Introduction to Electromagnetism
Orthogonal Coordinate Systems
(2-4, 2-5)

Yoonchan Jeong
School of Electrical Engineering, Seoul National University
Tel: +82 (0)2 880 1623, Fax: +82 (0)2 873 9953
Email: yoonchan@snu.ac.kr

Orthogonal Coordinate systems (1)

How many surfaces are required for determining the location of a point in a three-dimensional space?

A point can be defined as the intersection of three surfaces.

Orthogonal coordinate system:

If these three surfaces are mutually perpendicular to one another

Base vectors:

The unit vectors perpendicular to the three surfaces

Right-handed system

$$\rightarrow \begin{cases} \mathbf{a}_{u_1} \times \mathbf{a}_{u_2} = \mathbf{a}_{u_3} \\ \mathbf{a}_{u_2} \times \mathbf{a}_{u_3} = \mathbf{a}_{u_1} \\ \mathbf{a}_{u_3} \times \mathbf{a}_{u_1} = \mathbf{a}_{u_2} \end{cases}$$

$$\rightarrow \mathbf{a}_{u_1} \cdot \mathbf{a}_{u_2} = \mathbf{a}_{u_2} \cdot \mathbf{a}_{u_3} = \mathbf{a}_{u_3} \cdot \mathbf{a}_{u_1} = 0$$

$$\rightarrow \mathbf{a}_{u_1} \cdot \mathbf{a}_{u_1} = \mathbf{a}_{u_2} \cdot \mathbf{a}_{u_2} = \mathbf{a}_{u_3} \cdot \mathbf{a}_{u_3} = 1$$

Vector representation:

$$\rightarrow \mathbf{A} = \mathbf{a}_{u_1} A_{u_1} + \mathbf{a}_{u_2} A_{u_2} + \mathbf{a}_{u_3} A_{u_3}$$

$$\rightarrow A = |\mathbf{A}| = (A_{u_1}^2 + A_{u_2}^2 + A_{u_3}^2)^{1/2}$$

Orthogonal Coordinate systems (2)

Example 2-4: $$\mathbf{A} = \mathbf{a}_{u_1} A_{u_1} + \mathbf{a}_{u_2} A_{u_2} + \mathbf{a}_{u_3} A_{u_3},$$

$$\mathbf{B} = \mathbf{a}_{u_1} B_{u_1} + \mathbf{a}_{u_2} B_{u_2} + \mathbf{a}_{u_3} B_{u_3}.$$

Recall:

$$\rightarrow \begin{cases} \mathbf{a}_{u_1} \times \mathbf{a}_{u_2} = \mathbf{a}_{u_3} \\ \mathbf{a}_{u_2} \times \mathbf{a}_{u_3} = \mathbf{a}_{u_1} \\ \mathbf{a}_{u_3} \times \mathbf{a}_{u_1} = \mathbf{a}_{u_2} \end{cases} \quad \begin{array}{l} \rightarrow \mathbf{a}_{u_1} \cdot \mathbf{a}_{u_2} = \mathbf{a}_{u_2} \cdot \mathbf{a}_{u_3} = \mathbf{a}_{u_3} \cdot \mathbf{a}_{u_1} = 0 \\ \rightarrow \mathbf{a}_{u_1} \cdot \mathbf{a}_{u_1} = \mathbf{a}_{u_2} \cdot \mathbf{a}_{u_2} = \mathbf{a}_{u_3} \cdot \mathbf{a}_{u_3} = 1 \end{array}$$

Scalar product:

$$\mathbf{A} \cdot \mathbf{B} = (\mathbf{a}_{u_1} A_{u_1} + \mathbf{a}_{u_2} A_{u_2} + \mathbf{a}_{u_3} A_{u_3}) \cdot (\mathbf{a}_{u_1} B_{u_1} + \mathbf{a}_{u_2} B_{u_2} + \mathbf{a}_{u_3} B_{u_3})$$
$$= A_{u_1} B_{u_1} + A_{u_2} B_{u_2} + A_{u_3} B_{u_3}$$

