

Introduction to Electromagnetism

Vector Analysis

(2-1, 2-2, 2-3)

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Introduction

Quantities in electromagnetics (from a mathematical viewpoint)?

- **Scalar:** Completely specified by its **magnitude** (positive or negative, together with its unit)
- **Vector:** Required both a **magnitude** and a **direction** to specify

How do we specify the direction of a vector?

- In a three-dimensional space, three numbers are needed.
- These numbers depend on the choice of a coordinate system:
e.g.) **Cartesian coordinates, cylindrical coordinates, spherical coordinates, etc.**

However, physical laws and theorems certainly must hold irrespective of the coordinate system:

- The general expressions of the laws of electromagnetism do not require the specification of a coordinate system
- A particular coordinate system is chosen only when a problem of a given geometry is analyzed.

What we are going to learn on vector analysis

1. Vector algebra:

Addition, subtraction and multiplication of vectors

2. Orthogonal coordinate systems:

Cartesian, cylindrical, and spherical coordinates

3. Vector calculus:

Differentiation and integration of vectors;

Line, surface, and volume integrals;

“del” operator;

Gradient, divergence, and curl operations

A deficiency in vector analysis in the study of electromagnetics is similar to a deficiency in algebra and calculus in the study of physics.

“It is obvious that these deficiencies cannot yield fruitful outcomes!”

Vector: Magnitude and Direction

Magnitude and direction of a vector \mathbf{A} :

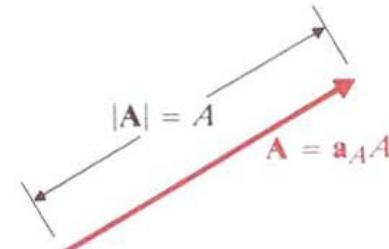
$$\mathbf{A} = \mathbf{a}_A A$$

Magnitude: $A = |\mathbf{A}|$

Dimensionless unit vector: $\mathbf{a}_A = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{A}}{A}$

Equality:

Two vectors are equal if they have the same **magnitude** and the same **direction**.



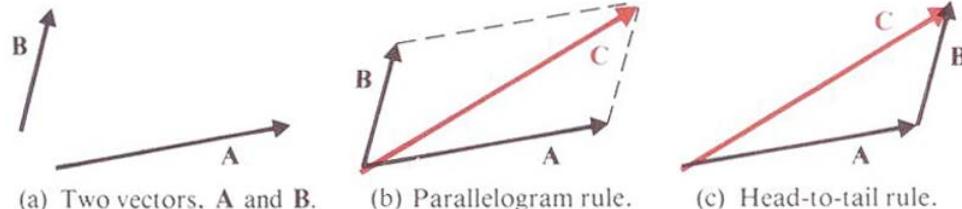
Graphical representation of vector \mathbf{A}

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Vector Addition and Subtraction

Vector addition: $\mathbf{C} = \mathbf{A} + \mathbf{B}$

1. By the parallelogram rule
2. By the head-to-tail rule



Rules for vector addition:

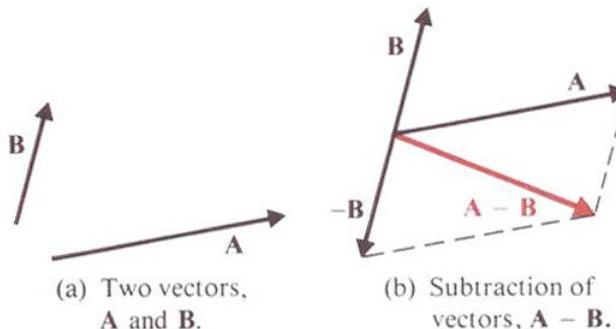
Commutative law: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

Associative law: $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$

Vector addition

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Vector subtraction: $\mathbf{C} = \mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$



$-\mathbf{B}$: Negative of vector \mathbf{B}

Having the same magnitude but with an opposite direction w.r.t. \mathbf{B}

$$-\mathbf{B} = (-\mathbf{a}_B)\mathbf{B}$$

Vector subtraction

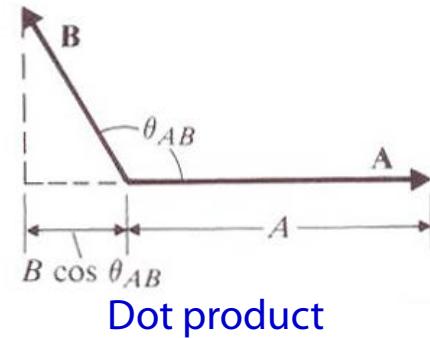
D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Scalar multiplication: $k\mathbf{A} = (kA)\mathbf{a}_A$

Scalar or Dot Product

Scalar or dot product of two vectors \mathbf{A} and \mathbf{B} :

$$\mathbf{A} \cdot \mathbf{B} \equiv AB \cos \theta_{AB}$$



where θ_{AB} is the smaller angle between \mathbf{A} and \mathbf{B}

- Note:
1. Magnitude: Less than or equal to the product of their magnitudes
 2. Either a positive or a negative quantity, depending on θ_{AB}
 3. Equal to the product of the magnitude of one vector and the **projection** of the other upon the first one
 4. **Zero** when the vectors are perpendicular to each other

Rules for dot product:

$$\mathbf{A} \cdot \mathbf{A} = A^2 \quad \text{or} \quad A = \sqrt{\mathbf{A} \cdot \mathbf{A}}$$

