Introduction to Electromagnetism
Static Electric Fields
(3-7, 3-8, 3-9)

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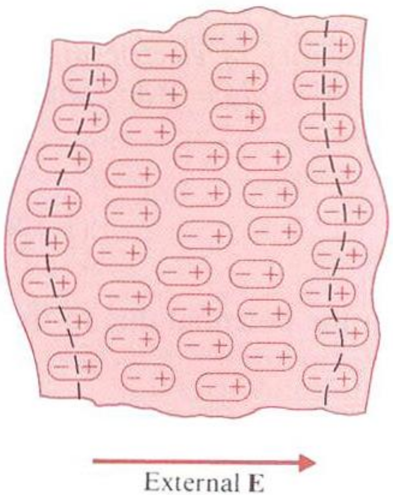
Dielectrics in Static Electric Field

Dielectrics:

No free charges but having bound charges



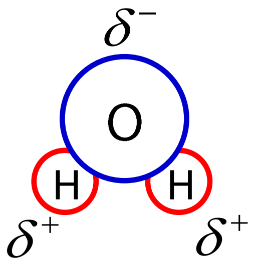
D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Induced electric dipoles:

Macroscopically neutral but
having small displacements of positive
and negative charges in opposite directions
(polarized)

Permanent dipole moments:

Polar molecules, e.g. H_2O (*cf.* nonpolar molecules)
Electrets (waxes, polymers, resins), e.g. HF microphones



Equivalent Charge Distribution of Polarized Dielectrics (1)

Polarization vector (macroscopic): Volume density of electric dipole moment

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \mathbf{p}_k}{\Delta v} \quad (\mathrm{C/m^2})$$

n: the number of molecules per unit volume

Differential dipole moment for an elemental volume dv':

$$\to \ d\mathbf{p} = \mathbf{P}dv' \qquad \to \ dV = \frac{\mathbf{P}\cdot\mathbf{a}_R}{4\pi\varepsilon_0 R^2} dv'$$

(See Eq. 3-53b)

Electrostatic potential:

$$\to \ V = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\mathbf{P}\cdot\mathbf{a}_R}{R^2} dv'$$

Note: $R^2 = (x-x')^2 + (y-y')^2 + (z-z')^2 \ \to \ \nabla'\left(\frac{1}{R}\right) = \frac{\mathbf{a}_R}{R^2}$

$$\to \ V = \frac{1}{4\pi\varepsilon_0} \int_{V'} \mathbf{P}\cdot\nabla'\left(\frac{1}{R}\right) dv' \ \to \ V = \frac{1}{4\pi\varepsilon_0}\left[\int_{V'} \nabla'\cdot\left(\frac{\mathbf{P}}{R}\right) dv' - \int_{V'} \frac{\nabla'\cdot\mathbf{P}}{R} dv'\right]$$

$$\therefore V = \frac{1}{4\pi\varepsilon_0} \oint_{S'} \frac{\mathbf{P}\cdot\mathbf{a}'_n}{R} ds' + \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{-\nabla'\cdot\mathbf{P}}{R} dv'$$

Equivalent Charge Distribution of Polarized Dielectrics (2)

Electrostatic potential:

$$V = \frac{1}{4\pi\varepsilon_0}\oint_{S'}\frac{\mathbf{P}\cdot\mathbf{a}'_n}{R}ds' + \frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{-\nabla'\cdot\mathbf{P}}{R}dv'$$

$$\rightarrow V = \frac{1}{4\pi\varepsilon_0}\oint_{S'}\frac{\rho_{ps}}{R}ds' + \frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\rho_p}{R}dv'$$

Equivalent polarization surface charge density: $\rho_{ps} = \mathbf{P}\cdot\mathbf{a}_n$

