Introduction to Electromagnetism
Solution of Electrostatic Problems
(4-5, 4-6, 4-7)

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BVPs in Cartesian Coordinates (1)

Boundary-value problems:

(1) Dirichlet problems: Potential specified on the boundaries
(2) Neumann problems: Normal derivative of the potential specified
(3) Mixed boundary-value problems

Laplace's equation in Cartesian coordinates:

$$\rightarrow \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$ *How to solve?*

Separation of variables:

$$\rightarrow V(x, y, z) = X(x)Y(y)Z(z)$$

$$\rightarrow Y(y)Z(z)\frac{d^2 X(x)}{dx^2} + X(x)Z(z)\frac{d^2 Y(y)}{dy^2} + X(x)Y(z)\frac{d^2 Z(z)}{dz^2} = 0$$

$$\rightarrow \frac{1}{X(x)}\frac{d^2 X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2 Y(y)}{dy^2} + \frac{1}{Z(z)}\frac{d^2 Z(z)}{dz^2} = 0$$

$$\rightarrow \frac{1}{X(x)}\frac{d^2 X(x)}{dx^2} = -k_x^2$$

BVPs in Cartesian Coordinates (2)

In result:

$$\rightarrow \frac{d^2X(x)}{dx^2} + k_x^2 X(x) = 0$$

$$\rightarrow \frac{d^2Y(y)}{dy^2} + k_y^2 Y(y) = 0$$

$$\rightarrow \frac{d^2Z(z)}{dz^2} + k_z^2 Z(z) = 0$$

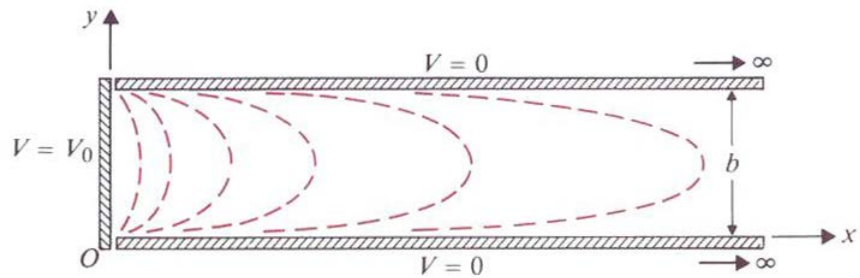
$$\rightarrow k_x^2 + k_y^2 + k_z^2 = 0$$

Possible solutions:

k_x^2	k_x	$X(x)$	Exponential forms† of $X(x)$
0	0	$A_0 x + B_0$	
+	k	$A_1 \sin kx + B_1 \cos kx$	$C_1 e^{jkx} + D_1 e^{-jkx}$
−	jk	$A_2 \sinh kx + B_2 \cosh kx$	$C_2 e^{kx} + D_2 e^{-kx}$

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Example 4-6 (1)



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With V independent of z:

$$\rightarrow V(x,y,z)=V(x,y)$$

Boundary conditions:

$$\rightarrow V(0,y)=V_0, \quad V(\infty,y)=0$$

$$\rightarrow V(x,0)=0, \quad V(x,b)=0$$

Solution form:

$$\rightarrow Z(z)=G_3 \;\rightarrow k_z=0 \;\rightarrow k_y^2=-k_x^2=k^2$$

$$\rightarrow X(x)=G_1 e^{-kx}$$

$$\rightarrow Y(y)=G_2 \sin ky$$

$$\rightarrow V_n(x,y)=G_1 G_2 G_3 e^{-kx}\sin ky = C_n e^{-kx}\sin ky$$

Boundary conditions:

$$\rightarrow V_n(x,b)=C_n e^{-kx}\sin kb = 0$$

$$\rightarrow k=\frac{n\pi}{b} \quad (n=1,2,3,\cdots)$$

Example 4-6 (2)

Boundary conditions:

$$\rightarrow V_n(x,y) = C_n e^{-n\pi x/b} \sin\frac{n\pi}{b}y$$

$$\rightarrow V(x,y) = \sum_{n=1}^{\infty} V_n(x,y)$$

$$\rightarrow V(0,y) = \sum_{n=1}^{\infty} V_n(0,y) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{b}y = V_0, \quad 0 < y < b$$

