Introduction to Electromagnetism
Static Magnetic Fields
(6-1, 6-2, 6-3)

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Introduction to Static Magnetic Fields

Basis of the electrostatic model:

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \times \mathbf{E} = 0$$

$$\mathbf{D} = \varepsilon \mathbf{E} \quad \leftarrow \text{Constitutive relation}$$

$$\boxed{\mathbf{F}_e = q\mathbf{E} \quad (\mathrm{N})} \quad \leftarrow \text{Electric force}$$

Test charge in motion in a "magnetic field":

Magnetic flux density

$$\boxed{\mathbf{F}_m = q\mathbf{u} \times \mathbf{B} \quad (\mathrm{N})} \quad \leftarrow \text{Magnetic force}$$

Electromagnetic force:

$$\boxed{\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \quad (\mathrm{N})}$$

$\rightarrow$ Lorentz's force equation

Postulates of Magnetostatics in Free Space (1)

Two fundamental postulates:

Right-hand rule

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad \rightarrow \quad \nabla \cdot \mathbf{J} = 0$$

$$\mu_0 = 4\pi \times 10^{-7} \quad \text{(H/m)}$$ → Permeability

Non-existence of isolated magnetic charges:

$$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$$

→ There are no magnetic flow sources: The magnetic flux lines always close upon themselves.

Ampère's circuital law:

$$\int_S \nabla \times \mathbf{B} \cdot d\mathbf{s} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{s}$$

$$\rightarrow \oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{s}$$

Postulates of Magnetostatics in Free Space (2)

Differential form:

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

Integral form:

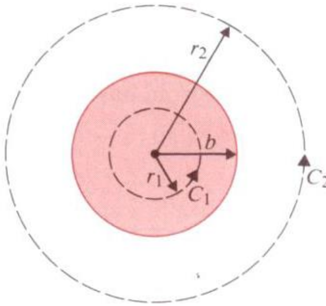
$$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$$

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{s}$$

Example 6-1

For an infinitely long, straight conductor:

a) Inside the conductor:



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

$$\oint_C \mathbf{B}\cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J}\cdot d\mathbf{s}$$

$$\rightarrow \mathbf{B}_1 = \mathbf{a}_\phi B_{\phi 1}, \quad d\mathbf{l} = \mathbf{a}_\phi r_1 d\phi$$

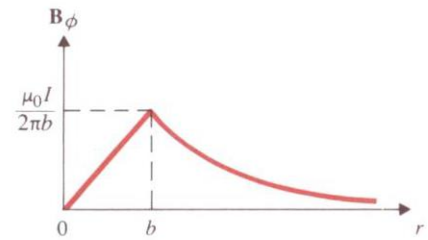
$$\rightarrow \oint_{C_1} \mathbf{B}_1 \cdot d\mathbf{l} = \oint_{C_1} B_{\phi 1} r_1 d\phi = 2\pi r_1 B_{\phi 1}$$

$$\rightarrow \mu_0 \int_{S_1} \mathbf{J}_1 \cdot d\mathbf{s} = \mu_0 I_1 = \mu_0 \frac{\pi r_1^2}{\pi b^2} I = \mu_0 \left(\frac{r_1}{b}\right)^2 I$$

$$\rightarrow \mathbf{B}_1 = \mathbf{a}_\phi B_{\phi 1} = \mathbf{a}_\phi \frac{\mu_0 r_1 I}{2\pi b^2}, \quad r \le b$$

b) Outside the conductor:

$$\rightarrow \mathbf{B}_2 = \mathbf{a}_\phi B_{\phi 2} = \mathbf{a}_\phi \frac{\mu_0 I}{2\pi r_2}, \quad r \ge b$$

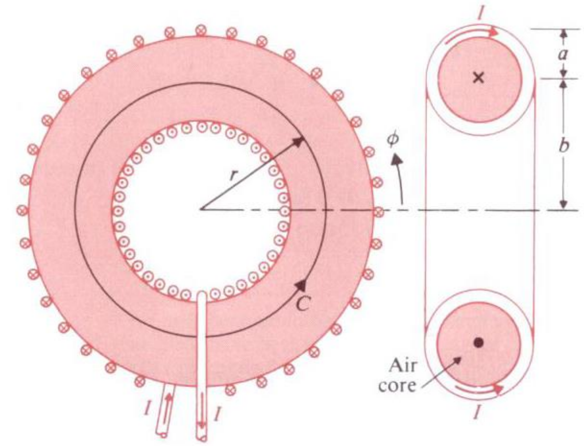


D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Example 6-2

For a toroidal coil:

a) Inside the toroid:



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{s}$$

$$\rightarrow \ \mathbf{B} = \mathbf{a}_\phi B_\phi, \quad d\mathbf{l} = \mathbf{a}_\phi r d\phi$$

$$\rightarrow \ \oint_C \mathbf{B} \cdot d\mathbf{l} = 2\pi r B_\phi = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{s} = \mu_0 NI$$

$$\rightarrow \ \mathbf{B} = \mathbf{a}_\phi B_\phi = \mathbf{a}_\phi \frac{\mu_0 NI}{2\pi r}, \quad (b-a) < r < (b+a)$$

b) Outside the toroid:

$$\rightarrow \ \mathbf{B} = 0, \quad r < (b-a), r > (b+a)$$

Why?

