Introduction to Electromagnetism
Vector Analysis
(2-1, 2-2, 2-3)

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Introduction

Quantities in electromagnetics (from a mathematical viewpoint)?

- Scalar: Completely specified by its magnitude (positive or negative, together with its unit)
- Vector: Required both a magnitude and a direction to specify

How do we specify the direction of a vector?

- In a three-dimensional space, three numbers are needed.
- These numbers depend on the choice of a coordinate system:

  e.g.) Cartesian coordinates, cylindrical coordinates, spherical coordinates, etc.

However, physical laws and theorems certainly must hold irrespective of the coordinate system:

- The general expressions of the laws of electromagnetism do not require the specification of a coordinate system
- A particular coordinate system is chosen only when a problem of a given geometry is analyzed.

What we are going to learn on vector analysis

1. Vector algebra:

   Addition, subtraction and multiplication of vectors

2. Orthogonal coordinate systems:

   Cartesian, cylindrical, and spherical coordinates

3. Vector calculus:

   Differentiation and integration of vectors;
   Line, surface, and volume integrals;
   "del" operator;
   Gradient, divergence, and curl operations

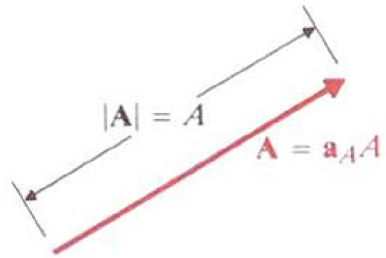
A deficiency in vector analysis in the study of electromagnetics is similar to a deficiency in algebra and calculus in the study of physics.

"It is obvious that these deficiencies cannot yield fruitful outcomes!"

Vector: Magnitude and Direction

Magnitude and direction of a vector **A**:

$$\mathbf{A} = \mathbf{a}_A A$$



Graphical representation of vector **A**

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Magnitude: $A = |\mathbf{A}|$

Dimensionless unit vector: $\mathbf{a}_A = \dfrac{\mathbf{A}}{|\mathbf{A}|} = \dfrac{\mathbf{A}}{A}$

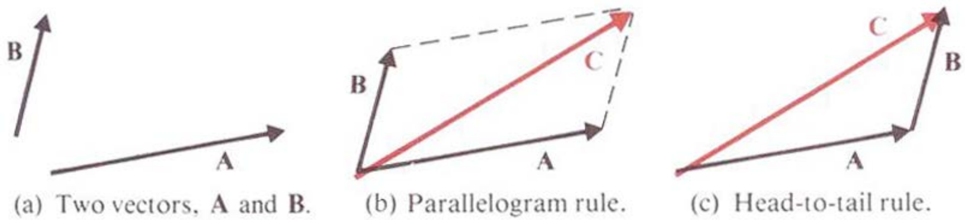
Equality:

Two vectors are equal if they have the same magnitude and the same direction.

Vector Addition and Subtraction

Vector addition: $\mathbf{C} = \mathbf{A} + \mathbf{B}$

1. By the parallelogram rule
2. By the head-to-tail rule



Vector addition

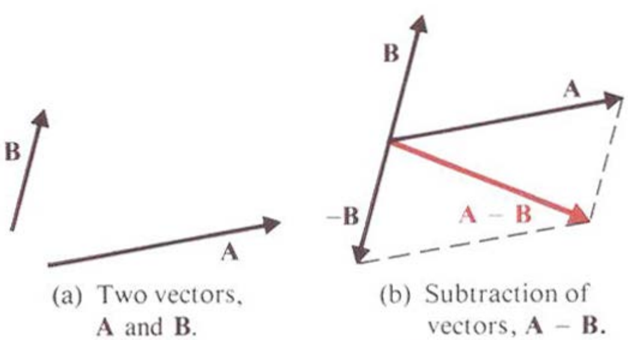
D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Rules for vector addition:

Commutative law: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

Associative law: $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$

Vector subtraction: $\mathbf{C} = \mathbf{A} - \mathbf{B} \;= \mathbf{A} + (-\mathbf{B})$



Vector subtraction

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

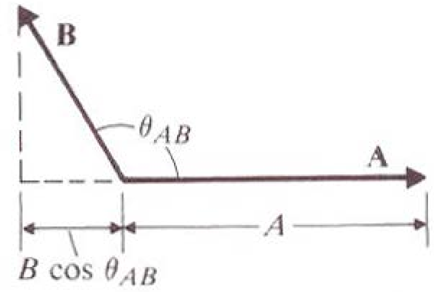
$-\mathbf{B}$: Negative of vector $\mathbf{B}$

Having the same magnitude but with an opposite direction w.r.t. $\mathbf{B}$

$-\mathbf{B} = (-\mathbf{a}_B)B$

Scalar multiplication: $k\mathbf{A} = (kA)\mathbf{a}_A$

Scalar or Dot Product



Dot product

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Scalar or dot product of two vectors $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{A}\cdot\mathbf{B} \equiv AB\cos\theta_{AB}$$

where θ_{AB} is the smaller angle between $\mathbf{A}$ and $\mathbf{B}$

Note: 1. Magnitude: Less than or equal to the product of their magnitudes

2. Either a positive or a negative quantity, depending on θ_{AB}

3. Equal to the product of the magnitude of one vector and the projection of the other upon the first one

4. Zero when the vectors are perpendicular to each other

Rules for dot product:

$$\mathbf{A}\cdot\mathbf{A} = A^2 \quad \text{or} \quad A = \sqrt{\mathbf{A}\cdot\mathbf{A}}$$

