

Introduction to Electromagnetism

Static Magnetic Fields

(6-4, 6-5, 6-6)

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Biot-Savart Law and Applications

Vector magnetic potential:

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J}}{R} dv'$$

$$\rightarrow \mathbf{J}dv' = JSd\mathbf{l}' = Id\mathbf{l}' \quad \leftarrow \text{For a wire with a cross-section } S$$

$$\rightarrow \mathbf{A} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\mathbf{l}'}{R} \quad (\text{Wb/m})$$

$$\begin{aligned} \rightarrow \mathbf{B} = \nabla \times \mathbf{A} &= \nabla \times \left[\frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\mathbf{l}'}{R} \right] = \frac{\mu_0 I}{4\pi} \oint_{C'} \nabla \times \left(\frac{d\mathbf{l}'}{R} \right) \\ &= \frac{\mu_0 I}{4\pi} \oint_{C'} \left[\frac{1}{R} \nabla \times d\mathbf{l}' + \left(\nabla \frac{1}{R} \right) \times d\mathbf{l}' \right] \\ &= \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\mathbf{l}' \times \mathbf{a}_R}{R^2} \quad (\text{T}) \quad \leftarrow \text{Bio-Savart law} \end{aligned}$$

$$\mathbf{B} = \oint_{C'} d\mathbf{B} \rightarrow d\mathbf{B} = \frac{\mu_0 I}{4\pi} \left(\frac{d\mathbf{l}' \times \mathbf{a}_R}{R^2} \right) = \frac{\mu_0 I}{4\pi} \left(\frac{d\mathbf{l}' \times \mathbf{R}}{R^3} \right)$$

Example 6-4

For a current-carrying straight wire:

a) By finding \mathbf{B} from \mathbf{A} :

$$\rightarrow \mathbf{A} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\mathbf{l}'}{R} \rightarrow \mathbf{B} = \nabla \times \mathbf{A}$$

b) By applying Biot-Savart law:

$$\rightarrow d\mathbf{B} = \frac{\mu_0 I}{4\pi} \left(\frac{d\mathbf{l}' \times \mathbf{R}}{R^3} \right)$$

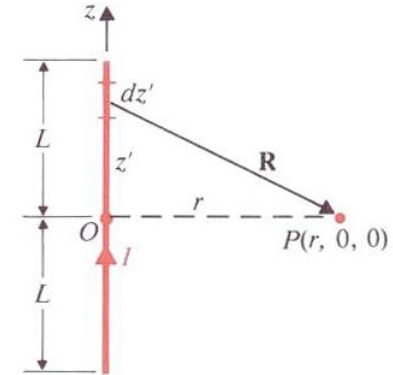
$$\rightarrow \mathbf{R} = \mathbf{a}_r r - \mathbf{a}_z z'$$

$$\rightarrow d\mathbf{l}' \times \mathbf{R} = \mathbf{a}_z dz' \times (\mathbf{a}_r r - \mathbf{a}_z z') = \mathbf{a}_\phi r dz'$$

$$\rightarrow \mathbf{B} = \int d\mathbf{B} = \mathbf{a}_\phi \frac{\mu_0 I}{4\pi} \int_{-L}^L \frac{r dz'}{(z'^2 + r^2)^{3/2}}$$

$$= \mathbf{a}_\phi \frac{\mu_0 I}{4\pi} \frac{1}{r} \left[\frac{z'}{\sqrt{z'^2 + r^2}} \right]_{-L}^L = \mathbf{a}_\phi \frac{\mu_0 I}{2\pi r} \frac{L}{\sqrt{L^2 + r^2}}$$

$$\rightarrow \mathbf{B} = \mathbf{a}_\phi \frac{\mu_0 I}{2\pi r} \quad (L \gg r)$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Magnetic Dipole (1)

For a small circular loop carrying current I :

Vector magnetic potential: $\mathbf{A} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\mathbf{l}'}{R_1}$

Pick a distant point ($R \gg b$):

$\rightarrow P(R, \theta, \pi/2), \quad R \gg b$

$\rightarrow d\mathbf{l}' = \mathbf{a}_\phi b d\phi' = (-\mathbf{a}_x \sin \phi' + \mathbf{a}_y \cos \phi') b d\phi'$

$\rightarrow \mathbf{A} = \frac{\mu_0 I}{4\pi} \oint_{C'} (-\mathbf{a}_x \frac{b \sin \phi'}{R_1}) d\phi'$

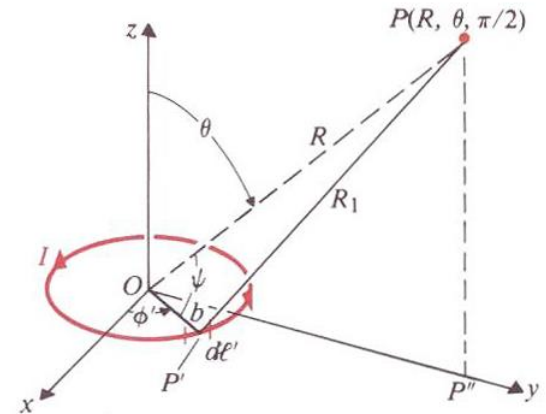
No contribution by symmetry

$\rightarrow \mathbf{A} = \mathbf{a}_\phi \frac{\mu_0 I b}{4\pi} \oint_{C'} \frac{\sin \phi'}{R_1} d\phi'$

$\rightarrow \mathbf{A} = \mathbf{a}_\phi \frac{\mu_0 I b}{2\pi R} \int_{-\pi/2}^{\pi/2} \left(1 + \frac{b}{R} \sin \theta \sin \phi' \right) \sin \phi' d\phi'$

