Introduction to Electromagnetism
Solution of Electrostatic Problems
(4-1, 4-2, 4-3)

Yoonchan Jeong
School of Electrical Engineering, Seoul National University
Tel: +82 (0)2 880 1623, Fax: +82 (0)2 873 9953
Email: yoonchan@snu.ac.kr

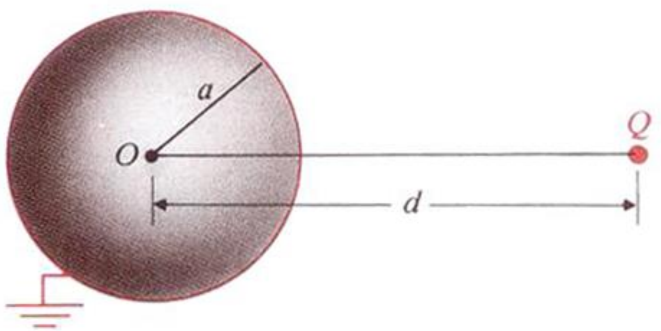
Electrostatic problems

How to deal with the electrostatics problems?

- Electric potential
- Electric field intensity
- Charge distribution

Boundary-value problems:

- Conductors
- Method of images



(a) Point charge and grounded conducting sphere.

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Poisson's and Laplace's Equations (1)

Master equations for electrostatics:

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \times \mathbf{E} = 0$$

$$\rightarrow \ \mathbf{E} = -\nabla V$$

For a linear and isotropic medium:

$$\mathbf{D} = \varepsilon \mathbf{E}$$

$$\rightarrow \ \nabla \cdot (\varepsilon \mathbf{E}) = \rho$$

$$\rightarrow \ \nabla \cdot (\varepsilon \nabla V) = -\rho$$

For a simple medium (linear/isotropic/homogeneous):

$$\rightarrow \boxed{\nabla^2 V = -\frac{\rho}{\varepsilon}}$$ *→ Poisson's equation*

$\nabla^2 :$ *→ Laplacian operator:*
"the divergence of the gradient of"

Poisson's and Laplace's Equations (2)

In Cartesian coordinates:

$$\nabla^2 V = \nabla \cdot \nabla V$$

$$= \left(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z} \right) \cdot \left(\mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z} \right)$$

$$\rightarrow \boxed{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\varepsilon} \quad (\mathrm{V/m^2})}$$

In cylindrical coordinates:

$$\nabla^2 V = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

In spherical coordinates:

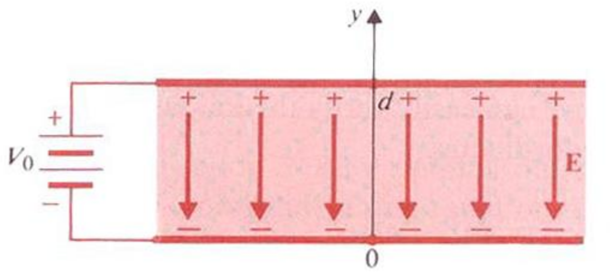
$$\nabla^2 V = \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial V}{\partial R} \right) + \frac{1}{R^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{R^2 \sin^2\theta} \frac{\partial^2 V}{\partial \phi^2}$$

Laplace's equation:

$$\nabla^2 V = 0 \qquad \leftarrow \textit{No free charge } (\rho = 0)$$

Example 4-1

A parallel-plate capacitor:



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Between the plates:

$$\nabla^2 V = 0$$

$$\rightarrow \frac{d^2V}{dy^2} = 0$$

$$\rightarrow V = C_1 y + C_2$$

Boundary conditions:

$$V(y=0) = 0, \quad V(y=d) = V_0$$

$$\rightarrow V = \frac{V_0}{d} y$$

Electric field:

$$\rightarrow \mathbf{E} = -\nabla V = -\mathbf{a}_y \frac{dV}{dy} = -\mathbf{a}_y \frac{V_0}{d}$$

Surface charge density:

$$\rightarrow E_n = \mathbf{a}_n \cdot \mathbf{E} = \frac{\rho_s}{\varepsilon} \qquad \rightarrow \rho_{su} = \frac{\varepsilon V_0}{d}, \quad \rho_{sl} = -\frac{\varepsilon V_0}{d}$$

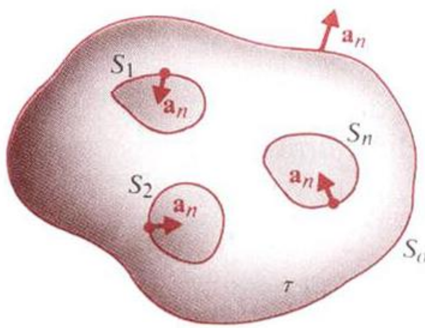
Uniqueness of Electrostatic Solutions

Uniqueness theorem:

A solution of Poisson's equation (or Laplace's equation) that satisfies the given boundary conditions is a unique solution.

Proof:

Let us suppose that there are two solutions:



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

$$\nabla^2 V_1 = -\frac{\rho}{\varepsilon}, \quad \nabla^2 V_2 = -\frac{\rho}{\varepsilon}$$

$$V_d = V_1 - V_2 \quad \rightarrow \quad \nabla^2 V_d = 0$$

$\leftarrow$ Boundary conditions: $V_d = 0$

$$\rightarrow \nabla \cdot (V_d \nabla V_d) = V_d \nabla^2 V_d + |\nabla V_d|^2$$

$$\rightarrow \oint_S (V_d \nabla V_d) \cdot \mathbf{a}_n ds = \int_\tau |\nabla V_d|^2 dv \quad \rightarrow \quad \int_\tau |\nabla V_d|^2 dv = 0$$

$$S = \{S_0, S_1, \cdots, S_n\} \qquad \rightarrow V_d = const.$$

$$= 0$$

$\therefore$ V_1 and V_2 must be identical.

Bare Copper Wires