Equivalent polarization volume charge density: $\rho_p = -\nabla\cdot\mathbf{P}$

Separation of bound charges to the surface:

$$\Delta Q = nq(\mathbf{d}\cdot\mathbf{a}_n)(\Delta s) \rightarrow \Delta Q = \mathbf{P}\cdot\mathbf{a}_n(\Delta s)$$

$$\rightarrow \frac{\Delta Q}{\Delta s} = \mathbf{P}\cdot\mathbf{a}_n = \rho_{ps}$$ $\rightarrow$ Surface charge density

Remaining charges within the volume:

$$\rightarrow Q = -\oint_S \mathbf{P}\cdot\mathbf{a}_n ds$$
$$= \int_V(-\nabla\cdot\mathbf{P})dv = \int_V \rho_p dv$$

$$\boxed{\text{Total charge} = \oint_S \rho_{ps}ds + \int_V \rho_p dv = 0}$$

Electric Flux Density and Dielectric Constant (1)

Modified divergence postulate:

"equivalent volume charge density"
"bounded charges"

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}(\rho + \rho_p)$$

$$\rightarrow \nabla \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \rho$$

$$\rightarrow \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \quad (\mathrm{C/m^2})$$ ← Electric flux density or electric displacement

$$\rightarrow \nabla \cdot \mathbf{D} = \rho \quad (\mathrm{C/m^3})$$

"free charge density"

Gauss's law:

$$\boxed{\begin{aligned} &\rightarrow \int_V \nabla \cdot \mathbf{D} dv = \int_V \rho dv \\ &\rightarrow \oint_S \mathbf{D} \cdot d\mathbf{s} = Q \end{aligned}}$$

→ The total outward flux of the electric displacement over any closed surface is equal to the total free charge enclosed in the surface.

Electric Flux Density and Dielectric Constant (2)

Electric susceptibility / Relative permittivity / Permittivity:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \quad \leftarrow \quad \mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$$

$$\rightarrow \ \mathbf{D} = \varepsilon_0 (1 + \chi_e) \mathbf{E}$$

$$= \varepsilon_0 \varepsilon_r \mathbf{E} = \varepsilon \mathbf{E}$$

Permittivity in general:

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

← Along the principal axes

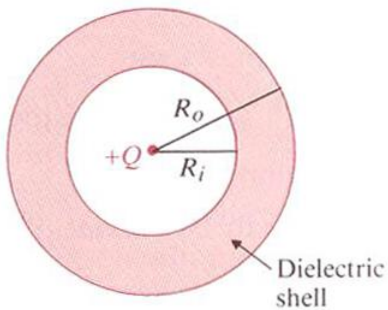
Isotropic: $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33}$

Uniaxial: $\varepsilon_{11} = \varepsilon_{22} \neq \varepsilon_{33}$

Biaxial: $\varepsilon_{11} \neq \varepsilon_{22} \neq \varepsilon_{33}$

Example 3-12

For a charge Q at the center of a spherical dielectric shell: $\mathbf{P} = \mathbf{D} - \varepsilon_0 \mathbf{E} = \varepsilon_0(\varepsilon_r - 1)\mathbf{E}$



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a) $R > R_o$

$$\rightarrow E_{R1} = \frac{Q}{4\pi\varepsilon_0 R^2} \qquad \rightarrow V_1 = \frac{Q}{4\pi\varepsilon_0 R}$$

$$\rightarrow D_{R1} = \varepsilon_0 E_{R1} = \frac{Q}{4\pi R^2} \qquad \rightarrow P_{R1} = 0$$

b) $R_i < R < R_o$

$$\rightarrow D_{R2} = \frac{Q}{4\pi R^2} \qquad \rightarrow E_{R2} = \frac{Q}{4\pi\varepsilon R^2}$$

$$\rightarrow P_{R2} = (1 - \frac{1}{\varepsilon_r})\frac{Q}{4\pi R^2} \qquad \rightarrow V_2 = -\int_\infty^R E dR = -\int_\infty^{R_o} E_{R1} dR - \int_{R_o}^R E_{R2} dR$$

