Introduction to Electromagnetism
Steady Electric Currents
(5-5, 5-6)

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Power Dissipation and Joule's Law

Thermal vibration (heat dissipation) by the drift motion of electrons:

→ Work done by E-field for a charge q:

$$p = \lim_{\Delta t \to 0} \frac{\Delta w}{\Delta t} = q\mathbf{E} \cdot \mathbf{u}$$

Total power delivered to all the charges:

$$dP = \sum_i p_i = \mathbf{E} \cdot \left(\sum_i N_i q_i \mathbf{u}_i \right) dv$$

$$\rightarrow \; dP = \sum_i p_i = \underline{\mathbf{E} \cdot \mathbf{J} dv}$$

→ Power density

Joule's law: $P = \int_V \mathbf{E} \cdot \mathbf{J} dv \quad \text{(W)}$

For a conductor of a constant cross section:

$$P = \int_L E dl \int_S J ds = VI = I^2 R \quad \text{(W)}$$

Boundary Conditions for Current Density

Governing equations for steady current density:

Differential form | Integral form

$$\nabla \cdot \mathbf{J} = 0 \quad \rightarrow \quad \oint_S \mathbf{J} \cdot d\mathbf{s} = 0$$

$$\nabla \times \left(\frac{\mathbf{J}}{\sigma} \right) = 0 \quad \rightarrow \quad \oint_C \frac{1}{\sigma} \mathbf{J} \cdot d\mathbf{l} = 0$$

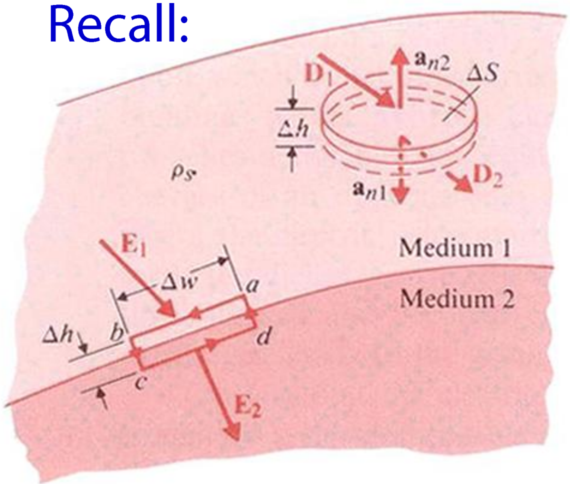
Normal component:

$$J_{1n} = J_{2n}$$

Tangential component:

$$\frac{J_{1t}}{\sigma_1} = \frac{J_{2t}}{\sigma_2} \rightarrow \frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}$$

Recall:



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Homogeneous Conducting Medium

For a homogeneous medium:

$$\nabla \cdot \mathbf{J} = 0$$

$$\nabla \times \left(\frac{\mathbf{J}}{\sigma} \right) = 0 \;\rightarrow\; \nabla \times \mathbf{J} = 0 \;\rightarrow\; \mathbf{J} = -\nabla \psi$$

$$\rightarrow\; \nabla^2 \psi = 0$$ ← Laplace's equation

Boundary conditions:

$$J_{1n} = J_{2n} \;\rightarrow\; \sigma_1 E_{1n} = \sigma_2 E_{2n}$$

$$\frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}$$

$$E_{1t} = E_{2t}$$

← Equivalent

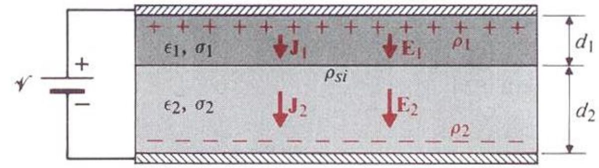
$$D_{1n} - D_{2n} = \rho_s \;\rightarrow\; \varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$$

$$\rightarrow\; \rho_s = \left(\varepsilon_1 \frac{\sigma_2}{\sigma_1} - \varepsilon_2 \right) E_{2n} = \left(\varepsilon_1 - \varepsilon_2 \frac{\sigma_1}{\sigma_2} \right) E_{1n}$$

$$\rightarrow\; \rho_s = \varepsilon_1 E_{1n} \quad (\sigma_2 >> \sigma_1)$$

Example 5-4

For a parallel plate with two lossy dielectrics:



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By Kirchhoff's voltage law:

$$V = (R_1 + R_2)I = \left(\frac{d_1}{\sigma_1 S} + \frac{d_2}{\sigma_2 S}\right)I$$

Current density: $\rightarrow J = \frac{I}{S} = \frac{\sigma_1\sigma_2 V}{\sigma_2 d_1 + \sigma_1 d_2}$

Electric fields:

$$\left.\begin{aligned} V &= E_1 d_1 + E_2 d_2 \\ \sigma_1 E_1 &= \sigma_2 E_2 \end{aligned}\right\} \rightarrow E_1 = \frac{\sigma_2 V}{\sigma_2 d_1 + \sigma_1 d_2}, \quad E_2 = \frac{\sigma_1 V}{\sigma_2 d_1 + \sigma_1 d_2}$$