$\rightarrow \mathbf{A} = \mathbf{a}_\phi \frac{\mu_0 I b^2}{4R^2} \sin \theta$

$\rightarrow R_1^2 = R^2 + b^2 - 2bR \cos \psi$
 $= R^2 + b^2 - 2bR \sin \theta \sin \phi'$
 $\rightarrow \frac{1}{R_1} = \frac{1}{R} \left(1 + \frac{b^2}{R^2} - \frac{2b}{R} \sin \theta \sin \phi' \right)^{-1/2}$
 $\cong \frac{1}{R} \left(1 + \frac{b}{R} \sin \theta \sin \phi' \right)$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Magnetic Dipole (2)

For a small circular loop carrying current I :

Vector magnetic potential:

$$\rightarrow \mathbf{A} = \mathbf{a}_\phi \frac{\mu_0 I b^2}{4R^2} \sin \theta$$

Magnetic flux density:

$$\rightarrow \mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu_0 I b^2}{4R^3} (\mathbf{a}_R 2 \cos \theta + \mathbf{a}_\theta \sin \theta)$$

Magnetic dipole moment:

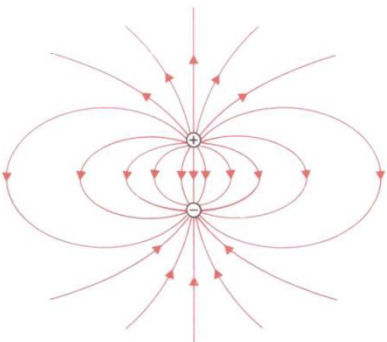
cf. Eq. (3-53b)

$$\rightarrow \mathbf{A} = \mathbf{a}_\phi \frac{\mu_0 (I \pi b^2)}{4\pi R^2} \sin \theta \quad \rightarrow V = \frac{\mathbf{p} \cdot \mathbf{a}_R}{4\pi \epsilon_0 R^2} \quad (\text{V})$$

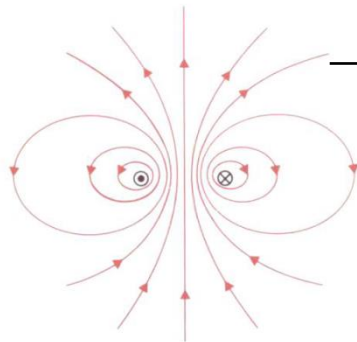
$$\rightarrow \mathbf{m} = \mathbf{a}_z I \pi b^2 = \mathbf{a}_z IS = \mathbf{a}_z m \quad (\text{A} \cdot \text{m}^2)$$

$$\rightarrow \mathbf{A} = \frac{\mu_0 \mathbf{m} \times \mathbf{a}_R}{4\pi R^2} \quad (\text{Wb/m})$$

$$\rightarrow \mathbf{B} = \frac{\mu_0 m}{4\pi R^3} (\mathbf{a}_R 2 \cos \theta + \mathbf{a}_\theta \sin \theta) \quad (\text{T})$$



(a) Electric dipole.



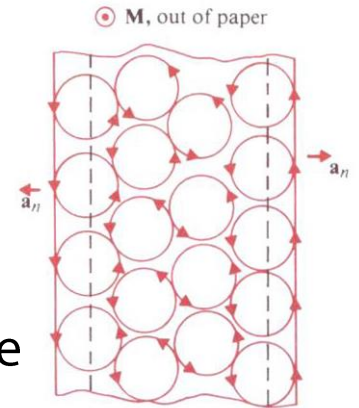
(b) Magnetic dipole.

Magnetization and Equivalent Current Densities

Magnetization vector (macroscopic):

$$\mathbf{M} = \lim_{\Delta v \rightarrow 0} \frac{\sum_{k=1}^{n\Delta v} \mathbf{m}_k}{\Delta v} \quad (\text{A/m})$$

← n : the number of atoms per unit volume



Differential dipole moment for an elemental volume dv' :

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

$$\rightarrow d\mathbf{m} = \mathbf{M}dv' \quad \rightarrow d\mathbf{A} = \frac{\mu_0 \mathbf{M} \times \mathbf{a}_R}{4\pi R^2} dv' = \frac{\mu_0}{4\pi} \mathbf{M} \times \nabla' \left(\frac{1}{R} \right) dv'$$

Vector magnetic potential:

$$\begin{aligned} \rightarrow \mathbf{A} &= \frac{\mu_0}{4\pi} \int_{V'} \mathbf{M} \times \nabla' \left(\frac{1}{R} \right) dv' = \frac{\mu_0}{4\pi} \int_{V'} \frac{\nabla' \times \mathbf{M}}{R} dv' - \frac{\mu_0}{4\pi} \int_{V'} \nabla' \times \left(\frac{\mathbf{M}}{R} \right) dv' \\ &= \frac{\mu_0}{4\pi} \int_{V'} \frac{\nabla' \times \mathbf{M}}{R} dv' + \frac{\mu_0}{4\pi} \oint_{S'} \frac{\mathbf{M} \times \mathbf{a}'_n}{R} ds' \quad \leftarrow \int_{V'} \nabla' \times \mathbf{F} dv' = - \oint_{S'} \mathbf{F} \times ds' \end{aligned}$$

Magnetization current densities:

$$\rightarrow \mathbf{J}_m = \nabla \times \mathbf{M} \quad (\text{A/m}^2) \quad \leftarrow \text{Volume current density}$$

$$\rightarrow \mathbf{J}_{ms} = \mathbf{M} \times \mathbf{a}_n \quad (\text{A/m}) \quad \leftarrow \text{Surface current density}$$