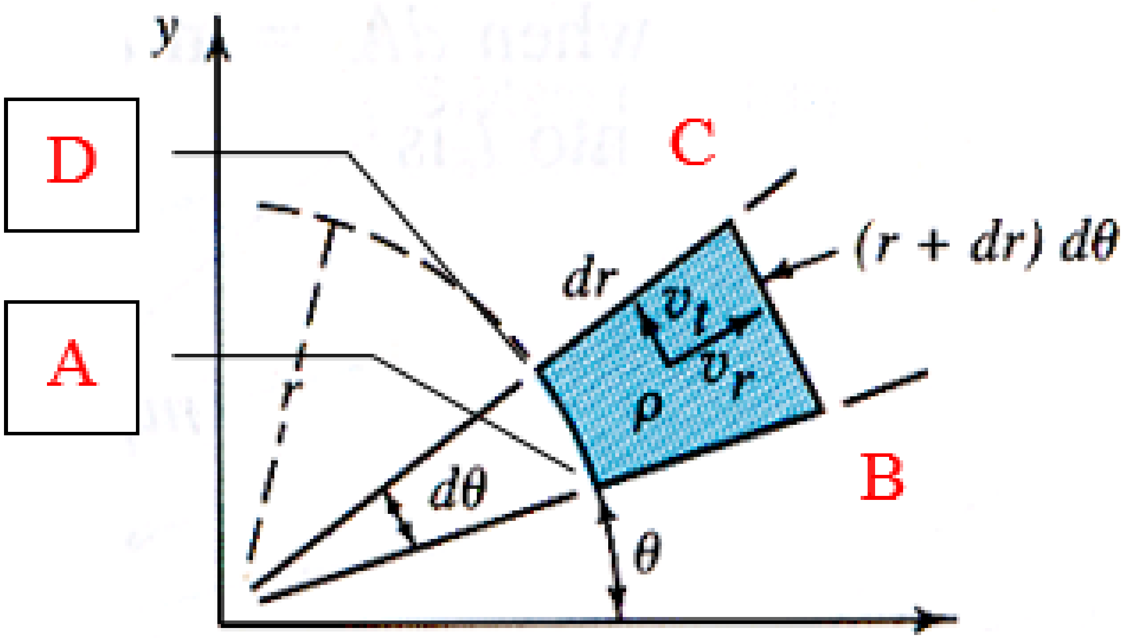
$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0$$

For steady 3-D flow of incompressible fluid

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

4.3 Continuity Equation for Two-Dimensional Steady Flow

- Continuity equation for polar coordinates



4.3 Continuity Equation for Two-Dimensional Steady Flow

Apply (4.9) to control volume ABCD

$$\int_{AB} \rho \vec{v} \cdot \vec{n} dA + \int_{BC} \rho \vec{V} \cdot \vec{n} dA + \int_{CD} \rho \vec{V} \cdot \vec{n} dA + \int_{DA} \rho \vec{V} \cdot \vec{n} dA = 0$$

$$\int_{AB} \rho \vec{V} \cdot \vec{n} dA \cong -\left(\rho - \frac{\partial \rho}{\partial \theta} \frac{d\theta}{2} \right) \left(v_t - \frac{\partial v_t}{\partial \theta} \frac{d\theta}{2} \right) dr$$

$$\int_{BC} \rho \vec{V} \cdot \vec{n} dA \cong \left(\rho + \frac{\partial \rho}{\partial r} \frac{dr}{2} \right) \left(v_r + \frac{\partial v_r}{\partial r} \frac{dr}{2} \right) (r + dr) d\theta$$

$$\int_{CD} \rho \vec{V} \cdot \vec{n} dA \cong \left(\rho + \frac{\partial \rho}{\partial \theta} \frac{d\theta}{2} \right) \left(v_t + \frac{v_t}{\partial \theta} \frac{d\theta}{2} \right) dr$$

$$\int_{DA} \rho \vec{V} \cdot \vec{n} dA \cong -\left(\rho - \frac{\partial \rho}{\partial r} \frac{dr}{2} \right) \left(v_r - \frac{\partial v_r}{\partial r} \frac{dr}{2} \right) r d\theta$$

