Introduction to Electromagnetism
Solution of Electrostatic Problems
(4-4)

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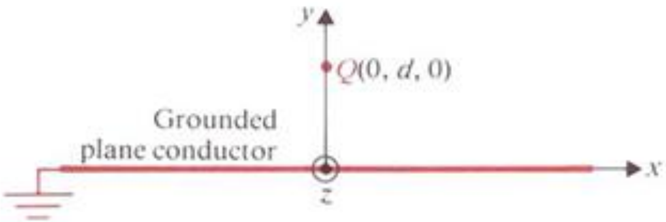
Method of Images

Boundary conditions for electrostatic problems:

Replaced by appropriate image (equivalent) charges

Example: Point charge and grounded plane conductor



(a) Physical arrangement.

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

$$\rightarrow V(x,y,z) = \frac{Q}{4\pi\varepsilon_0\sqrt{x^2+(y-d)^2+z^2}} + \frac{1}{4\pi\varepsilon_0}\int_S \frac{\rho_s}{R_1}ds$$

How to solve?

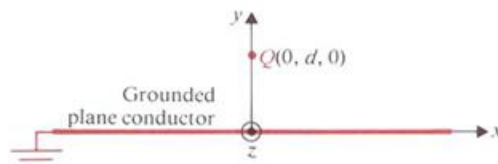
Point Charge and Grounded Plane Conductor

Laplace's equation:

$$\rightarrow \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Boundary conditions:

1. $V(x,0,z) = 0$
2. $V \rightarrow \frac{Q}{4\pi\varepsilon_0 R}$, as $R \rightarrow 0$
3. $V \rightarrow 0$, as $R \rightarrow \pm\infty$
4. $V(x,y,z) = V(-x,y,z)$,
   $V(x,y,z) = V(x,y,-z)$



(a) Physical arrangement.

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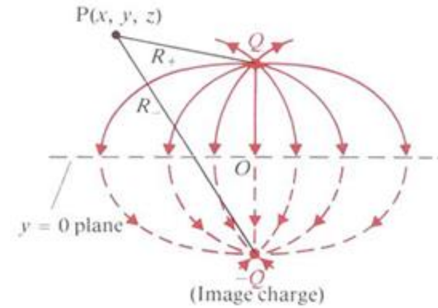


(b) Image charge and field lines.

$$\rightarrow \boxed{\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial V}{\partial R}\right)} + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

Method of images:

$$\rightarrow \boxed{V(x,y,z) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right)}$$

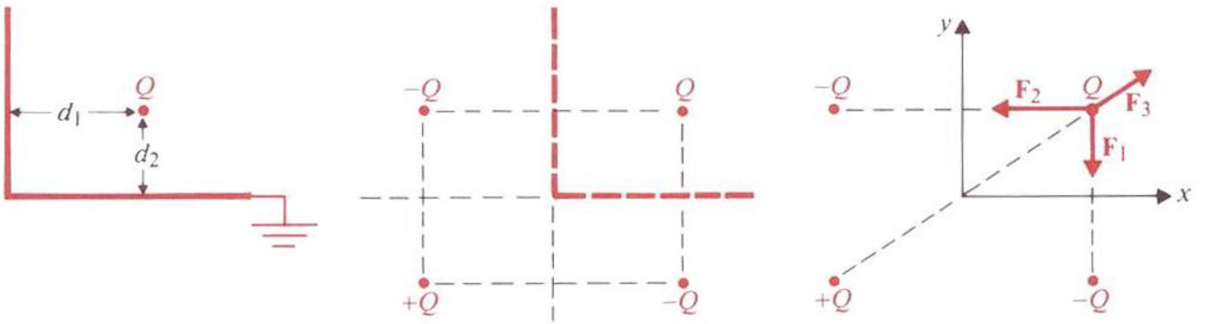
← Superposition of the solutions of Laplace's eq. satisfying the BCs

$$R_+ = [x^2 + (y-d)^2 + z^2]^{1/2}, \; R_- = [x^2 + (y+d)^2 + z^2]^{1/2}$$

Valid solution for $y < 0$?