Determination of the coefficients C_n:

$$\rightarrow \sum_{n=1}^{\infty} \int_0^b C_n \sin\frac{n\pi}{b}y \sin\frac{m\pi}{b}y\,dy = \int_0^b V_0 \sin\frac{m\pi}{b}y\,dy$$

$$\rightarrow \int_0^b V_0 \sin\frac{m\pi}{b}y\,dy = \begin{cases} \dfrac{2bV_0}{m\pi} & \text{if } m \text{ is odd} \\ 0 & \text{if } m \text{ is even} \end{cases}$$

$$\rightarrow \int_0^b C_n \sin\frac{n\pi}{b}y \sin\frac{m\pi}{b}y\,dy = \frac{C_n}{2}\int_0^b \left[\cos\frac{(n-m)\pi}{b}y - \cos\frac{(n+m)\pi}{b}y\right]dy$$

$$= \begin{cases} \dfrac{C_n}{2}b & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \qquad C_n = \begin{cases} \dfrac{4V_0}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

BVPs in Cylindrical Coordinates (1)

Laplace's equation:

$$\rightarrow \nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

← Bessel functions

In case:

$$\frac{\partial^2 V}{\partial z^2} = 0 \quad \rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Separation of variables:

$$\rightarrow V(r,\phi) = R(r)\Phi(\phi)$$

$$\rightarrow \frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) + \frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2} = 0$$

$$\rightarrow \frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) = k^2, \quad \frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2} = -k^2$$

Possible solution:

$$\rightarrow \frac{d^2\Phi(\phi)}{d\phi^2} + k^2\Phi(\phi) = 0 \quad \rightarrow \Phi(\phi) = A_\phi \sin n\phi + B_\phi \cos n\phi$$

← Periodic in ϕ

BVPs in Cylindrical Coordinates (2)

Solution form:

$$\rightarrow \Phi(\phi) = A_\phi \sin n\phi + B_\phi \cos n\phi$$

$$\rightarrow \; r^2 \frac{d^2 R(r)}{dr^2} + r\frac{dR(r)}{dr} - n^2 R(r) = 0$$

$$\rightarrow \; R(r) = A_r r^n + B_r r^{-n}$$

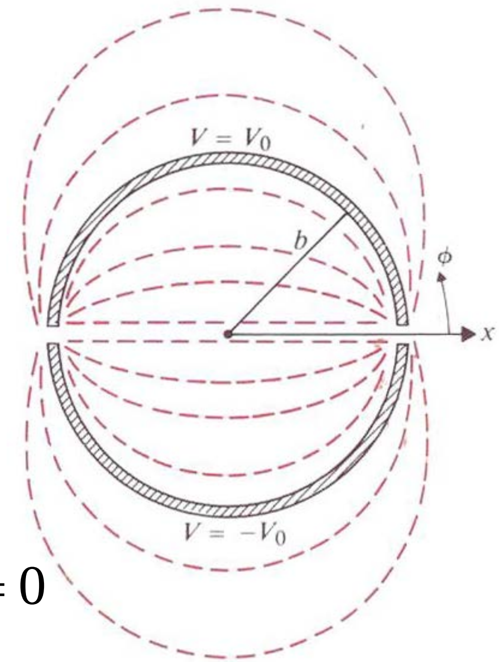
$$\rightarrow \; V_n(r,\phi) = r^n (A_n \sin n\phi + B_n \cos n\phi) + r^{-n}(A_n' \sin n\phi + B_n' \cos n\phi), \quad n \neq 0$$

What if $k = 0$:

$$\rightarrow \; \frac{d^2\Phi(\phi)}{d\phi^2} + k^2\Phi(\phi) = 0 \;\; \rightarrow \; \frac{d^2\Phi(\phi)}{d\phi^2} = 0 \;\; \rightarrow \; \Phi(\phi) = A_0\phi + B_0$$

$$\rightarrow \; \frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) = 0 \;\; \rightarrow \; \frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) = 0 \;\; \rightarrow \; R(r) = C_0 \ln r + D_0$$

Example 4-9 (1)



Boundary conditions:

$$V(b,\phi)=\begin{cases} V_0 & \text{for } 0<\phi<\pi \\ -V_0 & \text{for } \pi<\phi<2\pi \end{cases}$$