Commutative law: $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$

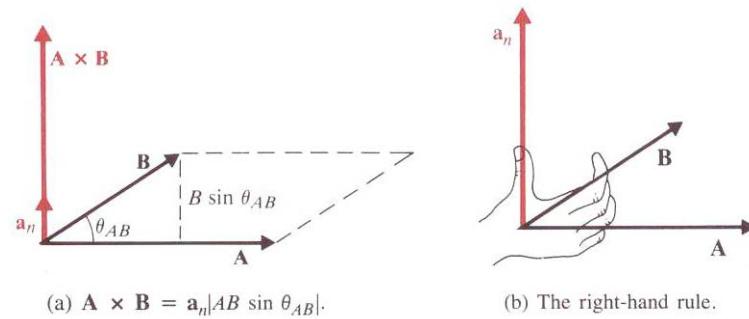
Distributive law: $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$ **Proof? HW**

Vector or Cross Product

Vector or cross product of two vectors \mathbf{A} and \mathbf{B} :

$$\mathbf{A} \times \mathbf{B} \equiv |AB \sin \theta_{AB}| \mathbf{a}_n$$

where \mathbf{a}_n is perpendicular to the plane containing \mathbf{A} and \mathbf{B} ;
its direction follows that of the thumb of the right hand when
the fingers rotate from \mathbf{A} to \mathbf{B} through the angle θ_{AB}
(the right-hand rule)



(a) $\mathbf{A} \times \mathbf{B} = \mathbf{a}_n |AB \sin \theta_{AB}|$.

(b) The right-hand rule.

Cross product

Rules for cross product:

Not commutative: $\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}$

Distributive law: $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$ Proof? HW

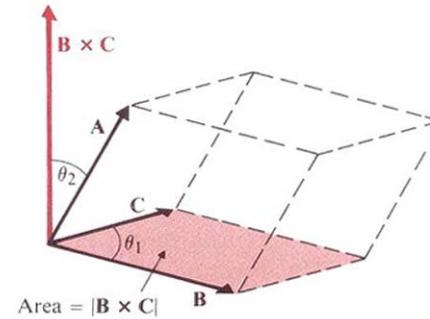
Not associative: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$

Product of Three Vectors

Scalar triple product:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

Cyclic permutation



Scalar triple product

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Vector triple product:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

"back-cab" rule

Vector Triple Product

Proof:

Let $\mathbf{A} = \mathbf{A}_{\parallel} + \mathbf{A}_{\perp}$ (w.r.t. $\mathbf{B} - \mathbf{C}$ plane)

$$\begin{aligned}\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (\mathbf{A}_{\parallel} + \mathbf{A}_{\perp}) \times (\mathbf{B} \times \mathbf{C}) \\ &= \mathbf{A}_{\parallel} \times (\mathbf{B} \times \mathbf{C}) \equiv \mathbf{D}\end{aligned}$$

$$D = \underline{\mathbf{D} \cdot \mathbf{a}_D} = A_{\parallel} BC \sin(\theta_1 - \theta_2)$$

$$= A_{\parallel} BC (\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2)$$

$$= (B \sin \theta_1)(A_{\parallel} C \cos \theta_2) - (C \sin \theta_2)(A_{\parallel} B \cos \theta_1)$$

$$= \underline{[\mathbf{B}(\mathbf{A}_{\parallel} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A}_{\parallel} \cdot \mathbf{B})] \cdot \mathbf{a}_D ?}$$

$$\text{Let } \mathbf{B}(\mathbf{A}_{\parallel} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A}_{\parallel} \cdot \mathbf{B}) = \mathbf{D} + k\mathbf{A}_{\parallel}$$

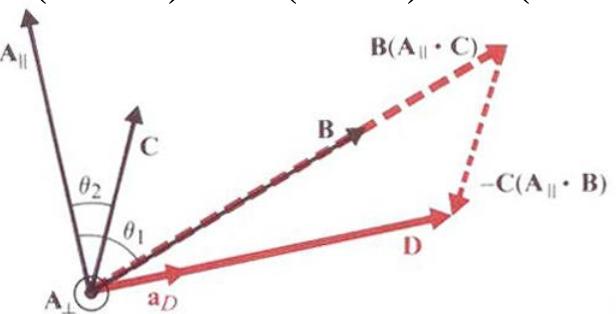
$$\rightarrow \mathbf{A}_{\parallel} \cdot [\mathbf{B}(\mathbf{A}_{\parallel} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A}_{\parallel} \cdot \mathbf{B})] = \mathbf{A}_{\parallel} \cdot (\mathbf{D} + k\mathbf{A}_{\parallel})$$

$$\rightarrow (\mathbf{A}_{\parallel} \cdot \mathbf{B})(\mathbf{A}_{\parallel} \cdot \mathbf{C}) - (\mathbf{A}_{\parallel} \cdot \mathbf{C})(\mathbf{A}_{\parallel} \cdot \mathbf{B}) = 0 = \cancel{\mathbf{A}_{\parallel} \cdot \mathbf{D}} + k\mathbf{A}_{\parallel}^2$$

$$\leftarrow \mathbf{A}_{\parallel} \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C}, \quad \mathbf{A}_{\parallel} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{B} \quad \rightarrow k = 0$$

$$\therefore \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$



Vector triple product

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Note that $\mathbf{A}_{\parallel}, \mathbf{B}$ & \mathbf{C} are on the same plane.