Point Charge and Perpendicular Conducting Planes

Example 4-3: Image charges



(a) Physical arrangement. (b) Equivalent image-charge arrangement. (c) Forces on charge Q.

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Force on *Q* caused by the induced charges:

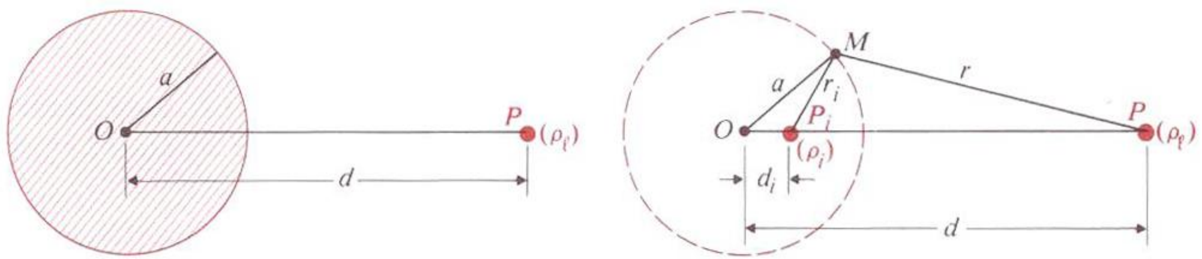
$$\mathbf{F}_1 = -\mathbf{a}_y \frac{Q^2}{4\pi\varepsilon_0 (2d_2)^2}$$

$$\mathbf{F}_2 = -\mathbf{a}_x \frac{Q^2}{4\pi\varepsilon_0 (2d_1)^2}$$

$$\mathbf{F}_3 = \frac{Q^2}{4\pi\varepsilon_0 [(2d_1)^2 + (2d_2)^2]^{3/2}} (\mathbf{a}_x 2d_1 + \mathbf{a}_y 2d_2)$$

$$\rightarrow \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$$

Line Charge and Parallel Conducting Cylinder



(a) Line charge and parallel conducting cylinder.

(b) Line charge and its image.

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Electric potential from a line charge:

$$V = -\int_{r_0}^{r} E_r dr = -\frac{\rho_l}{2\pi\varepsilon_0}\int_{r_0}^{r}\frac{1}{r}dr = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_0}{r}$$

Trial image charge:

$$\rightarrow \rho_i = -\rho_l \quad \text{at } d_i$$

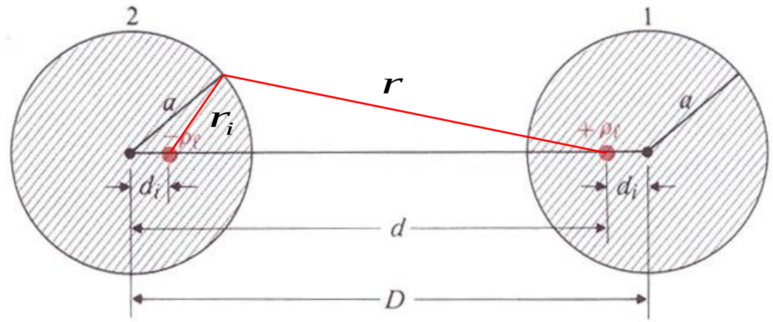
$$\rightarrow V_M = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_0}{r} - \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_0}{r_i} = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_i}{r} \leftarrow \text{Constant}$$

→ Triangles OMP_i and OPM must be similar!

$$\rightarrow \frac{\overline{P_iM}}{\overline{PM}} = \frac{\overline{OP_i}}{\overline{OM}} = \frac{\overline{OM}}{\overline{OP}} \quad \rightarrow \frac{r_i}{r} = \frac{d_i}{a} = \frac{a}{d} \quad \rightarrow \boxed{d_i = \frac{a^2}{d}}$$