4.3 Continuity Equation for Two-Dimensional Steady Flow

$$-\cancel{\rho v_t dr}+\rho\frac{\partial v_t}{\partial\theta}\frac{d\theta}{2}dr+v_t\frac{\partial\rho}{\partial\theta}\frac{d\theta}{2}dr-\frac{\partial\rho}{\partial\theta}\frac{\partial v_t}{\partial\theta}\cancel{\frac{(d\theta)^2}{4}}dr$$

$$+\cancel{\rho v_t dr}+\rho\frac{\partial v_t}{\partial\theta}\frac{d\theta}{2}dr+v_t\frac{\partial\rho}{\partial\theta}\frac{d\theta}{2}dr+\frac{\partial\rho}{\partial\theta}\frac{\partial v_t}{\partial\theta}\cancel{\frac{(d\theta)^2}{4}}dr$$

$$+\cancel{\rho v_r r d\theta}+\rho v_r dr d\theta+\rho\frac{\partial v_r}{\partial r}\frac{dr}{2}rd\theta+\rho\frac{\partial v_r}{\partial r}\frac{dr}{2}drd\theta$$

$$+v_r\frac{\partial\rho}{\partial r}\frac{dr}{2}rd\theta+v_r\frac{\partial\rho}{\partial r}\frac{dr}{2}drd\theta+\frac{\partial\rho}{\partial r}\cancel{\left(\frac{dr}{2}\right)^2}\frac{\partial v_r}{\partial r}rd\theta+\frac{\partial\rho}{\partial r}\left(\frac{dr}{2}\right)^2\frac{\partial v_r}{\partial r}drd\theta$$

$$-\cancel{\rho v_r r d\theta}+\rho\frac{\partial v_r}{\partial r}\frac{dr}{2}rd\theta+v_r\frac{\partial\rho}{\partial r}\frac{dr}{2}rd\theta-\frac{\partial\rho}{\partial r}\frac{\partial v_r}{\partial r}\cancel{\left(\frac{dr}{2}\right)^2}rd\theta=0$$

$$\rho\frac{\partial v_t}{\partial\theta}d\theta dr+v_t\frac{\partial\rho}{\partial\theta}d\theta dr+\rho\frac{\partial v_r}{\partial r}rdrd\theta+v_r\frac{\partial\rho}{\partial r}rdrd\theta$$

$$+\rho v_r drd\theta+\rho\frac{\partial v_r}{\partial r}\frac{1}{2}(dr)^2d\theta+v_r\frac{\partial\rho}{\partial r}\frac{1}{2}(dr)^2d\theta+\frac{\partial\rho}{\partial r}\frac{\partial v_r}{\partial r}\frac{1}{2}(dr)^3d\theta=0$$

4.3 Continuity Equation for Two-Dimensional Steady Flow

Divide by $dr d\theta$

$$\rho\frac{\partial v_t}{\partial\theta}+v_t\frac{\partial\rho}{\partial\theta}+\rho\frac{\partial v_r}{\partial r}+v_r\frac{\partial\rho}{\partial r}r+\rho v_r+\rho\frac{\partial v_r}{\partial r}\frac{1}{2}dr+v_r\frac{\partial\rho}{\partial r}\frac{1}{2}dr+\frac{\partial\rho}{\partial r}\frac{\partial v_r}{\partial r}\frac{1}{2}dr=0$$

$$\therefore\quad \rho\frac{\partial v_r}{\partial r}r+v_r\frac{\partial\rho}{\partial r}r+\rho v_r+\rho\frac{\partial v_t}{\partial\theta}+v_t\frac{\partial\rho}{\partial\theta}=0$$

