

# Introduction to Electromagnetism

## Static Magnetic Fields

### (6-4, 6-5, 6-6)

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# Biot-Savart Law and Applications

Vector magnetic potential:

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J}}{R} d\mathbf{v}'$$

→  $\mathbf{J}d\mathbf{v}' = JSd\mathbf{l}' = Id\mathbf{l}'$  ← For a wire with a cross-section  $S$

$$\rightarrow \boxed{\mathbf{A} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\mathbf{l}'}{R} \quad (\text{Wb/m})}$$

$$\rightarrow \mathbf{B} = \nabla \times \mathbf{A} = \nabla \times \left[ \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\mathbf{l}'}{R} \right] = \frac{\mu_0 I}{4\pi} \oint_{C'} \nabla \times \left( \frac{d\mathbf{l}'}{R} \right)$$

$$= \frac{\mu_0 I}{4\pi} \oint_{C'} \left[ \frac{1}{R} \cancel{\nabla \times} d\mathbf{l}' + \left( \nabla \frac{1}{R} \right) \times d\mathbf{l}' \right]$$

$$= \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\mathbf{l}' \times \mathbf{a}_R}{R^2} \quad (\text{T}) \quad \leftarrow \text{Biot-Savart law}$$

$$\mathbf{B} = \oint_{C'} d\mathbf{B} \rightarrow \boxed{d\mathbf{B} = \frac{\mu_0 I}{4\pi} \left( \frac{d\mathbf{l}' \times \mathbf{a}_R}{R^2} \right) = \frac{\mu_0 I}{4\pi} \left( \frac{d\mathbf{l}' \times \mathbf{R}}{R^3} \right)}$$

# Example 6-4

For a current-carrying straight wire ( $L \gg r$ ):

a) By finding  $\mathbf{B}$  from  $\mathbf{A}$ :

$$\rightarrow \mathbf{A} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\mathbf{l}'}{R} = \mathbf{a}_z \frac{\mu_0 I}{4\pi} \int_{-L}^L \frac{dz'}{\sqrt{z'^2 + r^2}}$$

$$\rightarrow \mathbf{B} = \nabla \times \mathbf{A}$$

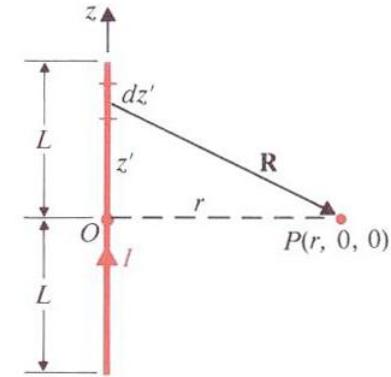
b) By applying Biot-Savart law:

$$\rightarrow d\mathbf{B} = \frac{\mu_0 I}{4\pi} \left( \frac{d\mathbf{l}' \times \mathbf{R}}{R^3} \right)$$

$$\rightarrow \mathbf{R} = \mathbf{a}_r r - \mathbf{a}_z z'$$

$$\rightarrow d\mathbf{l}' \times \mathbf{R} = \mathbf{a}_z dz' \times (\mathbf{a}_r r - \mathbf{a}_z z') = \mathbf{a}_\phi r dz'$$

$$\begin{aligned} \rightarrow \mathbf{B} = \int d\mathbf{B} &= \mathbf{a}_\phi \frac{\mu_0 I}{4\pi} \int_{-L}^L \frac{rdz'}{(z'^2 + r^2)^{3/2}} = \mathbf{a}_\phi \frac{\mu_0 I}{4\pi} \frac{1}{r} \left[ \frac{z'}{\sqrt{z'^2 + r^2}} \right]_{-L}^L \\ &= \mathbf{a}_\phi \frac{\mu_0 I}{2\pi r} \frac{L}{\sqrt{L^2 + r^2}} = \mathbf{a}_\phi \frac{\mu_0 I}{2\pi r} \quad (L \gg r) \end{aligned}$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