Surface charge densities:

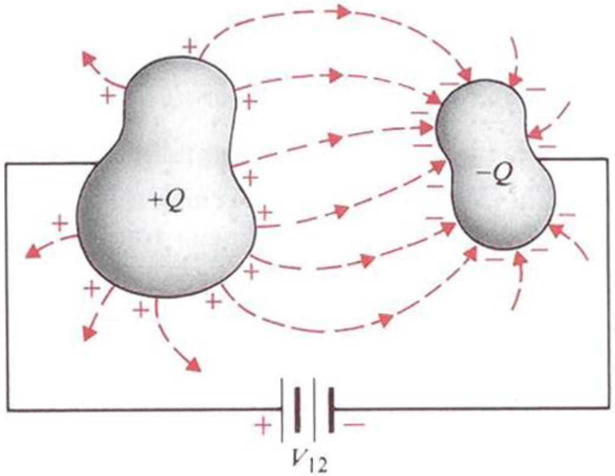
$$\rho_{s1} = \varepsilon_1 E_1, \quad \rho_{s2} = -\varepsilon_2 E_2$$

$$\rho_{si} = \left(\varepsilon_2 \frac{\sigma_1}{\sigma_2} - \varepsilon_1\right)E_1 = \frac{\varepsilon_1\varepsilon_2 V}{\sigma_2 d_1 + \sigma_1 d_2}\left(\frac{\sigma_1}{\varepsilon_1} - \frac{\sigma_2}{\varepsilon_2}\right)$$

$$\rightarrow \rho_{s1} + \rho_{s2} \neq 0 \quad \rightarrow \boxed{\rho_{s1} + \rho_{s2} + \rho_{si} = 0}$$

Resistance Calculations

Capacitance and resistance:



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$$C = \frac{Q}{V} = \frac{\oint_S \mathbf{D} \cdot d\mathbf{s}}{-\int_L \mathbf{E} \cdot d\mathbf{l}} = \frac{\oint_S \varepsilon \mathbf{E} \cdot d\mathbf{s}}{-\int_L \mathbf{E} \cdot d\mathbf{l}}$$

$$R = \frac{V}{I} = \frac{-\int_L \mathbf{E} \cdot d\mathbf{l}}{\oint_S \mathbf{J} \cdot d\mathbf{s}} = \frac{-\int_L \mathbf{E} \cdot d\mathbf{l}}{\oint_S \sigma \mathbf{E} \cdot d\mathbf{s}}$$

Relationship between capacitance and resistance:

$$RC = \frac{C}{G} = \frac{\varepsilon}{\sigma}$$

← In case the medium is homogeneous or
ε and σ are of the same space dependence.

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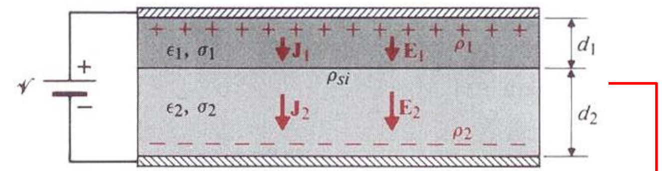


By Kirchhoff's voltage law:

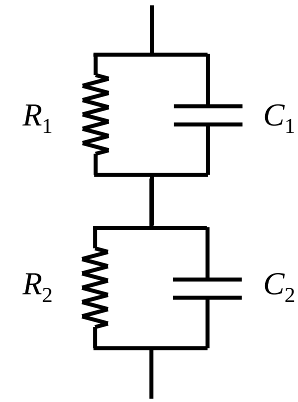
$$\mathcal{V} = (R_1 + R_2)I = \left(\frac{d_1}{\sigma_1 S} + \frac{d_2}{\sigma_2 S}\right)I$$

$$= \frac{q_1}{C_1} + \frac{q_2}{C_2}$$

$$\rightarrow \ C_1 = \frac{\varepsilon_1 S}{d_1}, \ C_2 = \frac{\varepsilon_2 S}{d_2}$$

Recall:

$$RC = \frac{C}{G} = \frac{\varepsilon}{\sigma}$$



Equivalent circuit

Recall the surface charge densities: $\rho_{s1} = \varepsilon_1 E_1, \ \rho_{s2} = -\varepsilon_2 E_2$

$$\rightarrow \ q_1 = \rho_{s1} S = \varepsilon_1 E_1 S, \ q_2 = -\rho_{s2} S = \varepsilon_2 E_2 S$$

$$\rightarrow \ \frac{q_1}{C_1} + \frac{q_2}{C_2} = \frac{d_1}{\varepsilon_1 S}\varepsilon_1 E_1 S + \frac{d_2}{\varepsilon_2 S}\varepsilon_2 E_2 S = E_1 d_1 + E_2 d_2 = \mathcal{V}$$