Two-Wire Line of a Radius

Example 4-4: Capacitance per unit length of a two-wire transmission line



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Recall:

$$V_M = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_i}{r}$$
$$\rightarrow \frac{r_i}{r} = \frac{a}{d}$$

By image charges:

$$V_2 = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{a}{d}, \quad V_1 = -\frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{a}{d}$$

$$\ln[x+\sqrt{x^2-1}] = \cosh^{-1}x$$

$$\rightarrow C = \frac{\rho_l}{V_1 - V_2} = \frac{\pi\varepsilon_0}{\ln(d/a)}$$
$$= \frac{\pi\varepsilon_0}{\ln[(D/2a)+\sqrt{(D/2a)^2-1}]} = \frac{\pi\varepsilon_0}{\cosh^{-1}(D/2a)}$$

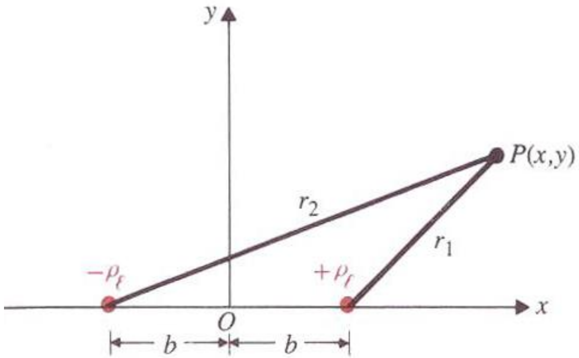
$$\rightarrow d = D - d_i = D - \frac{a^2}{d}$$

$$\rightarrow d = \frac{1}{2}(D+\sqrt{D^2-4a^2})$$

Two-Wire Line of Different Radii (1)

Potential at any point $P(x,y)$:

$$\rightarrow V_p = \frac{\rho_l}{2\pi\varepsilon_0} \ln \frac{r_2}{r_1}$$



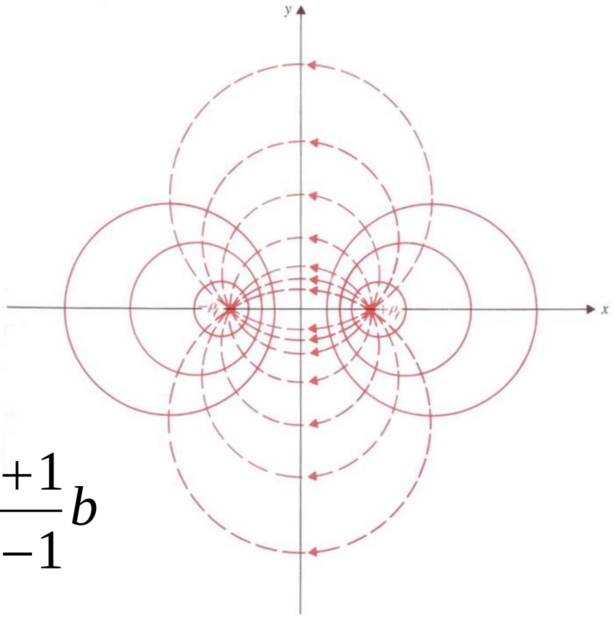
D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Equipoetntial line:

$$\rightarrow \frac{r_2}{r_1} = k \text{ (constant)}$$

$$\rightarrow \frac{r_2}{r_1} = \frac{\sqrt{(x+b)^2 + y^2}}{\sqrt{(x-b)^2 + y^2}} = k$$

$$\rightarrow (x - \frac{k^2+1}{k^2-1} b)^2 + y^2 = \left(\frac{2k}{k^2-1} b \right)^2$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Radius: $\rightarrow a = \left| \frac{2k}{k^2-1} b \right|$ Center: $\rightarrow c = \frac{k^2+1}{k^2-1} b$

$$\rightarrow c^2 = a^2 + b^2$$

$$\rightarrow +\rho_l \text{ for } k > 1, \quad -\rho_l \text{ for } k < 1$$

Two-Wire Line of Different Radii (2)