# Magnetic Dipole (1)

For a small circular loop carrying current  $I$ :

Vector magnetic potential:  $\mathbf{A} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\mathbf{l}'}{R_1}$

Pick a distant point ( $R \gg b$ ):

$$\rightarrow P(R, \theta, \pi/2), \quad R \gg b$$

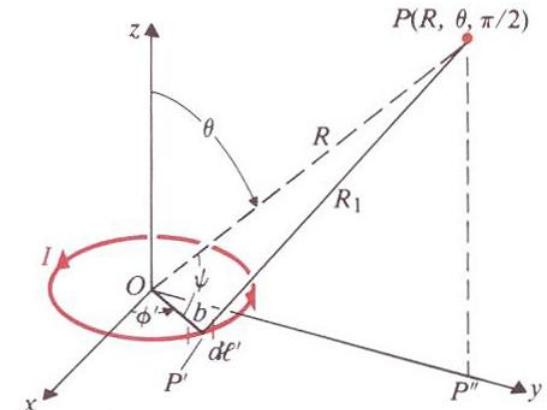
$$\rightarrow d\mathbf{l}' = \mathbf{a}_{\phi'} b d\phi' = (-\mathbf{a}_x \sin \phi' + \mathbf{a}_y \cos \phi') b d\phi'$$

$$\rightarrow \mathbf{A} = \frac{\mu_0 I}{4\pi} \oint_{C'} \left( -\mathbf{a}_x \frac{b \sin \phi'}{R_1} \right) d\phi'$$

$$\rightarrow \mathbf{A} = \mathbf{a}_\phi \frac{\mu_0 I b}{4\pi} \oint_{C'} \frac{\sin \phi'}{R_1} d\phi'$$

$$\rightarrow \mathbf{A} = \mathbf{a}_\phi \frac{\mu_0 I b}{2\pi R} \int_{-\pi/2}^{\pi/2} \left( 1 + \frac{b}{R} \sin \theta \sin \phi' \right) \sin \phi' d\phi'$$

$$\rightarrow \boxed{\mathbf{A} = \mathbf{a}_\phi \frac{\mu_0 I b^2}{4R^2} \sin \theta}$$



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No contribution by symmetry

$$\begin{aligned} \rightarrow R_1^2 &= R^2 + b^2 - 2bR \cos \psi \\ &= R^2 + b^2 - 2bR \sin \theta \sin \phi' \end{aligned}$$

$$\begin{aligned} \rightarrow \frac{1}{R_1} &= \frac{1}{R} \left( 1 + \frac{b^2}{R^2} - \frac{2b}{R} \sin \theta \sin \phi' \right)^{-1/2} \\ &\equiv \frac{1}{R} \left( 1 + \frac{b}{R} \sin \theta \sin \phi' \right) \end{aligned}$$

# Magnetic Dipole (2)

For a small circular loop carrying current  $I$ :

Vector magnetic potential:

$$\rightarrow \mathbf{A} = \mathbf{a}_\phi \frac{\mu_0 I b^2}{4R^2} \sin \theta$$

Magnetic flux density:

$$\rightarrow \mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu_0 I b^2}{4R^3} (\mathbf{a}_R 2 \cos \theta + \mathbf{a}_\theta \sin \theta)$$

Magnetic dipole moment:

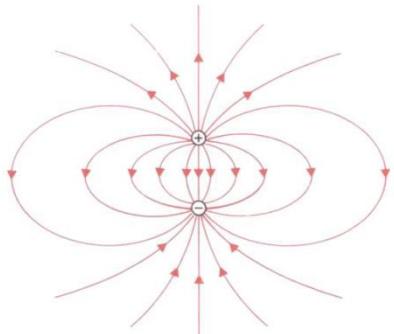
cf. Eq. (3-53b)

$$\rightarrow \mathbf{A} = \mathbf{a}_\phi \frac{\mu_0 (I \pi b^2)}{4 \pi R^2} \sin \theta \quad \rightarrow V = \frac{\mathbf{p} \cdot \mathbf{a}_R}{4 \pi \epsilon_0 R^2} \quad (\text{V})$$

