Introduction to Electromagnetism
Curl of a Vector Field, etc.
(2-9, 2-10, 2-11, 2-12)

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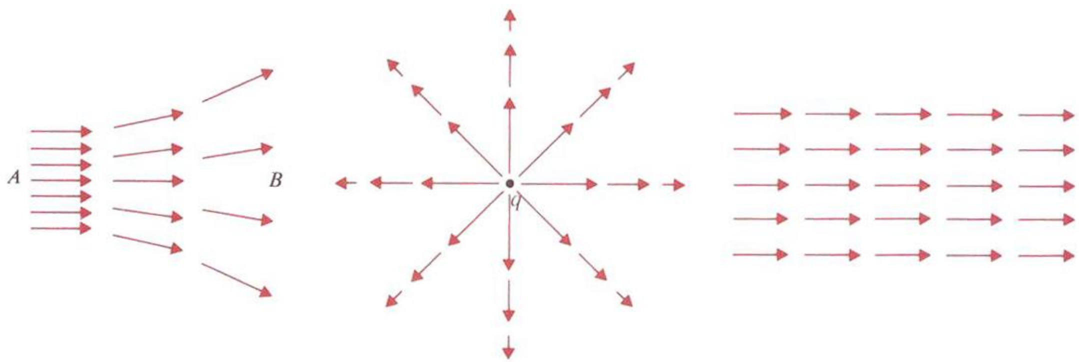
Curl of a Vector Field (1)

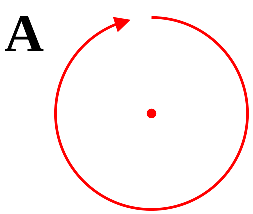
Directed field lines for vector fields: Flux lines or streamlines



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

(a) (b) (c)

$\rightarrow$ *Flow source*



What if there a vortex source that causes a circulation of a vector field around it?

Net circulation (or circulation) of a vector field around a closed path:

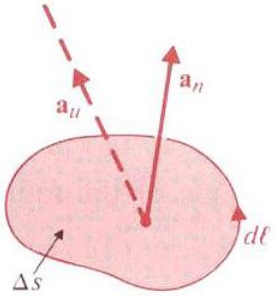
$$\rightarrow \textit{Circulation of } \mathbf{A} \textit{ around contour } C \equiv \oint_C \mathbf{A} \cdot d\mathbf{l}$$

Curl of a Vector Field (2)

Definition of the curl of a vector field **A** at a point:

A vector whose magnitude is the maximum net circulation of **A** per unit area as the area tends to zero and whose direction is the normal direction of the area (right-hand rule) when the area is oriented to make the net circulation maximum:

$$\text{curl}\,\mathbf{A} \equiv \nabla \times \mathbf{A} \equiv \lim_{\Delta s \to 0} \frac{1}{\Delta s}\left[\mathbf{a}_n \oint_C \mathbf{A} \cdot d\mathbf{l}\right]_{\max}$$



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Component of $\nabla \times \mathbf{A}$ in any other direction $\mathbf{a}_u$:

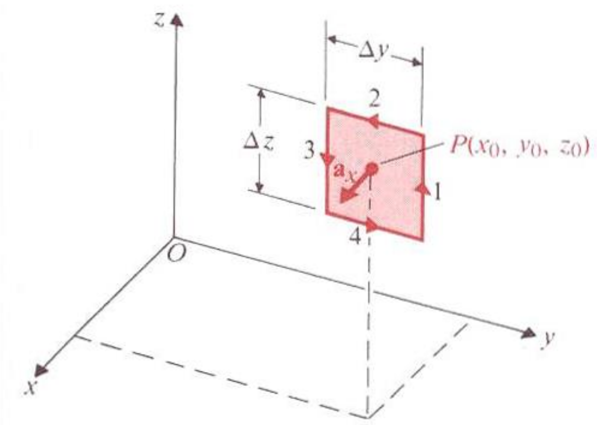
$$(\nabla \times \mathbf{A})_u = \mathbf{a}_u \cdot (\nabla \times \mathbf{A}) = \lim_{\Delta s_u \to 0} \frac{1}{\Delta s_u}\left(\oint_{C_u} \mathbf{A} \cdot d\mathbf{l}\right)$$