$$= V_1\Big|_{R=R_o} - \frac{Q}{4\pi\varepsilon}\int_{R_o}^R \frac{1}{R^2} dR = \frac{Q}{4\pi\varepsilon_0}\left[\left(1 - \frac{1}{\varepsilon_r}\right)\frac{1}{R_o} + \frac{1}{\varepsilon_r R}\right]$$

c) $R < R_i$

$$\rightarrow E_{R3} = \frac{Q}{4\pi\varepsilon_0 R^2} \qquad \rightarrow D_{R3} = \frac{Q}{4\pi R^2}$$

$$\rightarrow P_{R3} = 0 \qquad \rightarrow V_3 = V_2\Big|_{R=R_i} - \int_{R_i}^R E_{R3} dR = \frac{Q}{4\pi\varepsilon_0}\left[\left(1 - \frac{1}{\varepsilon_r}\right)\left(\frac{1}{R_o} - \frac{1}{R_i}\right) + \frac{1}{R}\right]$$

Example 3-12

Continued:

Equivalent polarization surface charge density:

$$\boxed{P_{R2} = (1 - \frac{1}{\varepsilon_r}) \frac{Q}{4\pi R^2}}$$

On the inner shell surface:

$$\rho_{ps}\Big|_{R=R_i} = \mathbf{P} \cdot (-\mathbf{a}_R)\Big|_{R=R_i} = -P_{R2}\Big|_{R=R_i}$$

$$= -\left(1 - \frac{1}{\varepsilon_r}\right) \frac{Q}{4\pi R_i^2}$$

On the outer shell surface:

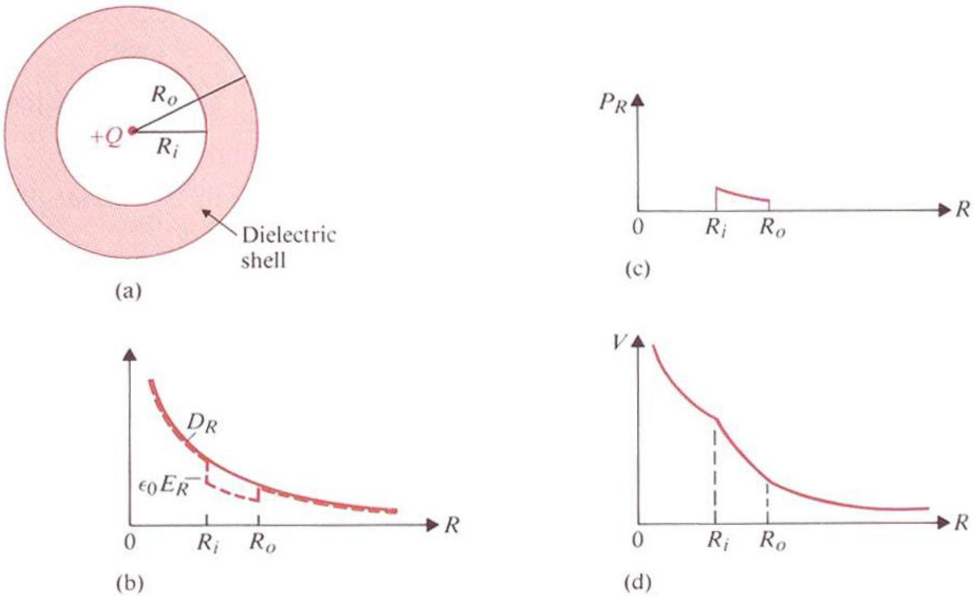
$$\rho_{ps}\Big|_{R=R_o} = \mathbf{P} \cdot \mathbf{a}_R\Big|_{R=R_o} = P_{R2}\Big|_{R=R_o}$$

$$= \left(1 - \frac{1}{\varepsilon_r}\right) \frac{Q}{4\pi R_o^2}$$

Inside the shell:

$$\rho_p = -\nabla \cdot \mathbf{P}$$

$$= -\frac{1}{R^2} \frac{\partial}{\partial R} (R^2 P_{R2}) \quad = 0$$



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Dielectric Strength

Dielectric breakdown:

Avalanche effect of ionization due to very strong electric field

Dielectric strength:

Maximum electric field intensity that a dielectric material can withstand without breakdown

e.g. air ~ 3 kV/mm

Lightening strike: Dielectric breakdown

Principle of a lightening rod:

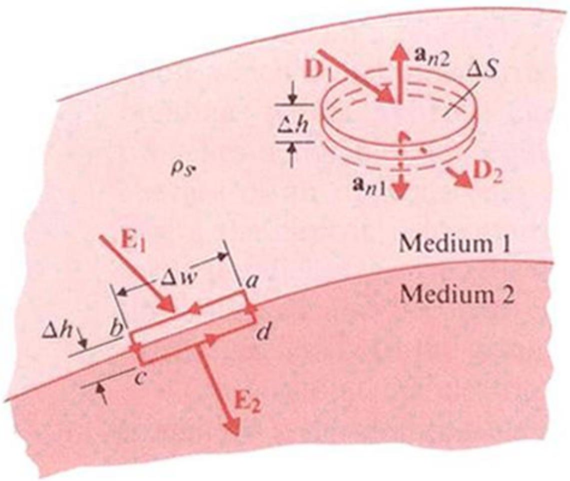
Electric field intensity much higher at sharp points → See: Example 3-13

Boundary Conditions for Electrostatic Fields

Tangential component of an E field:

$$\oint_{abcda} \mathbf{E} \cdot d\mathbf{l} = 0 \quad \rightarrow \quad E_{1t}\Delta w - E_{2t}\Delta w = 0$$

$$\boxed{E_{1t} = E_{2t}}$$



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Normal component of D field:

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = (\mathbf{D}_1 \cdot \mathbf{a}_{n2} + \mathbf{D}_2 \cdot \mathbf{a}_{n1})\Delta S$$

$$= \mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2)\Delta S$$

$$= \rho_s \Delta S$$

$$\boxed{\mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s}$$

or

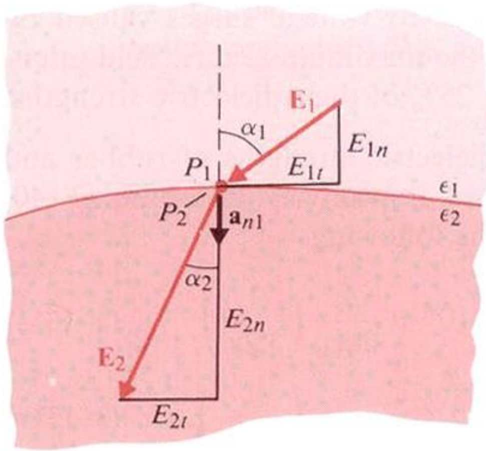
$$\boxed{D_{1n} - D_{2n} = \rho_s}$$

For dielectrics:

No free charges $\rightarrow \rho_s = 0$

Example 3-15

Two dielectric media with different permittivities:



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Boundary conditions:

$$E_2 \sin\alpha_2 = E_1 \sin\alpha_1 \quad \leftarrow \text{Tangential}$$

$$D_2 \cos\alpha_2 = D_1 \cos\alpha_1 \quad \leftarrow \text{Normal}$$

$$\rightarrow \varepsilon_2 E_2 \cos\alpha_2 = \varepsilon_1 E_1 \cos\alpha_1$$

$$\rightarrow \frac{\tan\alpha_2}{\tan\alpha_1} = \frac{\varepsilon_2}{\varepsilon_1}$$

$$\rightarrow E_2 = \sqrt{E_{2t}^2 + E_{2n}^2} = \left[(E_1 \sin\alpha_1)^2 + (\frac{\varepsilon_1}{\varepsilon_1} E_1 \cos\alpha_1)^2 \right]^{1/2}$$

$$\rightarrow E_2 = E_1 \left[(\sin\alpha_1)^2 + (\frac{\varepsilon_1}{\varepsilon_1} \cos\alpha_1)^2 \right]^{1/2}$$