Divide by r

$$\rho\frac{\partial v_r}{\partial r}+v_r\frac{\partial\rho}{\partial r}+\rho\frac{v_r}{r}+\rho\frac{\partial v_t}{r\partial\theta}+v_t\frac{\partial\rho}{r\partial\theta}=0$$

$$\therefore\quad \frac{\partial(\rho v_r)}{\partial r}+\frac{\rho v_r}{r}+\frac{\partial(\rho v_t)}{r\partial\theta}=0 \qquad (4.12)$$

For incompressible fluid

$$\boxed{\frac{v_r}{r}+\frac{\partial v_r}{\partial r}+\frac{\partial v_t}{r\partial\theta}=0} \qquad (4.13)$$

4.3 Continuity Equation for Two-Dimensional Steady Flow

[IP 4.4] p. 117

A mixture of ethanol and gasoline, called "gasohol," is created by pumping the two liquids into the "wye" pipe junction. Find Q_{eth} and V_{eth}

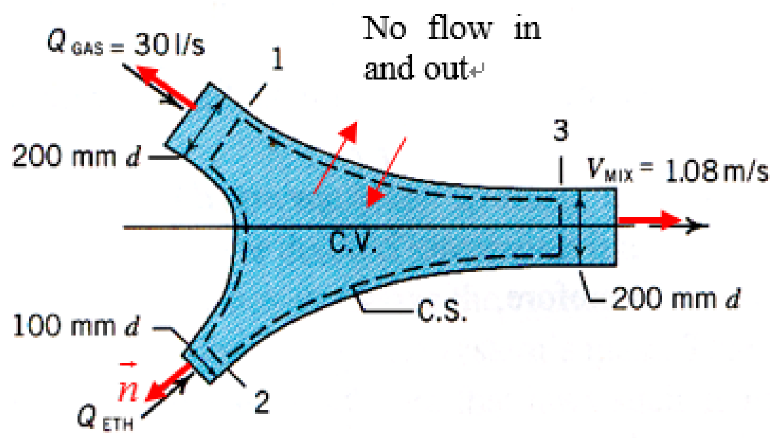
$$\rho_{mix} = 691.1\ \mathrm{kg/m^3}$$

$$V_{mix} = 1.08\ \mathrm{m/s}$$

$$Q_{gas} = 30\ l/s = 30 \times 10^{-3}\ \mathrm{m^3/s}$$

$$\rho_{gas} = 680.3\ \mathrm{kg/m^3}$$

$$\rho_{eth} = 788.6\ \mathrm{kg/m^3}$$



4.3 Continuity Equation for Two-Dimensional Steady Flow

[Sol]

$$A_1 = \frac{\pi}{4}(0.2)^2 = 0.031\text{m}^2 \quad A_2 = 0.0079\,\text{m}^2 \quad A_3 = 0.031\text{m}^2$$

$$V_1 = 30\times10^{-3} / 0.031 = 0.97 \text{ m/s}$$

$$\boxed{\int_1 \rho\vec{v}\cdot\vec{n}\,dA + \int_2 \rho\vec{v}\cdot\vec{n}\,dA + \int_3 \rho\vec{v}\cdot\vec{n}\,dA = 0}$$

$$\int_1 \rho\vec{v}\cdot\vec{n}\,dA = -680.3\times0.97\times0.031 = -20.4 \text{ kg/s}$$

$$\int_2 \rho\vec{v}\cdot\vec{n}\,dA = -788.6\times V_2\times0.0079 = -6.23\,V_2$$

$$\int_3 \rho\vec{v}\cdot\vec{n}\,dA = 691.1\times1.08\times0.031 = 23.1 \text{ kg/s}$$

