Introduction to Electromagnetism
Gradient and Divergence
(2-6, 2-7, 2-8)

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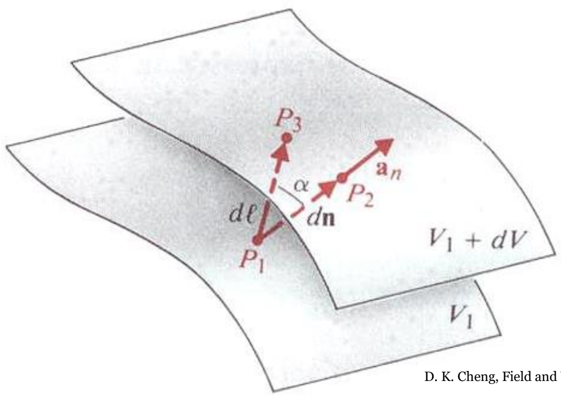
Gradient of a Scalar Field (1)

Let us consider a scalar function of space coordinates: $V(u_1, u_2, u_3)$

Suppose that there are two surfaces on which V is constant:

$$\rightarrow\ V = V_1\ \&\ V = V_1 + dV$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Normal vector

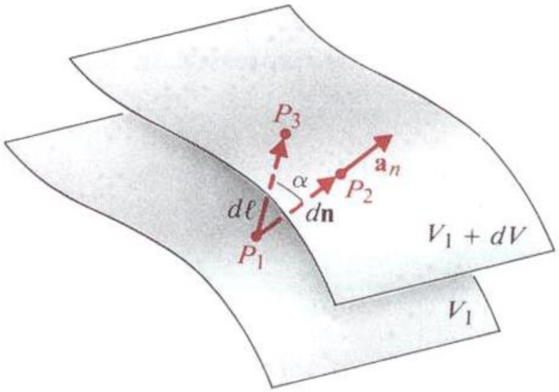
Space rate of change: dV/dl $\rightarrow$ Maximum when: $d\mathbf{l} \,//\, d\mathbf{n}$

Gradient of a scalar: The vector of the maximum space rate of increase of the scalar

$$\mathbf{grad}\, V \equiv \mathbf{a}_n \frac{dV}{dn} \quad \leftarrow \quad \mathbf{a}_n : \text{Unit normal vector}$$

$$\rightarrow \nabla V \equiv \mathbf{a}_n \frac{dV}{dn}$$

Gradient of a Scalar Field (2)



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Directional derivative along $d\mathbf{l}$:

$$\frac{dV}{dl} = \frac{dV}{dn}\frac{dn}{dl} = \frac{dV}{dn}\cos\alpha$$

$$= \frac{dV}{dn}\mathbf{a}_n \cdot \mathbf{a}_l = (\nabla V)\cdot \mathbf{a}_l$$

$$\rightarrow dV = (\nabla V)\cdot d\mathbf{l} \qquad \leftarrow d\mathbf{l} = \mathbf{a}_{u_1} dl_1 + \mathbf{a}_{u_2} dl_2 + \mathbf{a}_{u_3} dl_3$$

$$= \mathbf{a}_{u_1}(h_1 du_1) + \mathbf{a}_{u_2}(h_2 du_2) + \mathbf{a}_{u_3}(h_3 du_3)$$

Expressed in terms of the differential changes in coordinates:

$$\rightarrow dV = \frac{\partial V}{\partial l_1} dl_1 + \frac{\partial V}{\partial l_2} dl_2 + \frac{\partial V}{\partial l_3} dl_3 \quad \leftarrow \text{Chain rule}$$

$$= \left(\mathbf{a}_{u_1}\frac{\partial V}{\partial l_1} + \mathbf{a}_{u_2}\frac{\partial V}{\partial l_2} + \mathbf{a}_{u_3}\frac{\partial V}{\partial l_3}\right)\cdot\left(\mathbf{a}_{u_1} dl_1 + \mathbf{a}_{u_2} dl_2 + \mathbf{a}_{u_3} dl_3\right)$$

$$= \left(\mathbf{a}_{u_1}\frac{\partial V}{\partial l_1} + \mathbf{a}_{u_2}\frac{\partial V}{\partial l_2} + \mathbf{a}_{u_3}\frac{\partial V}{\partial l_3}\right)\cdot d\mathbf{l}$$

$$\therefore \nabla V = \mathbf{a}_{u_1}\frac{\partial V}{\partial l_1} + \mathbf{a}_{u_2}\frac{\partial V}{\partial l_2} + \mathbf{a}_{u_3}\frac{\partial V}{\partial l_3} = \mathbf{a}_{u_1}\frac{\partial V}{h_1\partial u_1} + \mathbf{a}_{u_2}\frac{\partial V}{h_2\partial u_2} + \mathbf{a}_{u_3}\frac{\partial V}{h_3\partial u_3}$$

Gradient of a Scalar Field (3)