Curl of a Vector Field (3)

Consider a differential rectangle in Cartesian coordinates:

$$\mathbf{A} \equiv \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z$$

$$(\nabla \times \mathbf{A})_x = \mathbf{a}_x \cdot (\nabla \times \mathbf{A}) = \lim_{\Delta y \Delta z \to 0} \frac{1}{\Delta y \Delta z} \left(\oint_{\substack{\text{sides} \\ 1,2,3,4}} \mathbf{A} \cdot d\mathbf{l} \right)$$



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Side 1: $d\mathbf{l} = \mathbf{a}_z \Delta z$ ← Taylor series expansion

$$\rightarrow \mathbf{A} \cdot d\mathbf{l} = A_z(x_0, y_0 + \frac{\Delta y}{2}, z_0)\Delta z$$

$$= \left\{ A_z(x_0, y_0, z_0) + \frac{\Delta y}{2} \frac{\partial A_z}{\partial y}\bigg|_{(x_0, y_0, z_0)} + \text{H.O.T.} \right\} \Delta z$$

Side 3: $d\mathbf{l} = -\mathbf{a}_z \Delta z$ ← Taylor series expansion

$$\rightarrow \mathbf{A} \cdot d\mathbf{l} = A_z(x_0, y_0 - \frac{\Delta y}{2}, z_0)(-\Delta z)$$

$$= \left\{ A_z(x_0, y_0, z_0) - \frac{\Delta y}{2} \frac{\partial A_z}{\partial y}\bigg|_{(x_0, y_0, z_0)} + \text{H.O.T.} \right\} (-\Delta z)$$

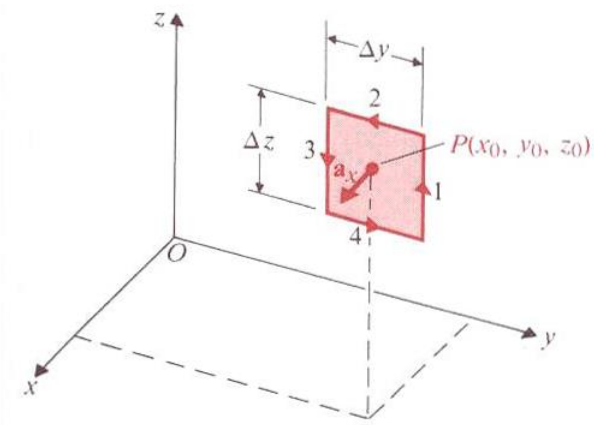
$$\rightarrow \oint_{\substack{\text{sides} \\ 1\&3}} \mathbf{A} \cdot d\mathbf{l} = \Delta y \Delta z \left(\frac{\partial A_z}{\partial y} + \text{H.O.T.} \right)\bigg|_{(x_0, y_0, z_0)}$$

Curl of a Vector Field (4)

Continued:

$$\mathbf{A} \equiv \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z$$

$$(\nabla \times \mathbf{A})_x = \mathbf{a}_x \cdot (\nabla \times \mathbf{A}) = \lim_{\Delta y \Delta z \to 0} \frac{1}{\Delta y \Delta z} \left(\oint_{\substack{\text{sides} \\ 1,2,3,4}} \mathbf{A} \cdot d\mathbf{l} \right)$$



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Side 2: $d\mathbf{l} = -\mathbf{a}_y \Delta y$

$$\to \mathbf{A} \cdot d\mathbf{l} = A_y(x_0, y_0, z_0 + \frac{\Delta z}{2})(-\Delta y)$$

$$= \left\{ A_y(x_0, y_0, z_0) + \frac{\Delta z}{2} \frac{\partial A_y}{\partial z} \bigg|_{(x_0, y_0, z_0)} + \text{H.O.T.} \right\} (-\Delta y)$$