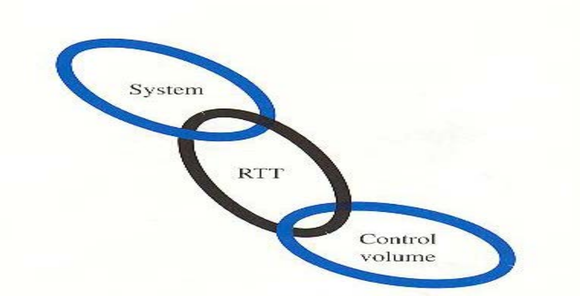
$$\therefore \quad \oint_{c.s} \rho\vec{v}\cdot\vec{n}dA = -20.4 - 6.23\,V_2 + 23.1 = 0$$

$$V_2 = 0.43 \text{ m/s}$$

$$\rightarrow \quad Q_{eth} = V_2\,A_2 = (0.43)(0.0079) = 3.4\times10^{-3}\ \text{m}^3/\text{s} = 3.4\ l/\text{s}$$

4.4 The Reynolds Transport Theorem

- **Reynolds Transport Theorem (RTT; 레이놀즈 수송정리)**
  - A general relationship that converts the laws such as mass conservation and Newton's 2nd law from the system (Lagrangian approach) to the control volume (Eulerian approach)
  - Most principles of fluid mechanics are adopted from solid mechanics, where the physical laws dealing with the time rates of change of extensive properties are expressed for systems.
  - There is a need to relate the changes in a control volume to the changes in a system.



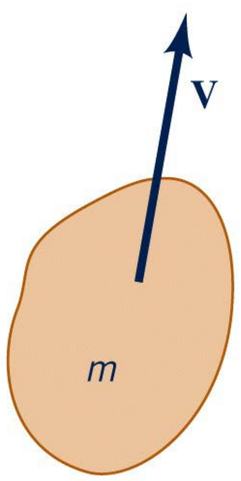
4.4 The Reynolds Transport Theorem

- Consider two types of properties

Extensive properties (종량상태량) (E): total system mass, momentum, energy

Intensive properties (강성상태량) (i): mass, momentum, energy <u>per unit mass</u>

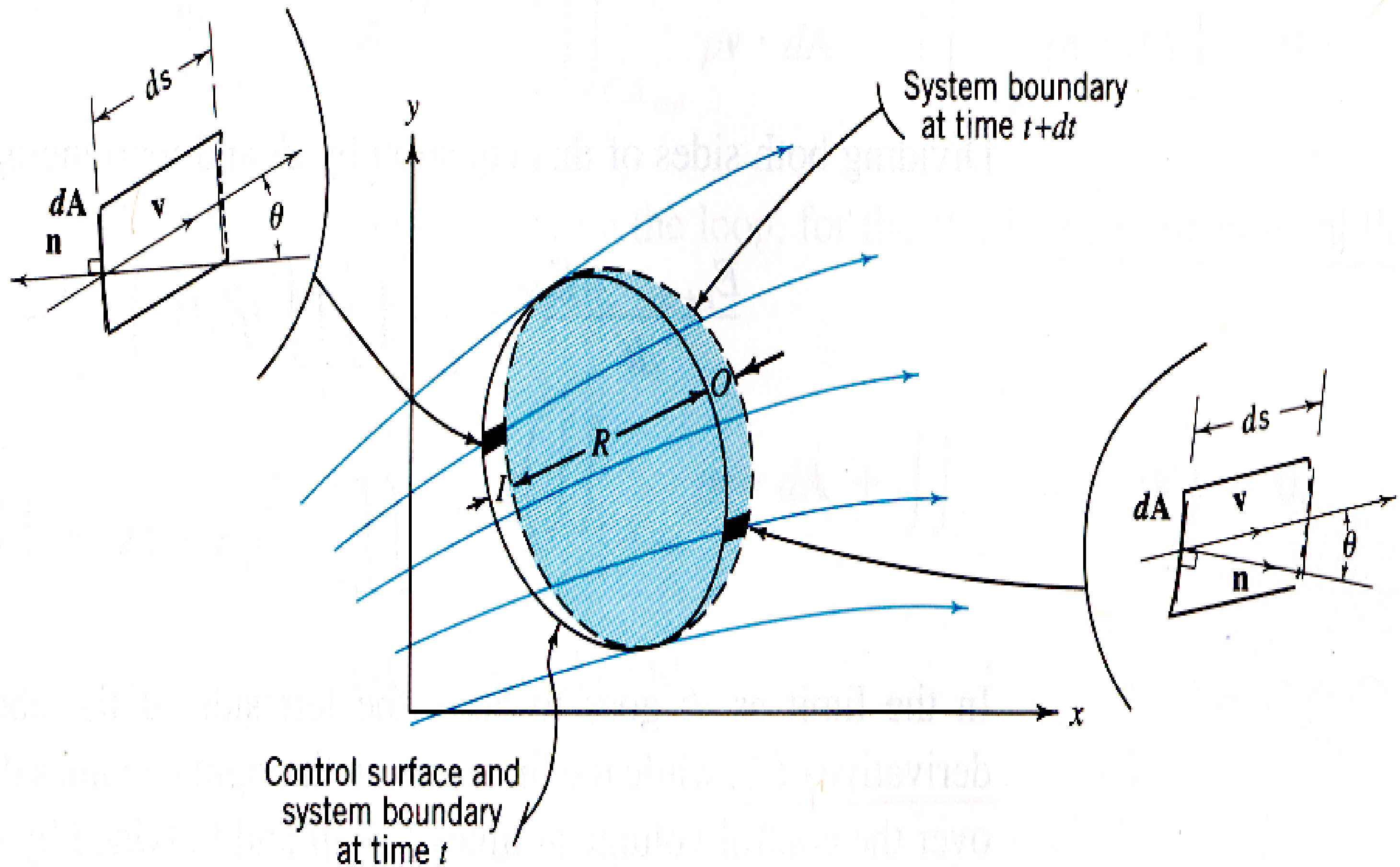
$\underline{E}$	$\underline{i}$
system mass, m	1
system momentum, $m\vec{v}$	$\vec{v}$
system energy, $m(\vec{v})^2$	$(\vec{v})^2$