Recall:

$$\nabla V = \mathbf{a}_{u_1}\frac{\partial V}{\partial l_1} + \mathbf{a}_{u_2}\frac{\partial V}{\partial l_2} + \mathbf{a}_{u_3}\frac{\partial V}{\partial l_3} = \mathbf{a}_{u_1}\frac{\partial V}{h_1 \partial u_1} + \mathbf{a}_{u_2}\frac{\partial V}{h_2 \partial u_2} + \mathbf{a}_{u_3}\frac{\partial V}{h_3 \partial u_3}$$

Vector differential operator (del):

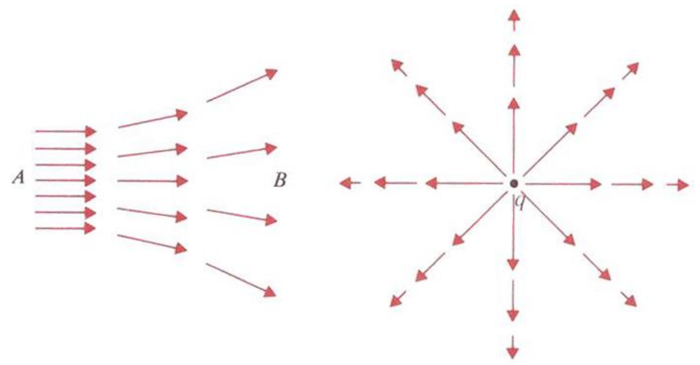
$$\nabla \equiv \mathbf{a}_{u_1}\frac{\partial}{h_1 \partial u_1} + \mathbf{a}_{u_2}\frac{\partial}{h_2 \partial u_2} + \mathbf{a}_{u_3}\frac{\partial}{h_3 \partial u_3}$$

In Cartesian coordinates:

$$\nabla \equiv \mathbf{a}_x\frac{\partial}{\partial x} + \mathbf{a}_y\frac{\partial}{\partial y} + \mathbf{a}_y\frac{\partial}{\partial z}$$

Divergence of a Vector Field (1)

Directed field lines for vector fields: Flux lines or streamlines



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(a) (b) (c)

Consider a volume with an enclosed surface:

1. An excess of outward flow through the surface:

   $\rightarrow$ A net positive divergence $\rightarrow$ The volume contains a source.

2. An excess of inward flow through the surface:

   $\rightarrow$ A net negative divergence $\rightarrow$ The volume contains a sink.

3. No excess of outward or inward flow through the surface:

   $\rightarrow$ A net zero divergence $\rightarrow$ Neither a source nor a sink

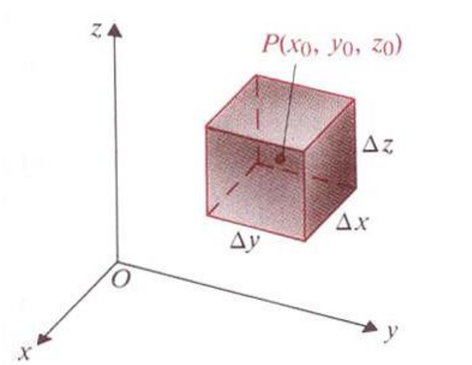
Divergence of a Vector Field (2)

Definition of the divergence a vector field $\mathbf{A}$ at a point:

The net outward flux of $\mathbf{A}$ per unit volume as the volume about the point tends to zero:

$$\text{div }\mathbf{A} \equiv \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A}\cdot d\mathbf{s}}{\Delta v}$$

Consider a differential volume of sides Δx, Δy and Δz in Cartesian coordinates:



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$$\mathbf{A} \equiv \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z$$

$$\oint_S \mathbf{A}\cdot d\mathbf{s} = \left[\int_{\substack{\text{front}\\\text{face}}} + \int_{\substack{\text{back}\\\text{face}}} + \int_{\substack{\text{right}\\\text{face}}} + \int_{\substack{\text{left}\\\text{face}}} + \int_{\substack{\text{top}\\\text{face}}} + \int_{\substack{\text{bottom}\\\text{face}}}\right] \mathbf{A}\cdot d\mathbf{s}$$

$$\int_{\substack{\text{front}\\\text{face}}} \mathbf{A}\cdot d\mathbf{s} = \mathbf{A}_{\substack{\text{front}\\\text{face}}} \cdot \Delta\mathbf{s}_{\substack{\text{front}\\\text{face}}} = \mathbf{A}_{\substack{\text{front}\\\text{face}}} \cdot \mathbf{a}_x (\Delta y \Delta z)$$

$$= A_x\left(x_0 + \frac{\Delta x}{2}, y_0, z_0\right)\Delta y \Delta z$$

$\rightarrow$ Taylor series expansion:

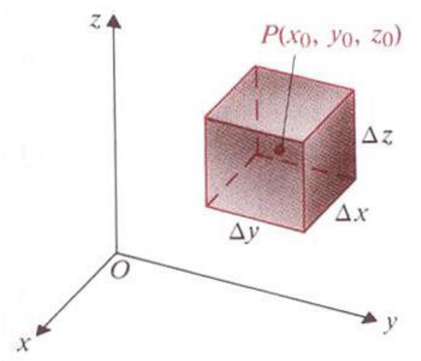
$$A_x\left(x_0 + \frac{\Delta x}{2}, y_0, z_0\right) = A_x(x_0, y_0, z_0) + \frac{\Delta x}{2}\left.\frac{\partial A_x}{\partial x}\right|_{(x_0, y_0, z_0)} + \text{H.O.T.}$$