Vector product:

$$\mathbf{A} \times \mathbf{B} = (\mathbf{a}_{u_1} A_{u_1} + \mathbf{a}_{u_2} A_{u_2} + \mathbf{a}_{u_3} A_{u_3}) \times (\mathbf{a}_{u_1} B_{u_1} + \mathbf{a}_{u_2} B_{u_2} + \mathbf{a}_{u_3} B_{u_3})$$
$$= \mathbf{a}_{u_1} (A_{u_2} B_{u_3} - A_{u_3} B_{u_2}) + \mathbf{a}_{u_2} (A_{u_3} B_{u_1} - A_{u_1} B_{u_3}) + \mathbf{a}_{u_3} (A_{u_1} B_{u_2} - A_{u_2} B_{u_1})$$
$$= \begin{vmatrix} \mathbf{a}_{u_1} & \mathbf{a}_{u_2} & \mathbf{a}_{u_3} \\ A_{u_1} & A_{u_2} & A_{u_3} \\ B_{u_1} & B_{u_2} & B_{u_3} \end{vmatrix}$$

Conversion factors

Some of the coordinates, say u_i (i = 1, 2, or 3), may not be a length:

A conversion factor is needed to convert a differential change du_i into a change in length dl_i:

$$dl_i = h_i du_i$$

where h_i is called a "metric coefficient"

Directed differential length change: $d\mathbf{l} = \mathbf{a}_{u_1} dl_1 + \mathbf{a}_{u_2} dl_2 + \mathbf{a}_{u_3} dl_3$

$$\rightarrow d\mathbf{l} = \mathbf{a}_{u_1}(h_1 du_1) + \mathbf{a}_{u_2}(h_2 du_2) + \mathbf{a}_{u_3}(h_3 du_3)$$

Differential volume change: $dv = dl_1 dl_2 dl_3$

$$\rightarrow dv = h_1 h_2 h_3 du_1 du_2 du_3$$

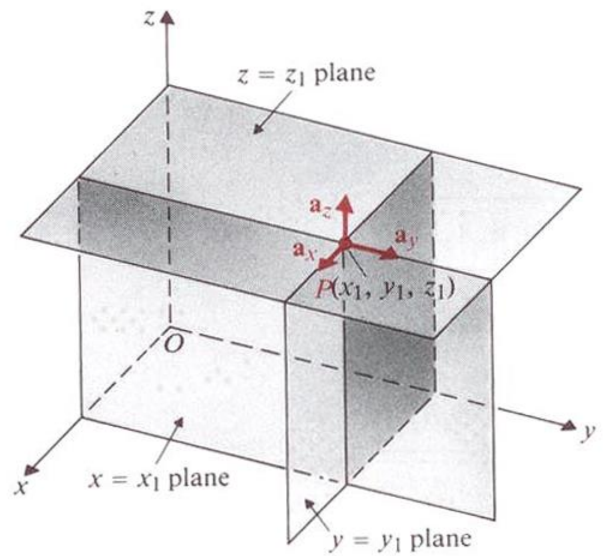
Differential area change: $d\mathbf{s} = \mathbf{a}_n ds \quad \rightarrow \; ds_1 = dl_2 dl_3$

$$\rightarrow ds_1 = h_2 h_3 du_2 du_3$$

$$\rightarrow ds_2 = h_1 h_3 du_1 du_3$$

$$\rightarrow ds_3 = h_1 h_2 du_1 du_2$$

Cartesian (or Rectangular) Coordinates (1)



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Notation: $(u_1, u_2, u_3) = (x, y, z)$

$$\rightarrow \begin{cases} \mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z \\ \mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x \\ \mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y \end{cases}$$

$$\rightarrow \overrightarrow{OP} = \mathbf{a}_x x_1 + \mathbf{a}_y y_1 + \mathbf{a}_z z_1$$

Vector representation:

$$\rightarrow \boxed{\mathbf{A} = \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z}$$

Dot product:

$$\rightarrow \boxed{\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_y B_y}$$

Cross product:

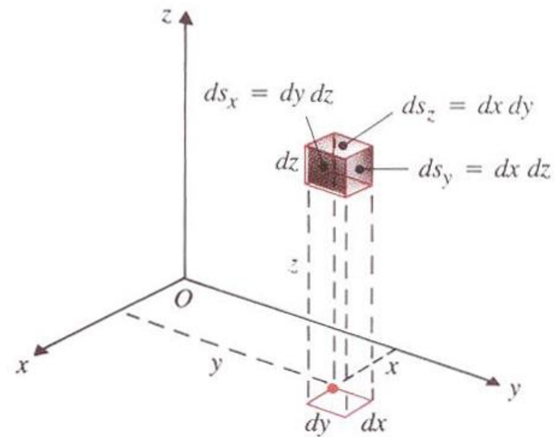
$$\rightarrow \mathbf{A} \times \mathbf{B} = \mathbf{a}_x (A_y B_z - A_z B_y) + \mathbf{a}_y (A_z B_x - A_x B_z) + \mathbf{a}_z (A_x B_y - A_y B_x)$$

$$= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Cartesian (or Rectangular) Coordinates (2)

Metric coefficients:

$$\rightarrow h_1 = h_2 = h_3 = 1$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Directed differential length change: $d\mathbf{l} = \mathbf{a}_x dx + \mathbf{a}_y dy + \mathbf{a}_z dz$

Differential area change:

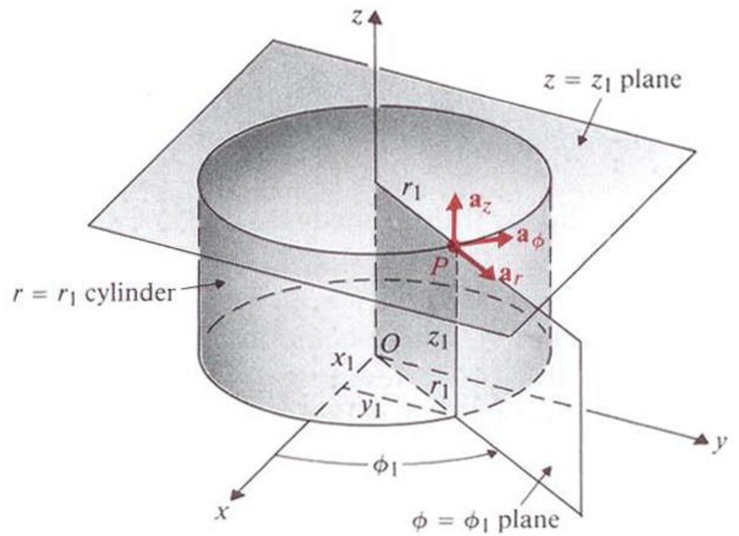
$$ds_x = dydz$$
$$ds_y = dxdz$$
$$ds_z = dxdy$$

Differential volume change: $dv = dxdydz$

Cylindrical Coordinates (1)

Notation: $(u_1, u_2, u_3) = (r, \phi, z)$

$$\rightarrow \begin{cases} \mathbf{a}_r \times \mathbf{a}_\phi = \mathbf{a}_z \\ \mathbf{a}_\phi \times \mathbf{a}_z = \mathbf{a}_r \\ \mathbf{a}_z \times \mathbf{a}_r = \mathbf{a}_\phi \end{cases}$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Vector representation:

$$\rightarrow \boxed{\mathbf{A} = \mathbf{a}_r A_r + \mathbf{a}_\phi A_\phi + \mathbf{a}_z A_z}$$

Dot product:

$$\rightarrow \mathbf{A} \cdot \mathbf{B} = A_r B_r + A_\phi B_\phi + A_z B_z$$

Cross product:

$$\rightarrow \mathbf{A} \times \mathbf{B} = \mathbf{a}_r (A_\phi B_z - A_z B_\phi) + \mathbf{a}_\phi (A_z B_r - A_r B_z) + \mathbf{a}_z (A_r B_\phi - A_\phi B_r)$$