$$E = \iiint_{system} i \, dm = \iiint_{system} i\rho \, dvol \qquad (4.14)$$

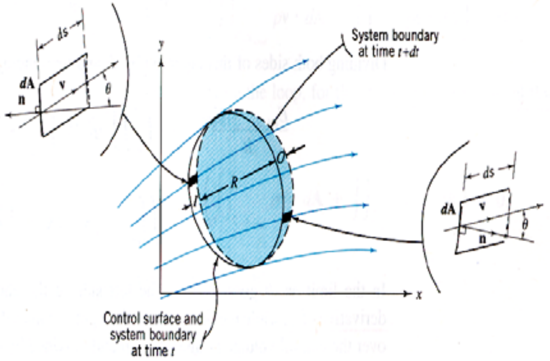
4.4 The Reynolds Transport Theorem

- Derivation of RTT



4.4 The Reynolds Transport Theorem

Consider time rate of change of a system property, *E*



$$E_{t+dt} - E_t = (E_R + E_0)_{t+dt} - (E_R + E_I)_t \qquad \text{(a)}$$

$$(E_0)_{t+dt} = dt \iint_{c.s.out} i\rho \vec{v} \cdot \overrightarrow{dA} \qquad \text{(b.1)}$$

$$(E_I)_t = dt\left(-\iint_{c.s.in} i\rho \vec{v} \cdot \overrightarrow{dA} \right) \qquad \text{(b.2)}$$

$$(E_R)_{t+dt} = \left(\iiint_R i\rho \, dvol \right)_{t+dt} \qquad \text{(b.3)}$$

$$(E_R)_t = \left(\iiint_R i\rho \, dvol \right)_t \qquad \text{(b.4)}$$

4.4 The Reynolds Transport Theorem

Substitute (b) into (a) and divide by dt

$$\therefore \frac{E_{t+dt}-E_t}{dt}=\frac{1}{dt}\left\{\left(\iiint_R i\rho\, dvol\right)_{t+dt}-\left(\iiint_R i\rho\, dvol\right)_t\right\}$$

$$+\iint_{c.s.out} i\rho\,\vec{v}\cdot\overrightarrow{dA}+\iint_{c.s.in} i\rho\,\vec{v}\cdot\overrightarrow{dA}$$