Divergence of a Vector Field (3)

Continued:

$$\int_{\substack{\text{back}\\\text{face}}} \mathbf{A}\cdot d\mathbf{s} = \mathbf{A}_{\substack{\text{back}\\\text{face}}} \cdot \Delta\mathbf{s}_{\substack{\text{back}\\\text{face}}} = \mathbf{A}_{\substack{\text{back}\\\text{face}}} \cdot (-\mathbf{a}_x \Delta y \Delta z)$$

$$= -A_x\left(x_0 - \frac{\Delta x}{2}, y_0, z_0\right)\Delta y \Delta z$$



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$$A_x\left(x_0 - \frac{\Delta x}{2}, y_0, z_0\right) = A_x(x_0, y_0, z_0) + \left(-\frac{\Delta x}{2}\right)\left.\frac{\partial A_x}{\partial x}\right|_{(x_0, y_0, z_0)} + \text{H.O.T.}$$

$$\rightarrow \left[\int_{\substack{\text{front}\\\text{face}}} + \int_{\substack{\text{back}\\\text{face}}}\right] \mathbf{A}\cdot d\mathbf{s} = \left.\left(\frac{\partial A_x}{\partial x} + \text{H.O.T.}\right)\right|_{(x_0, y_0, z_0)} \Delta x \Delta y \Delta z$$

Similarly:

$$\rightarrow \left[\int_{\substack{\text{right}\\\text{face}}} + \int_{\substack{\text{left}\\\text{face}}}\right] \mathbf{A}\cdot d\mathbf{s} = \left.\left(\frac{\partial A_y}{\partial y} + \text{H.O.T.}\right)\right|_{(x_0, y_0, z_0)} \Delta x \Delta y \Delta z$$

$$\rightarrow \left[\int_{\substack{\text{top}\\\text{face}}} + \int_{\substack{\text{botton}\\\text{face}}}\right] \mathbf{A}\cdot d\mathbf{s} = \left.\left(\frac{\partial A_z}{\partial z} + \text{H.O.T.}\right)\right|_{(x_0, y_0, z_0)} \Delta x \Delta y \Delta z$$

In result:

$$\oint_S \mathbf{A}\cdot d\mathbf{s} = \left.\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\right|_{(x_0, y_0, z_0)} \Delta x \Delta y \Delta z + \text{H.O.T.}$$

Divergence of a Vector Field (4)

Recall:

$$\text{div }\mathbf{A} \equiv \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{s}}{\Delta v}$$

$$\text{div }\mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{s} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right)\Bigg|_{(x_0, y_0, z_0)} \Delta x \Delta y \Delta z + \text{H.O.T.}}{\Delta v}$$

$$\boxed{\text{div }\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}} \quad \leftarrow \quad \nabla = \mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_y \frac{\partial}{\partial z}$$

$$\boxed{\nabla \cdot \mathbf{A} \equiv \text{div }\mathbf{A}}$$

In general orthogonal curvilinear coordinates:

$$\boxed{\nabla \cdot \mathbf{A} \equiv \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_1 h_3 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3) \right]}$$

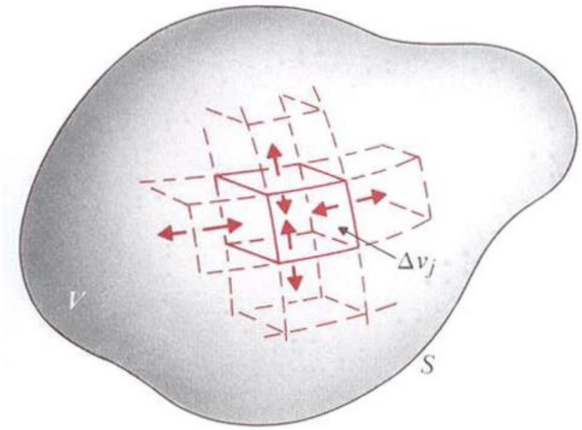
$\rightarrow$ HW

Divergence Theorem

The volume integral of the divergence of a vector field equals the total outward flux of the vector through the surface that bounds the volume:

$$\int_V \nabla \cdot \mathbf{A} dv = \oint_S \mathbf{A} \cdot d\mathbf{s}$$ ← **A**: Smooth or piecewise smooth!

Divide the volume into small differential volume elements:



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$$(\nabla \cdot \mathbf{A})_j \Delta v_j = \oint_{S_j} \mathbf{A} \cdot d\mathbf{s}$$ ← By definition

$$\int_V \nabla \cdot \mathbf{A} dv = \lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} (\nabla \cdot \mathbf{A})_j \Delta v_j \right]$$

$$= \lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} \oint_{S_j} \mathbf{A} \cdot d\mathbf{s} \right]$$

$$= \oint_S \mathbf{A} \cdot d\mathbf{s}$$

What if not continuously smooth?

A volume integral of the divergence of a vector can be converted into a closed surface integral of the vector, and vice versa!