Side 4: $d\mathbf{l} = \mathbf{a}_y \Delta y$

$$\to \mathbf{A} \cdot d\mathbf{l} = A_y(x_0, y_0, z_0 - \frac{\Delta z}{2}) \Delta y$$

$$= \left\{ A_y(x_0, y_0, z_0) - \frac{\Delta z}{2} \frac{\partial A_y}{\partial z} \bigg|_{(x_0, y_0, z_0)} + \text{H.O.T.} \right\} \Delta y$$

$$\to \oint_{\substack{\text{sides} \\ 2\&4}} \mathbf{A} \cdot d\mathbf{l} = -\Delta y \Delta z \left(\frac{\partial A_y}{\partial z} + \text{H.O.T.} \right) \bigg|_{(x_0, y_0, z_0)}$$

Curl of a Vector Field (5)

Recall:

$$(\nabla \times \mathbf{A})_x = \mathbf{a}_x \cdot (\nabla \times \mathbf{A}) = \lim_{\Delta y \Delta z \to 0} \frac{1}{\Delta y \Delta z} \left(\oint_{\substack{\text{sides} \\ 1,2,3,4}} \mathbf{A} \cdot d\mathbf{l} \right)$$

$$= \lim_{\Delta y \Delta z \to 0} \frac{1}{\Delta y \Delta z} \left\{ \Delta y \Delta z \left(\frac{\partial A_z}{\partial y} + \text{H.O.T.} \right) \Bigg|_{(x_0, y_0, z_0)} - \Delta y \Delta z \left(\frac{\partial A_y}{\partial z} + \text{H.O.T.} \right) \Bigg|_{(x_0, y_0, z_0)} \right\}$$

$$\rightarrow (\nabla \times \mathbf{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

Similarly:

$$(\nabla \times \mathbf{A})_y = \mathbf{a}_y \cdot (\nabla \times \mathbf{A}) = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}$$

$$(\nabla \times \mathbf{A})_z = \mathbf{a}_z \cdot (\nabla \times \mathbf{A}) = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

In result:

$$\nabla \times \mathbf{A} = \mathbf{a}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{a}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{a}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

Curl of a Vector Field (6)

Recall:

$$\nabla \times \mathbf{A} = \mathbf{a}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{a}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{a}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

Determinantal form:

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

In general orthogonal curvilinear coordinates:

$$\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \mathbf{a}_{u_1} h_1 & \mathbf{a}_{u_2} h_2 & \mathbf{a}_{u_3} h_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

$\rightarrow$ HW2-6

Stokes's Theorem

For a very small differential area Δs_j bounded by a contour C_j:

$$(\nabla \times \mathbf{A})_j \cdot (\Delta \mathbf{s}_j) = \oint_{C_j} \mathbf{A} \cdot d\mathbf{l}$$

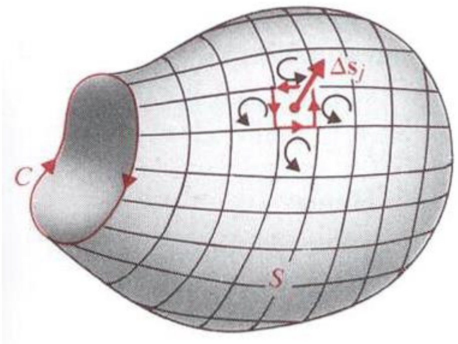
$\leftarrow$ **A**: Continuously differentiable!

Add the contributions of all the differential area to the flux:

$$\lim_{\Delta s_j \to 0} \sum_{j=1}^{N} (\nabla \times \mathbf{A})_j \cdot (\Delta \mathbf{s}_j) = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s}$$

Sum up the line integrals around the contours of all the differential elements:

$$\rightarrow \lim_{\Delta s_j \to 0} \sum_{j=1}^{N} \left(\oint_{c_j} \mathbf{A} \cdot d\mathbf{l} \right) = \oint_C \mathbf{A} \cdot d\mathbf{l}$$



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Stokes's theorem:

$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_C \mathbf{A} \cdot d\mathbf{l}$$