Inside the tube:

$$\rightarrow V_n(r,\phi)=r^n(A_n\sin n\phi+B_n\cos n\phi)$$
$$+r^{-n}(A_n'\sin n\phi+B_n'\cos n\phi),\quad n\neq 0$$

$$\rightarrow V(r,\phi)=\sum_{n=1}^{\infty}A_n r^n\sin n\phi$$

$$\rightarrow V(b,\phi)=\sum_{n=1}^{\infty}A_n b^n\sin n\phi=\begin{cases} V_0 & \text{for } 0<\phi<\pi \\ -V_0 & \text{for } \pi<\phi<2\pi \end{cases}$$

$$\rightarrow A_n=\begin{cases} \dfrac{4V_0}{n\pi b^n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$\rightarrow V(r,\phi)=\frac{4V_0}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{1}{n}\left(\frac{r}{b}\right)^n\sin n\phi,\quad r<b$$

Example 4-9 (2)

Outside the tube:

$$\rightarrow V_n(r,\phi) = r^n(A_n \sin n\phi + B_n \cos n\phi) + r^{-n}(A_n' \sin n\phi + B_n' \cos n\phi), \quad n \neq 0$$

$$\rightarrow V_n(r,\phi) = A_n' r^{-n} \sin n\phi$$

$$\rightarrow V(r,\phi) = \sum_{n=1}^{\infty} A_n' r^{-n} \sin n\phi$$

$$\rightarrow V(b,\phi) = \sum_{n=1}^{\infty} A_n' b^{-n} \sin n\phi = \begin{cases} V_0 & \text{for } 0 < \phi < \pi \\ -V_0 & \text{for } \pi < \phi < 2\pi \end{cases}$$

$$\rightarrow A_n' = \begin{cases} \dfrac{4V_0 b^n}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$\rightarrow V(r,\phi) = \frac{4V_0}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{1}{n}\left(\frac{b}{r}\right)^n \sin n\phi, \quad r > b$$

BVPs in Spherical Coordinates (1)

Laplace's equation:

$$\rightarrow \nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial r}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

In case:

$$\frac{\partial^2 V}{\partial\phi^2} = 0 \qquad \rightarrow \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial r}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0$$

Separation of variables:

$$\rightarrow V(R,\theta) = \Gamma(R)\Theta(\theta)$$

$$\rightarrow \frac{1}{\Gamma(R)}\frac{d}{dR}\left(R^2\frac{d\Gamma(R)}{dR}\right) + \frac{1}{\Theta(\theta)\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right) = 0$$

$$\rightarrow \frac{1}{\Gamma(R)}\frac{d}{dR}\left(R^2\frac{d\Gamma(R)}{dR}\right) = k^2, \quad \frac{1}{\Theta(\theta)\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right) = -k^2$$

Solution form:

$$\rightarrow R^2\frac{d^2\Gamma(R)}{dR^2} + 2R\frac{d\Gamma(R)}{dR} - k^2\Gamma(R) = 0$$

$$\rightarrow \Gamma_n(R) = A_n R^n + B_n R^{-(n+1)}, \quad n(n+1) = k^2$$

BVPs in Spherical Coordinates (2)

Solution form:

$$\rightarrow \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta(\theta)}{d\theta}\right) + n(n+1)\Theta(\theta)\sin\theta = 0$$ ← *Legendre's equation*

$\rightarrow \Theta(\theta)$: *Legendre functions*

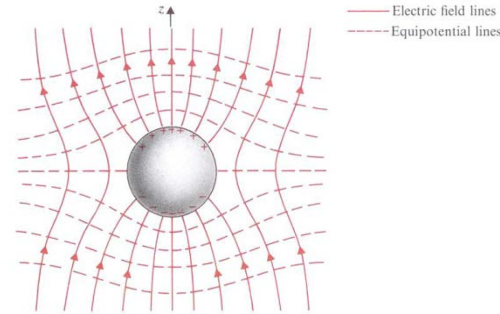
→ *Legendre polynomials for integral* n

$$\rightarrow \Theta(\theta) = P_n(\cos\theta)$$

$$\rightarrow \Gamma_n(R) = A_n R^n + B_n R^{-(n+1)}, \quad n(n+1) = k^2$$

$$\rightarrow V_n(R,\theta) = \left[A_n R^n + B_n R^{-(n+1)}\right] P_n(\cos\theta)$$