Two cylindrical conductors separated by a distance D:

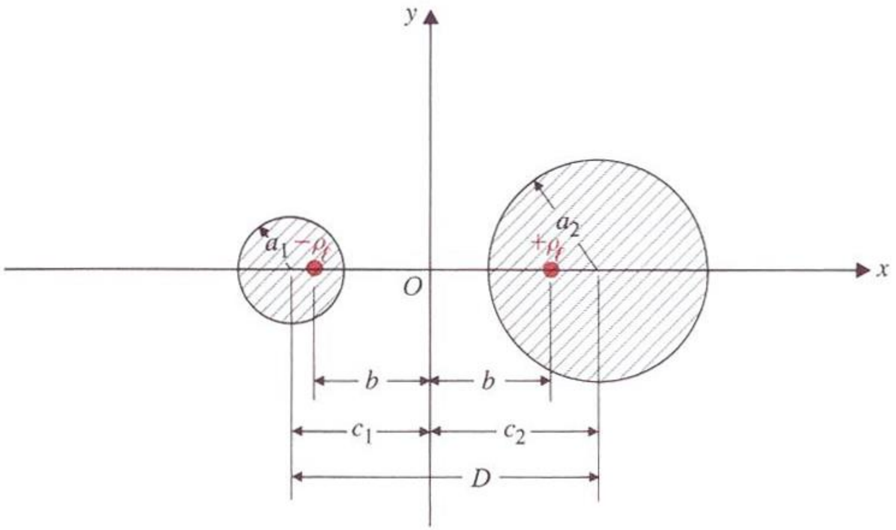
$$\rightarrow b^2 = c_1^2 - a_1^2$$

$$\rightarrow b^2 = c_2^2 - a_2^2$$

$$\rightarrow c_1 + c_2 = D$$

$$\rightarrow c_1 = \frac{1}{2D}(D^2 + a_1^2 - a_2^2)$$

$$\rightarrow c_2 = \frac{1}{2D}(D^2 + a_2^2 - a_1^2)$$



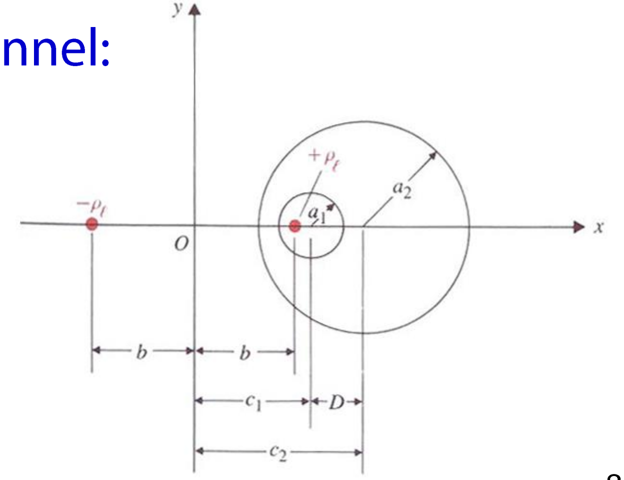
D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

An off-center conductor inside a cylindrical tunnel:

$$\rightarrow c_2 - c_1 = D$$

$$\rightarrow c_1 = \frac{1}{2D}(a_2^2 - a_1^2 - D^2)$$

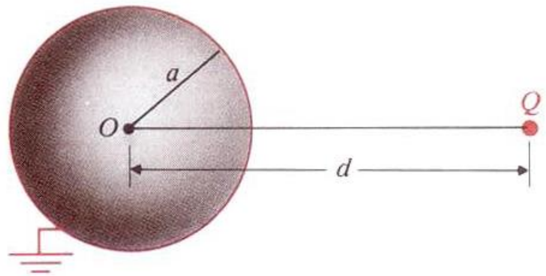
$$\rightarrow c_2 = \frac{1}{2D}(a_2^2 - a_1^2 + D^2)$$



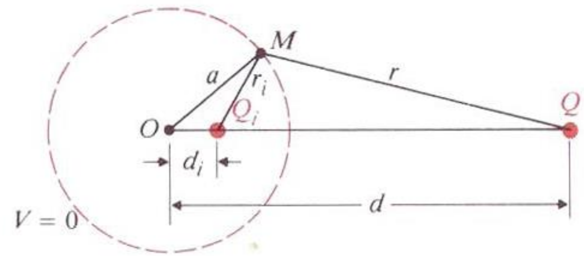
D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Point Charge and Conducting Sphere

Potential on the spherical surface grounded:



(a) Point charge and grounded conducting sphere.