$$\boxed{\frac{dE}{dt}=\frac{d}{dt}\left(\iiint_{sys} i\rho\, dvol\right)=\frac{\partial}{\partial t}\left(\iiint_{c.v.} i\rho\, dvol\right)+\oint\!\!\oint_{c.s.} i\rho\,\vec{v}\cdot\overrightarrow{dA}} \qquad (4.15)$$

time rate of change of E in the system

time rate change within the control volume
→ unsteady term

fluxes of E across the control surface
→ advective term

4.4 The Reynolds Transport Theorem

- Application of RTT to conservation of mass

For application of RTT to the conservation of mass,

in Eq. (4.15), $E = m, \quad i = 1$ and $\dfrac{dm}{dt} = 0$ because mass is conserved.

$$\frac{dm}{dt} = 0 = \frac{\partial}{\partial t}\left(\iiint_{c.v.} \rho \, dvol\right) + \oint\oint_{c.s.} \rho \vec{v} \cdot \overrightarrow{dA}$$

$$\therefore \frac{\partial}{\partial t}\left(\iiint_{c.v.} \rho \, dvol\right) = -\oint\oint_{c.s.} \rho \vec{v} \cdot \overrightarrow{dA} = -\left(\iint_{c.s.out} \rho \vec{v} \cdot \overrightarrow{dA} + \iint_{c.s.in} \rho \vec{v} \cdot \overrightarrow{dA}\right) \quad (4.16)$$

Unsteady flow: mass within the control volume may change if the density changes

4.4 The Reynolds Transport Theorem

For flow of uniform density or steady flow, (4.16) becomes

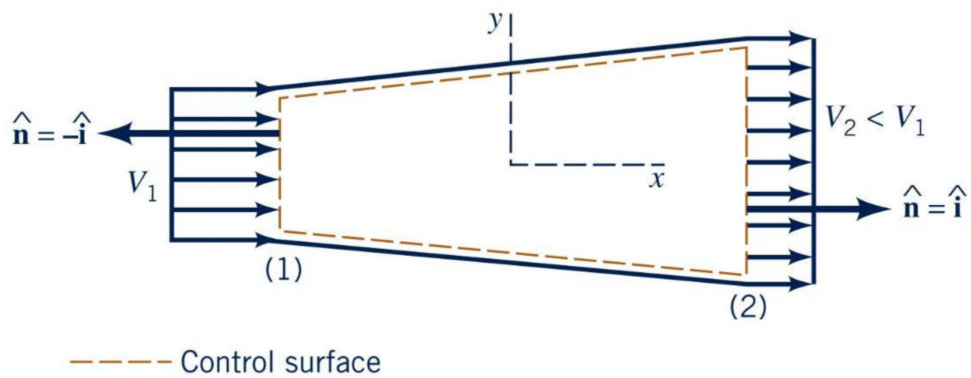
$$\iint_{c.s.out} \rho \vec{v} \cdot \overrightarrow{dA} + \iint_{c.s.in} \rho \vec{v} \cdot \overrightarrow{dA} = 0$$ ~ same as Eq. (4.9)

For one-dimensional flow

$$\iint_{c.s.out} \rho \vec{v} \cdot \overrightarrow{dA} = \rho_2 V_2 A_2$$

$$\iint_{c.s.in} \rho \vec{v} \cdot \overrightarrow{dA} = -\rho_1 V_1 A_1$$

$$\therefore \rho_1 A_1 V_1 = \rho_2 A_2 V_2 \qquad (4.1)$$



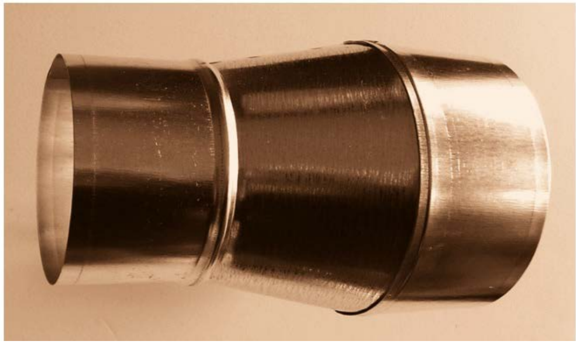
• In Ch. 5 & 6, RTT is also used to derive the work-energy, impulse-momentum, and moment of momentum principles.