Example 4-10 (1)



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Boundary conditions:

$$V(b,\theta)=0$$

$$V(R,\theta)=-E_0 z=-E_0 R\cos\theta, \quad \text{for } R>>b$$

Solution:

$$V_n(R,\theta)=\left[A_n R^n+B_n R^{-(n+1)}\right]P_n(\cos\theta), \quad R\geq b$$

$$\rightarrow V(R,\theta)=\underline{-E_0 R P_1(\cos\theta)}+\sum_{n=0}^{\infty} B_n R^{-(n+1)} P_n(\cos\theta), \quad R\geq b$$

← Asymptotic for $R>>b$

$$=\frac{B_0}{R}+\left(\frac{B_1}{R^2}-E_0 R\right)\cos\theta+\sum_{n=2}^{\infty} B_n R^{-(n+1)} P_n(\cos\theta)$$

No net charge in the sphere, i.e., $V(b,\theta)=0$

Boundary condition at $R=b$:

$$\rightarrow 0=\left(\frac{B_1}{b^2}-E_0 b\right)\cos\theta+\sum_{n=2}^{\infty} B_n b^{-(n+1)} P_n(\cos\theta)$$

$$\rightarrow B_1=E_0 b^3, \quad B_n=0 \quad (n\geq 2)$$

$$\rightarrow V(R,\theta)=-E_0\left[1-\left(\frac{b}{R}\right)^3\right]R\cos\theta, \quad R\geq b$$

TABLE 4–2
Several Legendre Polynomials

n	$P_n(\cos\theta)$
0	1
1	$\cos\theta$
2	$\frac{1}{2}(3\cos^2\theta-1)$
3	$\frac{1}{2}(5\cos^3\theta-3\cos\theta)$

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Example 4-10 (2)

Electric potential:

$$\rightarrow V(R,\theta) = -E_0\left[1-\left(\frac{b}{R}\right)^3\right]R\cos\theta, \quad R \geq b$$

Electric field:

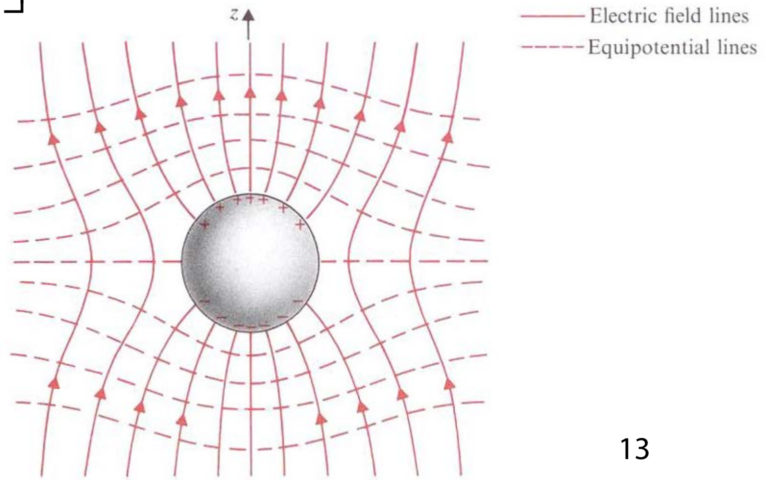
$$\rightarrow \mathbf{E}(R,\theta) = -\nabla V = \mathbf{a}_R E_R + \mathbf{a}_\theta E_\theta, \quad R \geq b$$

$$\rightarrow E_R = -\frac{\partial V}{\partial R} = E_0\left[1+2\left(\frac{b}{R}\right)^3\right]\cos\theta, \quad R \geq b$$

$$\rightarrow E_\theta = -\frac{\partial V}{R\partial \theta} = -E_0\left[1-\left(\frac{b}{R}\right)^3\right]\sin\theta, \quad R \geq b$$

Surface charge density:

$$\rightarrow \rho_s(\theta) = \varepsilon_0 E_R\big|_{R=b} = 3\varepsilon_0 E_0 \cos\theta$$



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