Example 6-3

For an infinitely long solenoid:

a) Outside the solenoid:

$$\mathbf{B} = 0$$

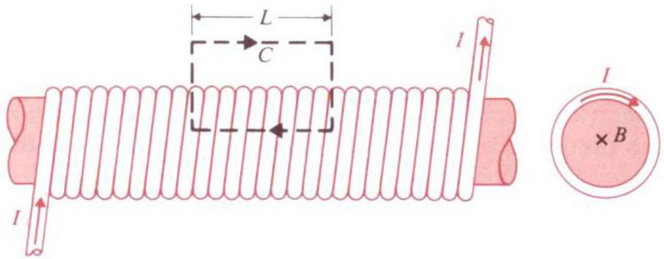
b) Inside the solenoid:

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{s}$$

$$\rightarrow \quad BL = \mu_0 nLI$$

$$\rightarrow \quad B = \mu_0 nI$$

Turns per unit length



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Vector Magnetic Potential

From the divergence-free postulate of **B**:

$$\nabla \cdot \mathbf{B} = 0$$

$$\rightarrow \boxed{\mathbf{B} = \nabla \times \mathbf{A} \quad (\mathrm{T})} \quad \rightarrow \text{Vector magnetic potential}$$

Recall: Helmholtz's theorem

$$\left.\begin{array}{l} \nabla \times \mathbf{A} = \mathbf{B} \\ \nabla \cdot \mathbf{A} = ? \end{array}\right\} \rightarrow \text{In order to define } \mathbf{A}$$

$$\nabla^2 \mathbf{A} = \mathbf{a}_x \nabla^2 A_x + \mathbf{a}_y \nabla^2 A_y + \mathbf{a}_z \nabla^2 A_z$$

$$\rightarrow \nabla \times \mathbf{B} = \nabla \times \nabla \times \mathbf{A} = \mu_0 \mathbf{J}$$

$$\rightarrow \nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$$

Coulomb condition (or Coulomb gauge):

$$\rightarrow \boxed{\nabla \cdot \mathbf{A} = 0}$$

Vector Poisson's equation:

$$\rightarrow \boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}} \qquad \text{Recall: } \nabla^2 V = -\frac{\rho}{\varepsilon_0} \rightarrow V = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho}{R} dv'$$

$$\rightarrow \boxed{\mathbf{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J}}{R} dv' \quad (\mathrm{Wb/m})}$$

Magnetic Flux

Magnetic flux:

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s}$$

$$\rightarrow \ \Phi = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_C \mathbf{A} \cdot d\mathbf{l} \quad (\text{Wb})$$

→ "**A**" does have physical significance.

Aharonov-Bohm effect:

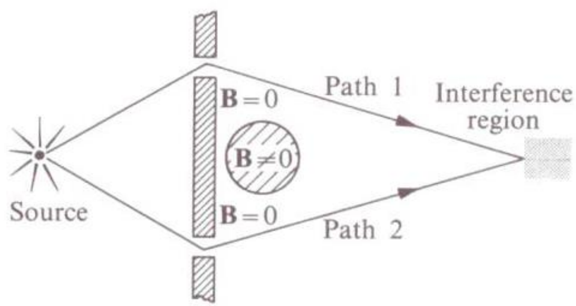


Fig. 1–2. An idealized experimental arrangement to illustrate the Aharonov-Bohm effect.

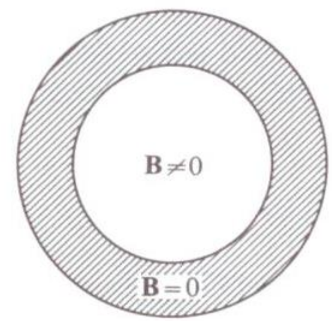


Fig. 1–3. Magnetic flux trapped by a superconductor ring.

J. J. Sakurai, Advanced Quantum Mechanics, Addison-Wesley, 1967.

$$\rightarrow \ \psi = \psi_1^{(0)} \exp\left[\frac{iq}{\hbar c}\int_{Path\,1} \mathbf{A} \cdot d\mathbf{l}\right] + \psi_2^{(0)} \exp\left[\frac{iq}{\hbar c}\int_{Path\,2} \mathbf{A} \cdot d\mathbf{l}\right]$$

→ *Interference dependent on the strength of the magnetic field inside the solenoid even with* $\mathbf{B} = 0$ *everywhere outside!*