4.4 The Reynolds Transport Theorem

- Consider system of flow through a variable area pipe

At time t: System = CV

At time $t+\delta t$: System=CV-I+II



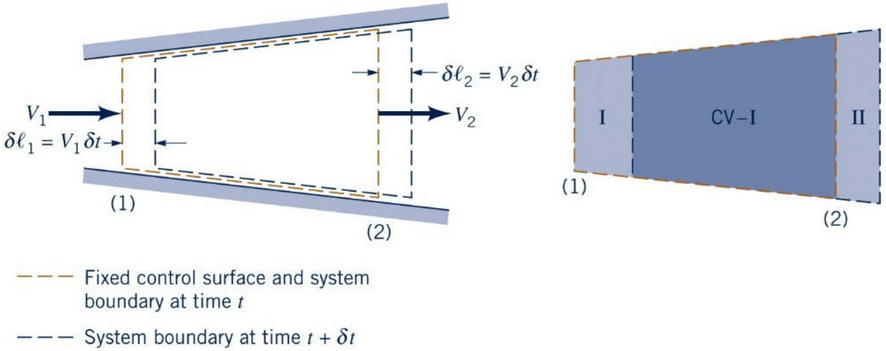
$$E_{sys}(t) = E_{CV}(t)$$

$$E_{sys}(t+dt) = E_{CV}(t+dt) - E_I(t+dt) + E_{II}(t+dt)$$

$$\frac{\delta E_{sys}}{\delta t} = \frac{E_{sys}(t+dt) - E_{sys}(t)}{\delta t}$$

$$= \frac{[E_{CV}(t+dt) + E_{II}(t+dt)] - [E_{CV}(t) + E_I(t+dt)]}{\delta t}$$

$$= \frac{[E_{CV}(t+dt) - E_{CV}(t)] + [E_{II}(t+dt) - E_I(t+dt)]}{\delta t}$$



4.4 The Reynolds Transport Theorem

In the limit $\delta t \rightarrow 0$

$$\lim_{\delta t \to 0} \frac{E_{CV}(t+dt) - E_{CV}(t)}{\delta t} = \frac{\partial E_{CV}}{\partial t}$$

$$\lim_{\delta t \to 0} \frac{E_{II}(t+dt)}{\delta t} = \dot{E}_{out} = \rho_2 A_2 V_2$$

$$\lim_{\delta t \to 0} \frac{E_{I}(t+dt)}{\delta t} = \dot{E}_{in} = \rho_1 A_1 V_1$$

$$\boxed{\frac{\delta E_{SYS}}{\delta t} = \frac{\partial E_{CV}}{\partial t} + \dot{E}_{out} - \dot{E}_{in}}$$

- 레이놀즈 수송정리: 시스템에서 *E*의 시간변화율은 검사체적내에서의 *E*의 시간변화율과 (유출량-유입량)의 합과 같다.

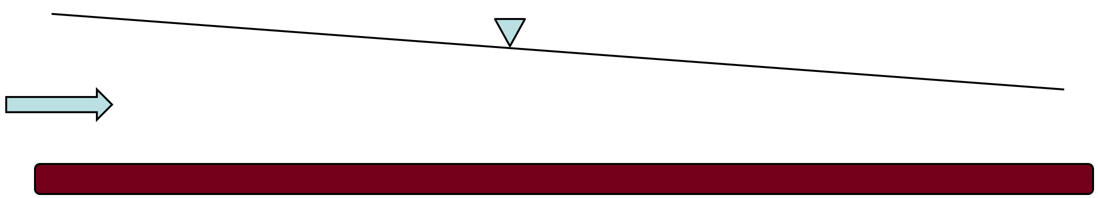
Homework Assignment # 4

Homework Assignment # 4

Due: 1 week from today

1. (Prob. 4.9)