(b) Point charge and its image.

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

$$\rightarrow V_M = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{Q_i}{r_i}\right) = 0 \quad \leftarrow \quad Q_i = -Q\,?$$

$$\rightarrow \frac{r_i}{r} = -\frac{Q_i}{Q} = \frac{a}{d}$$

$$\rightarrow \boxed{Q_i = -\frac{a}{d}Q} \quad \rightarrow \boxed{d_i = \frac{a^2}{d}}$$

Recall:

$$\rightarrow \frac{r_i}{r} = \frac{d_i}{a} = \frac{a}{d}$$

For a conducting sphere that is electrically neutral and is not grounded:

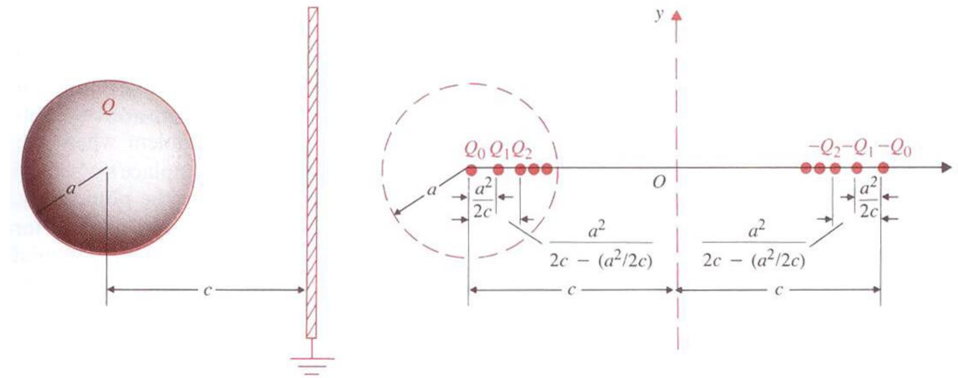
Additional image charge to the center: $\rightarrow Q' = -Q_i = \dfrac{a}{d}Q$

Charged Sphere and Grounded Plane

Boundary conditions:

(1) Equipotential on the conducting sphere

(2) Zero potential at $x = 0$



(a) Physical arrangement. (b) Two groups of image point charges.

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Image charge for the B.C. at $x = 0$

$$\rightarrow Q_0 \;\; at\ (-c_0, 0) \qquad \rightarrow Q_0 \;\; at\ (+c_0, 0)$$

Image charge for the B.C. on the sphere

$$\rightarrow Q_1 \;\; at\ (-c_1, 0) \qquad \rightarrow Q_1 \;\; at\ (+c_1, 0)$$

$$\rightarrow Q_2 \;\; at\ (-c_2, 0) \qquad \rightarrow Q_2 \;\; at\ (+c_2, 0)$$

$$\vdots \qquad\qquad \vdots$$

$$\rightarrow Q = \sum_{k=0}^{\infty} Q_k = Q_0\left(1 + \alpha + \frac{\alpha^2}{1-\alpha^2} + \cdots\right) \leftarrow \alpha = \frac{a}{2c}$$

Capacitance: $\rightarrow V_0 = \dfrac{Q_0}{4\pi\varepsilon_0 a} \quad \rightarrow C = \dfrac{Q}{V_0} = 4\pi\varepsilon_0 a\left(1 + \alpha + \dfrac{\alpha^2}{1-\alpha^2} + \cdots\right)$