Commutative law: $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$

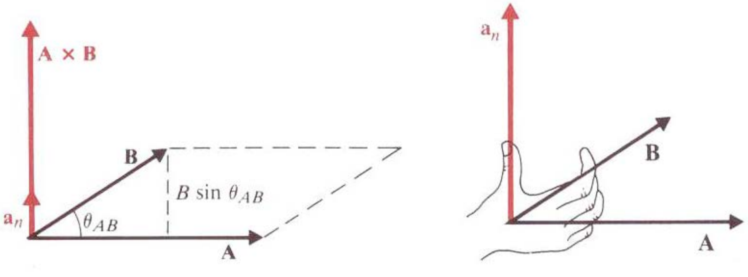
Distributive law: $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$ Proof? HW

Vector or Cross Product

Vector or cross product of two vectors $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{A} \times \mathbf{B} \equiv \left| AB \sin\theta_{AB} \right| \mathbf{a}_n$$

where $\mathbf{a}_n$ is perpendicular to the plane containing $\mathbf{A}$ and $\mathbf{B}$; its direction follows that of the thumb of the right hand when the fingers rotate from $\mathbf{A}$ to $\mathbf{B}$ through the angle θ_{AB}
(the right-hand rule)



(a) $\mathbf{A} \times \mathbf{B} = \mathbf{a}_n |AB \sin\theta_{AB}|$. (b) The right-hand rule.

Cross product

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Rules for cross product:

Not commutative: $\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}$

Distributive law: $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$ Proof? HW

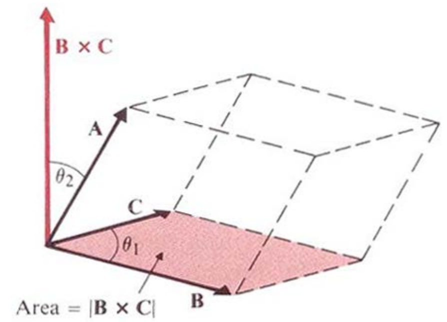
Not associative: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$

Product of Three Vectors

Scalar triple product:

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})=\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

Cyclic permutation



Scalar triple product

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

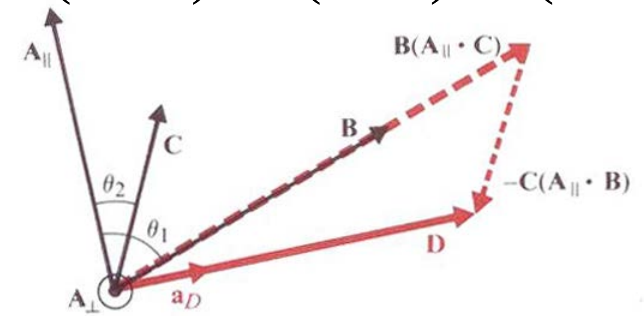
Vector triple product:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

“back-cab” rule

Vector Triple Product

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$



Vector triple product

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Proof:

Let $\mathbf{A}=\mathbf{A}_{//}+\mathbf{A}_{\perp}$ (w.r.t. $\mathbf{B}-\mathbf{C}$ plane)

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}_{//}+\cancel{\mathbf{A}_{\perp}})\times(\mathbf{B}\times\mathbf{C})$$

$$=\mathbf{A}_{//}\times(\mathbf{B}\times\mathbf{C})\ \equiv\mathbf{D}$$

Note that $\mathbf{A}_{//},\mathbf{B}\,\&\,\mathbf{C}$ are on the same plane.

$$D=\underline{\mathbf{D}\cdot\mathbf{a}_D}=A_{//}BC\sin(\theta_1-\theta_2)$$

$$=A_{//}BC(\sin\theta_1\cos\theta_2-\cos\theta_1\sin\theta_2)$$

$$=(B\sin\theta_1)(A_{//}C\cos\theta_2)-(C\sin\theta_2)(A_{//}B\cos\theta_1)$$

$$=\underline{[\mathbf{B}(\mathbf{A}_{//}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}_{//}\cdot\mathbf{B})]\cdot\mathbf{a}_D}\ ?$$

Let $\mathbf{B}(\mathbf{A}_{//}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}_{//}\cdot\mathbf{B})=\mathbf{D}+k\mathbf{A}_{//}$

$$\rightarrow\mathbf{A}_{//}\cdot[\mathbf{B}(\mathbf{A}_{//}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}_{//}\cdot\mathbf{B})]=\mathbf{A}_{//}\cdot(\mathbf{D}+k\mathbf{A}_{//})$$

$$\rightarrow(\mathbf{A}_{//}\cdot\mathbf{B})(\mathbf{A}_{//}\cdot\mathbf{C})-(\mathbf{A}_{//}\cdot\mathbf{C})(\mathbf{A}_{//}\cdot\mathbf{B})=\cancel{\mathbf{A}_{//}\cdot\mathbf{D}}+kA_{//}^2=0$$

$$\leftarrow\ \mathbf{A}_{//}\cdot\mathbf{C}=\mathbf{A}\cdot\mathbf{C},\ \ \mathbf{A}_{//}\cdot\mathbf{B}=\mathbf{A}\cdot\mathbf{B}\qquad\rightarrow k=0$$

$$\boxed{\therefore\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})}$$