Note: $\oint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = 0$

Two Null Identities

Identity I: $\nabla \times (\nabla V) \equiv 0$

Proof:

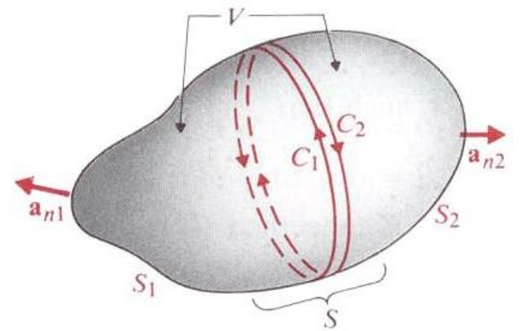
$$\int_S [\nabla \times (\nabla V)] \cdot d\mathbf{s} = \oint_C (\nabla V) \cdot d\mathbf{l}$$

Recall: $dV = (\nabla V) \cdot d\mathbf{l}$ $\rightarrow$ Eq. (2-88)

$$= \oint_C dV = 0$$

$\rightarrow$ For any S

Curl-free vector: $\nabla \times \mathbf{E} = 0 \;\rightarrow\; \mathbf{E} = -\nabla V$ $\rightarrow$ Irrotational (conservative)



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Identity II: $\nabla \cdot (\nabla \times \mathbf{A}) \equiv 0$

Proof:

$$\int_V \nabla \cdot (\nabla \times \mathbf{A}) dv = \oint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_{S_1} (\nabla \times \mathbf{A}) \cdot \mathbf{a}_{n_1} ds + \oint_{S_2} (\nabla \times \mathbf{A}) \cdot \mathbf{a}_{n_2} ds$$

$$= \oint_{C_1} \mathbf{A} \cdot d\mathbf{l} + \oint_{C_2} \mathbf{A} \cdot d\mathbf{l} = 0$$

$\rightarrow$ For any V

Divergenceless vector: $\nabla \cdot \mathbf{B} = 0 \;\rightarrow\; \mathbf{B} = \nabla \times \mathbf{A}$ $\rightarrow$ Solenoidal

Helmholtz's Theorem (1)

1. Solenoidal and irrotational if:

$$\nabla \cdot \mathbf{F} = 0 \quad \text{and} \quad \nabla \times \mathbf{F} = 0$$

2. Solenoidal but not irrotational if:

$$\nabla \cdot \mathbf{F} = 0 \quad \text{and} \quad \nabla \times \mathbf{F} \neq 0$$

3. Irrotational but not solenoidal if:

$$\nabla \times \mathbf{F} = 0 \quad \text{and} \quad \nabla \cdot \mathbf{F} \neq 0$$

4. Neither solenoidal nor irrotational if:

$$\nabla \cdot \mathbf{F} \neq 0 \quad \text{and} \quad \nabla \times \mathbf{F} \neq 0$$

Helmholtz's Theorem (2)

Helmholtz's theorem:

A vector field (vector point function) is determined to within an additive constant if both its divergence and its curl are specified everywhere.

Divergence of a vector: A measure of the strength of the flow source

Curl of a vector: A measure of the strength of the vortex source

Decomposition of a vector into an irrotational part and a solenoidal part:

$$\mathbf{F} = \mathbf{F}_i + \mathbf{F}_s$$

$$\text{with} \quad \begin{cases} \nabla \times \mathbf{F}_i = 0 \\ \nabla \cdot \mathbf{F}_i = g \end{cases} \quad \& \quad \begin{cases} \nabla \cdot \mathbf{F}_s = 0 \\ \nabla \times \mathbf{F}_s = G \end{cases}$$

$$\nabla \cdot \mathbf{F} = \nabla \cdot \mathbf{F}_i = g$$

$$\nabla \times \mathbf{F} = \nabla \times \mathbf{F}_s = G$$

$\rightarrow$ When g and $\mathbf{G}$ are specified, the vector function $\mathbf{F}$ is determined!

$$\rightarrow \mathbf{F} = -\nabla V + \nabla \times \mathbf{A}$$