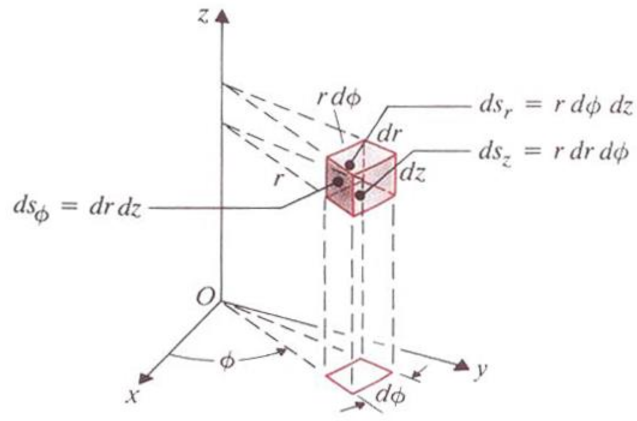
$$= \begin{vmatrix} \mathbf{a}_r & \mathbf{a}_\phi & \mathbf{a}_z \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$$

Cylindrical Coordinates (2)

Metric coefficients:

$$\rightarrow\ h_1 = h_3 = 1$$

$$\rightarrow\ h_2 = r$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Directed differential length change: $d\mathbf{l} = \mathbf{a}_r dr + \mathbf{a}_\phi r d\phi + \mathbf{a}_z dz$

Differential area change:

$$ds_r = r d\phi dz$$
$$ds_\phi = dr dz$$
$$ds_z = r dr d\phi$$

Differential volume change: $dv = r dr d\phi dz$

Cylindrical Coordinates (3)

Transformation from cylindrical coordinates to Cartesian coordinates:

$$\mathbf{A} = \mathbf{a}_r A_r + \mathbf{a}_\phi A_\phi + \mathbf{a}_z A_z$$

$$\rightarrow A_x = \mathbf{A} \cdot \mathbf{a}_x$$

$$= A_r \mathbf{a}_r \cdot \mathbf{a}_x + A_\phi \mathbf{a}_\phi \cdot \mathbf{a}_x$$

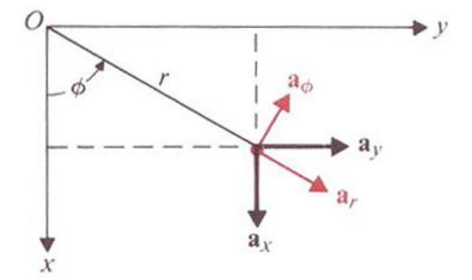
$$= A_r \cos\phi - A_\phi \sin\phi$$

$$\rightarrow A_y = \mathbf{A} \cdot \mathbf{a}_y$$

$$= A_r \mathbf{a}_r \cdot \mathbf{a}_y + A_\phi \mathbf{a}_\phi \cdot \mathbf{a}_y$$

$$= A_r \sin\phi + A_\phi \cos\phi$$

$$\rightarrow A_z = A_z$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

$$\rightarrow \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}$$

Conversion formulas:

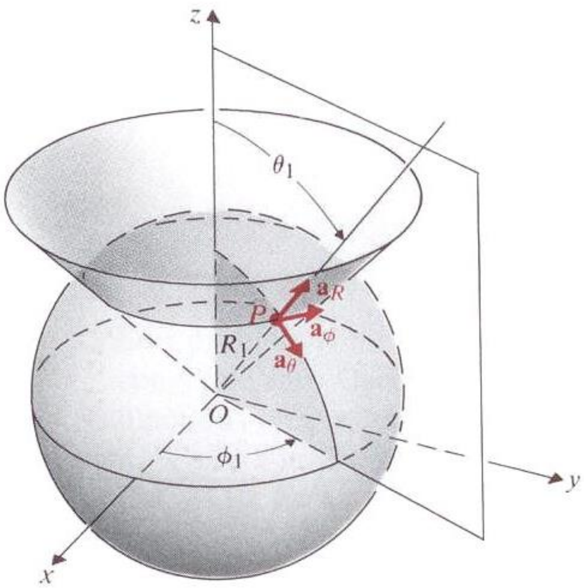
$$x = r\cos\phi \qquad y = r\sin\phi \qquad z = z$$

$$r = \sqrt{x^2 + y^2} \qquad \phi = \tan^{-1}\frac{y}{x} \qquad z = z$$

Spherical Coordinates (1)

