Introduction to Electromagnetism
Static Magnetic Fields
(6-6, 6-7, 6-8, 6-9)

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Magnetic Field Intensity

For free space:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad \rightarrow \quad \frac{1}{\mu_0} \nabla \times \mathbf{B} = \mathbf{J}$$

In the presence of a magnetic material:

$$\mathbf{J}_m = \nabla \times \mathbf{M} \quad (\mathrm{A/m^2})$$ ← Equivalent volume current density

$$\rightarrow \frac{1}{\mu_0} \nabla \times \mathbf{B} = \mathbf{J} + \mathbf{J}_m = \mathbf{J} + \nabla \times \mathbf{M}$$

$$\rightarrow \nabla \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M} \right) = \mathbf{J} \quad \rightarrow \boxed{\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \quad (\mathrm{A/m})}$$

← Magnetic field intensity

$$\rightarrow \boxed{\nabla \times \mathbf{H} = \mathbf{J} \quad (\mathrm{A/m^2})}$$

$$\rightarrow \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{s} = \int_S \mathbf{J} \cdot d\mathbf{s}$$

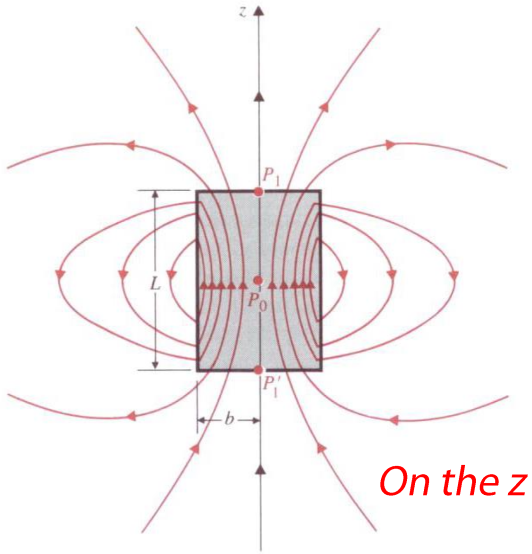
$$\rightarrow \boxed{\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{s} = I \quad (\mathrm{A})}$$ ← Ampère's circuital law

Example 6-6/6-8/6-13

For a uniform axial magnetization: $\mathbf{M} = \mathbf{a}_z M_0$

$$\rightarrow \mathbf{J}_{ms} = \mathbf{M} \times \mathbf{a}'_n = (\mathbf{a}_z M_0) \times \mathbf{a}_r = \mathbf{a}_\phi M_0$$

$$\rightarrow \mathbf{J}_m = \nabla' \times \mathbf{M} = 0$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Biot-Sarvart law:

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi}\left(\frac{d\mathbf{l}' \times \mathbf{R}}{R^3}\right) = \frac{\mu_0 J_{ms} dz'}{4\pi}\left(\frac{d\mathbf{l}' \times \mathbf{R}}{R^3}\right)$$

$$\rightarrow d\mathbf{l}' = \mathbf{a}_\phi b d\phi'$$

On the z axis $\rightarrow \mathbf{R} = \mathbf{a}_z(z - z') - \mathbf{a}_r b \quad \rightarrow R = [(z - z')^2 + b^2]^{1/2}$

$$\rightarrow d\mathbf{l}' \times \mathbf{R} = \mathbf{a}_\phi b d\phi' \times [\mathbf{a}_z(z - z') - \mathbf{a}_r b]$$

Cylindrical symmetry

$$= \mathbf{a}_r b(z - z') d\phi' + \mathbf{a}_z b^2 d\phi'$$

$$\rightarrow \mathbf{B}\Big|_{r=0} = \frac{\mu_0}{4\pi}\int_0^L \int_0^{2\pi} \frac{M_0 d\mathbf{l}' \times \mathbf{R}}{R^3} dz' = \mathbf{a}_z \frac{\mu_0}{2}\int_0^L \frac{M_0 b^2 dz'}{[(z - z')^2 + b^2]^{3/2}} = \mathbf{a}_z \frac{\mu_0 M_0}{2}\left[\frac{-(z - z')}{[(z - z')^2 + b^2]^{1/2}}\right]_0^L$$

$$= \mathbf{a}_z \frac{\mu_0 M_0}{2}\left[\frac{z}{\sqrt{z^2 + b^2}} - \frac{z - L}{\sqrt{(z - L)^2 + b^2}}\right]$$