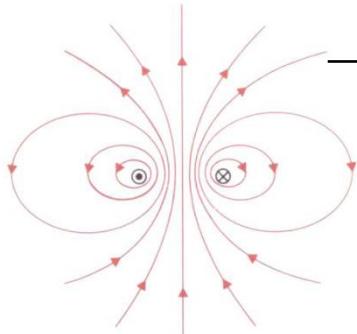
$$\rightarrow \mathbf{m} = \mathbf{a}_z I \pi b^2 = \mathbf{a}_z I S = \mathbf{a}_z m \quad (\text{A} \cdot \text{m}^2)$$

$$\rightarrow \boxed{\mathbf{A} = \frac{\mu_0 \mathbf{m} \times \mathbf{a}_R}{4 \pi R^2} \quad (\text{Wb/m})}$$

$$\rightarrow \boxed{\mathbf{B} = \frac{\mu_0 m}{4 \pi R^3} (\mathbf{a}_R 2 \cos \theta + \mathbf{a}_\theta \sin \theta) \quad (\text{T})}$$



(a) Electric dipole.



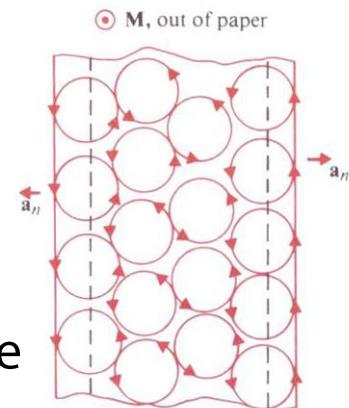
(b) Magnetic dipole.

# Magnetization and Equivalent Current Densities

Magnetization vector (macroscopic):

$$\mathbf{M} = \lim_{\Delta v \rightarrow 0} \frac{\sum_{k=1}^{n\Delta v} \mathbf{m}_k}{\Delta v} \quad (\text{A/m})$$

←  $n$ : the number of atoms per unit volume



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Differential dipole moment for an elemental volume  $dv'$ :

$$\rightarrow d\mathbf{m} = \mathbf{M} dv' \rightarrow d\mathbf{A} = \frac{\mu_0 \mathbf{M} \times \mathbf{a}_R}{4\pi R^2} dv' = \frac{\mu_0}{4\pi} \mathbf{M} \times \nabla' \left( \frac{1}{R} \right) dv'$$

Vector magnetic potential:

$$\begin{aligned} \rightarrow \mathbf{A} &= \frac{\mu_0}{4\pi} \int_{V'} \mathbf{M} \times \nabla' \left( \frac{1}{R} \right) dv' = \frac{\mu_0}{4\pi} \int_{V'} \frac{\nabla' \times \mathbf{M}}{R} dv' - \frac{\mu_0}{4\pi} \int_{V'} \nabla' \times \left( \frac{\mathbf{M}}{R} \right) dv' \\ &= \frac{\mu_0}{4\pi} \int_{V'} \frac{\nabla' \times \mathbf{M}}{R} dv' + \frac{\mu_0}{4\pi} \oint_{S'} \frac{\mathbf{M} \times \mathbf{a}'_n}{R} ds' \quad \leftarrow \boxed{\int_{V'} \nabla' \times \mathbf{F} dv' = - \oint_{S'} \mathbf{F} \times d\mathbf{s}'} \end{aligned}$$

Magnetization current densities:

$$\rightarrow \boxed{\mathbf{J}_m = \nabla \times \mathbf{M} \quad (\text{A/m}^2)} \quad \leftarrow \text{Volume current density}$$

$$\rightarrow \boxed{\mathbf{J}_{ms} = \mathbf{M} \times \mathbf{a}_n \quad (\text{A/m})} \quad \leftarrow \text{Surface current density}$$