Notation: $(u_1, u_2, u_3) = (R, \theta, \phi)$

$$\rightarrow \begin{cases} \mathbf{a}_R \times \mathbf{a}_\theta = \mathbf{a}_\phi \\ \mathbf{a}_\theta \times \mathbf{a}_\phi = \mathbf{a}_R \\ \mathbf{a}_\phi \times \mathbf{a}_R = \mathbf{a}_\theta \end{cases}$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Vector representation:

$$\rightarrow \boxed{\mathbf{A} = \mathbf{a}_R A_R + \mathbf{a}_\theta A_\theta + \mathbf{a}_\phi A_\phi}$$

Dot product:

$$\rightarrow \mathbf{A} \cdot \mathbf{B} = A_R B_R + A_\theta B_\theta + A_\phi B_\phi$$

Cross product:

$$\rightarrow \mathbf{A} \times \mathbf{B} = \mathbf{a}_R (A_\theta B_\phi - A_\phi B_\theta) + \mathbf{a}_\theta (A_\phi B_R - A_R B_\phi) + \mathbf{a}_\phi (A_R B_\theta - A_\theta B_R)$$

$$= \begin{vmatrix} \mathbf{a}_R & \mathbf{a}_\theta & \mathbf{a}_\phi \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$$

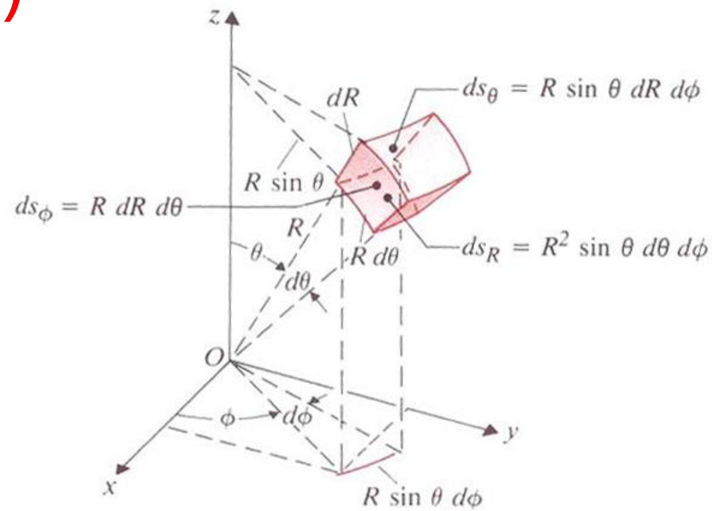
Spherical Coordinates (2)

Metric coefficients:

$\rightarrow h_1 = 1$

$\rightarrow h_2 = R$

$\rightarrow h_3 = R\sin\theta$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Directed differential length change: $d\mathbf{l} = \mathbf{a}_R dR + \mathbf{a}_\theta R d\theta + \mathbf{a}_\phi R\sin\theta d\phi$

Differential area change:

$$ds_R = R^2 \sin\theta d\theta d\phi$$
$$ds_\theta = R\sin\theta dR d\phi$$
$$ds_\phi = R dR d\theta$$

Differential volume change: $dv = R^2 \sin\theta dR d\theta d\phi$

Spherical Coordinates (3)