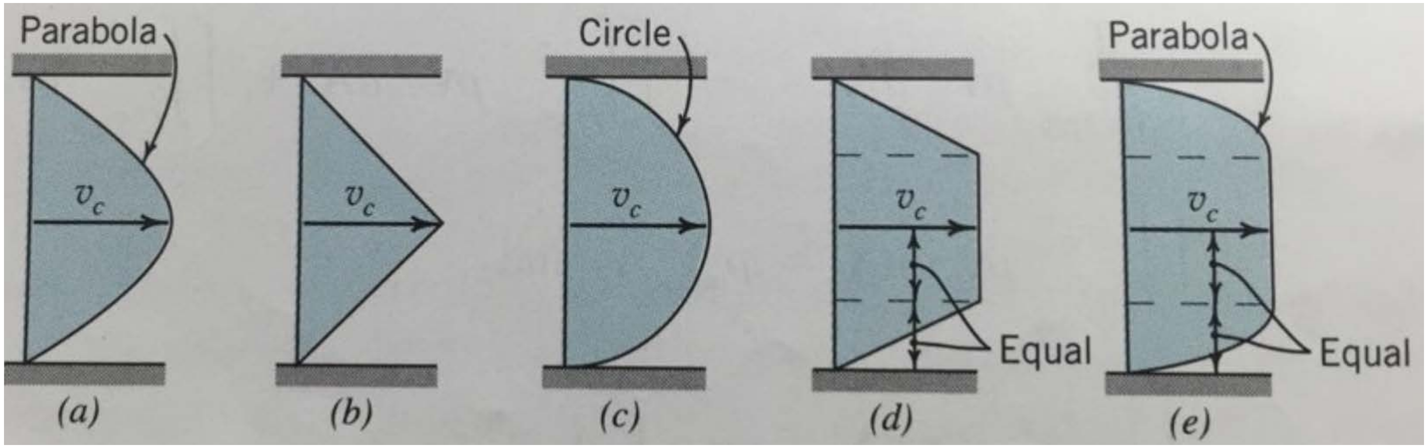
At a point in a two-dimensional fluid flow, two streamlines are parallel and 75 mm apart. At another point these streamlines are parallel but only 25 mm apart. If the velocity at the first point is 3 m/s, calculate the velocity at the second.



Homework Assignment # 4

2. (Prob. 4.12)

Calculate the mean velocities for these two-dimensional velocity profiles if $\upsilon_c = 3\ m/s$.



Homework Assignment # 4

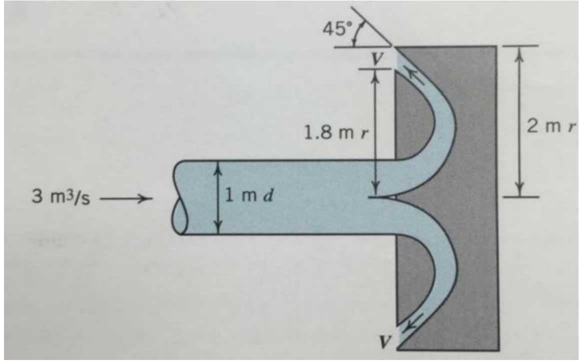
3. (Prob. 4.14)

If the velocity profile in a passage of width $2R$ is given by the equation $\upsilon / \upsilon_c = (y / R)^{1/n}$, derive an expression for V / υ_c in terms of n: (a) for a two-dimensional passage, and (b) for a cylindrical passage.

Homework Assignment # 4

4. (Prob. 4.20)

Find V for this mushroom cap (shower head) on a pipeline.



5. (Prob. 4.31)

For a <u>differential control volume</u>, show that Eq. 4.16 reduces to Eq. 4.11 for a steady flow with a uniform constant density.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$