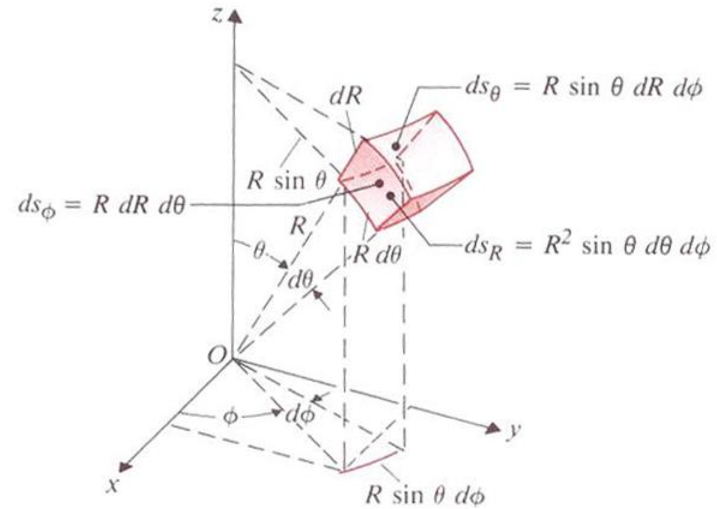
Transformation from cylindrical coordinates to Cartesian coordinates:

See Example 2-11 (HW2-3)

Conversion formulas:

$$x = R\sin\theta\cos\phi$$
$$y = R\sin\theta\sin\phi$$
$$z = R\cos\theta$$

$$R = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}$$
$$\phi = \tan^{-1}\frac{y}{x}$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Three Basic Orthogonal Coordinate Systems

Coordinate System Relations		Cartesian Coordinates (x, y, z)	Cylindrical Coordinates (r, ϕ, z)	Spherical Coordinates (R, θ, ϕ)
Base vectors	$\mathbf{a}_{u_1}$	$\mathbf{a}_x$	$\mathbf{a}_r$	$\mathbf{a}_R$
	$\mathbf{a}_{u_2}$	$\mathbf{a}_y$	$\mathbf{a}_\phi$	$\mathbf{a}_\theta$
	$\mathbf{a}_{u_3}$	$\mathbf{a}_z$	$\mathbf{a}_z$	$\mathbf{a}_\phi$
Metric coefficients	h_1	1	1	1
	h_2	1	r	R
	h_3	1	1	$R \sin \theta$
Differential volume	dv	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2 \sin \theta\,dR\,d\theta\,d\phi$

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Integrals Containing Vector Functions

Common forms of integrals containing vector functions:

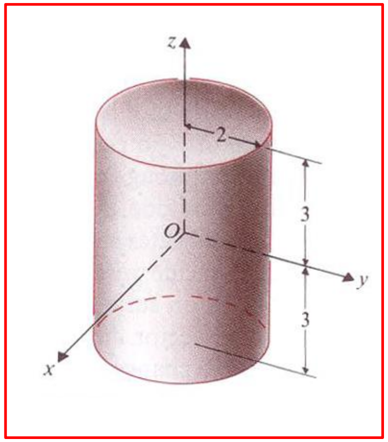
$$\int_V \mathbf{F}\,dv$$

$$\int_C V\,d\mathbf{l}$$

$$\int_C \mathbf{F}\cdot d\mathbf{l}$$

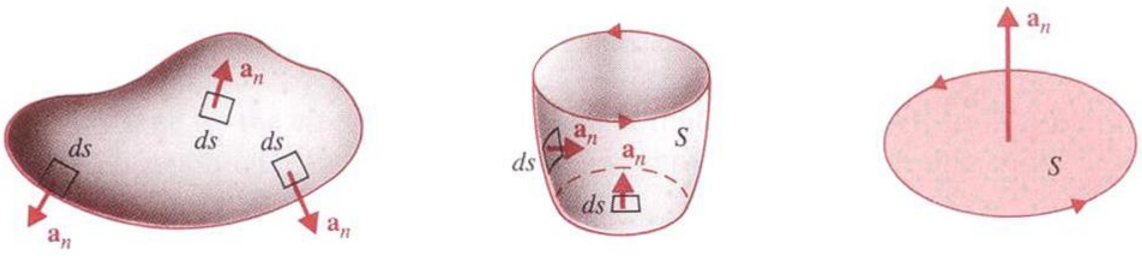
$$\int_S \mathbf{A}\cdot d\mathbf{s}$$

Surface normal vectors?



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Defining the surface normal vector:



(a) A closed surface. (b) An open surface. (c) A disk.

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.