Inside

$$\rightarrow \boxed{\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}}$$

Outside

$$\rightarrow \boxed{\mathbf{H} = \frac{\mathbf{B}}{\mu_0}}$$

Relative Permeability

Magnetic susceptibility:

$$\mathbf{M} = \chi_m \mathbf{H} \quad \leftarrow \textit{For linear and isotropic}$$

$$\rightarrow \mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$$

$$\rightarrow \boxed{\begin{aligned} \mathbf{B} &= \mu_0 (1 + \chi_m) \mathbf{H} \\ &= \mu_0 \mu_r \mathbf{H} = \mu \mathbf{H} \quad (\mathrm{Wb/m^2}) \end{aligned}}$$

$$\rightarrow \boxed{\mathbf{H} = \frac{1}{\mu} \mathbf{B} \quad (\mathrm{A/m})}$$

Relative permeability:

$$\rightarrow \boxed{\mu_r = 1 + \chi_m = \frac{\mu}{\mu_0}}$$

Analogous relations:

Electrostatics	Magnetostatics
$\mathbf{E}$	$\mathbf{B}$
$\mathbf{D}$	$\mathbf{H}$
ϵ	$\frac{1}{\mu}$
$\mathbf{P}$	$-\mathbf{M}$
ρ	$\mathbf{J}$
V	$\mathbf{A}$
$\cdot$	$\times$
$\times$	$\cdot$

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Behavior of Magnetic Materials

Classification of magnetic materials:

Diamagnetic: $\mu_r \leq 1$ (χ_m *is a very small negative number.*) $\rightarrow \ \chi_m \sim -10^{-5}$

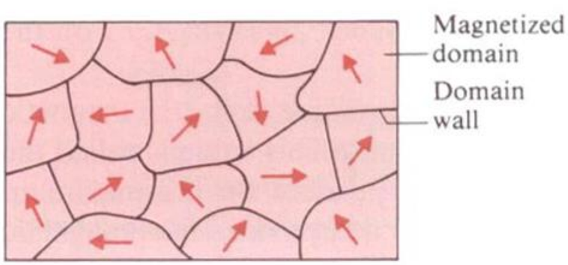
(*bismuth, copper, lead, mercury, germanium, silver, gold, diamond, etc.*)

Paramagnetic: $\mu_r \geq 1$ (χ_m *is a very small positive number.*) $\rightarrow \ \chi_m \sim +10^{-5}$

(*aluminum, magnesium, titanium, tungsten, etc.*)

Ferromagnetic: $\mu_r >> 1$ (χ_m *is a large positive number.*) $\rightarrow \ \chi_m > 10^3$

(*cobalt, nickel, iron, etc.*)



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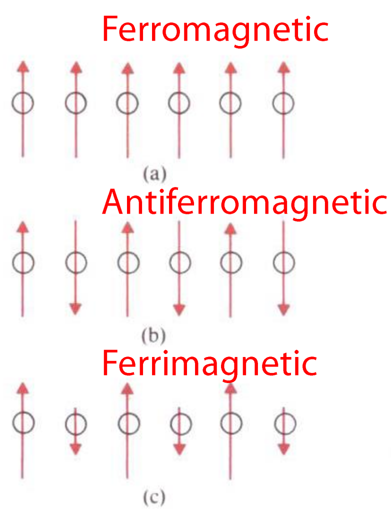
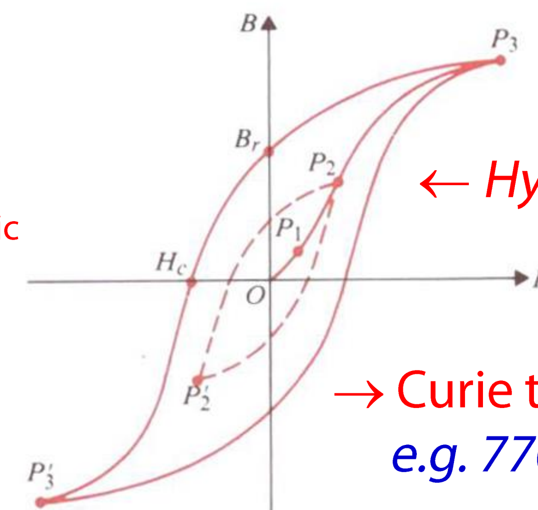
Domain size:

$\mu m \sim mm$





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← *Hysteresis loop*

→ Curie temperature

e.g. 770 ℃ for iron

Magnetic Circuits (1)

Two basic equations for magnetostatics:

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{H} = \mathbf{J}$$

Magnetomotive force (mmf):

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{s} = NI = V_m$$

Example 6-10: $\leftarrow$ Fringing effect neglected

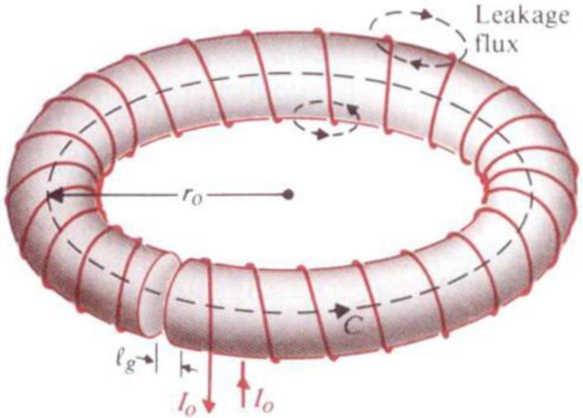
$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{s} = NI_o$$

$$\rightarrow \mathbf{B}_f = \mathbf{B}_g = \mathbf{a}_\phi B_f$$ $\leftarrow$ Why?

$$\rightarrow \mathbf{H}_f = \mathbf{a}_\phi \frac{B_f}{\mu} \rightarrow \mathbf{H}_g = \mathbf{a}_\phi \frac{B_f}{\mu_0}$$

$$\rightarrow \frac{B_f}{\mu}(2\pi r_o - l_g) + \frac{B_f}{\mu_0} l_g = NI_o$$

$$\rightarrow \mathbf{B}_f = \mathbf{a}_\phi \frac{\mu_0 \mu N I_o}{\mu_0 (2\pi r_o - l_g) + \mu l_g}$$



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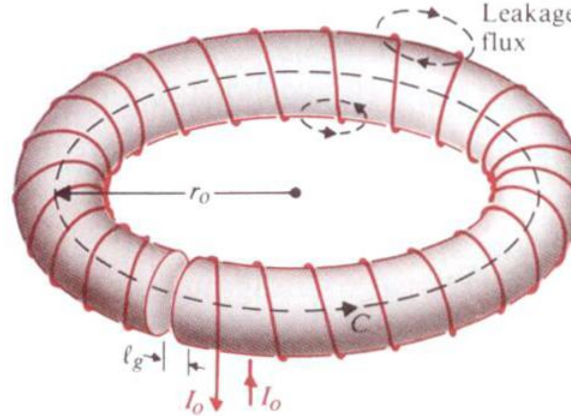
Magnetic Circuits (2)



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Magnetic flux:

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} \cong BS$$

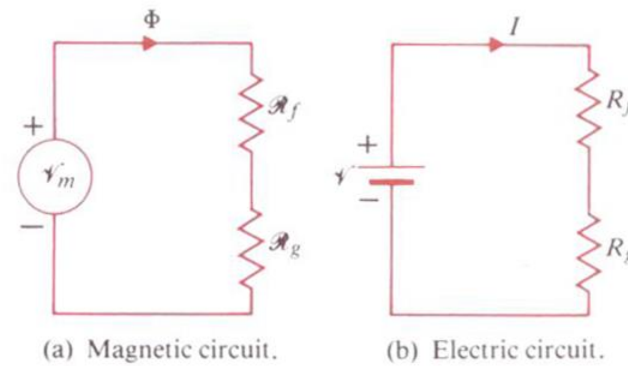
$$\rightarrow \ \Phi = \frac{NI_o}{(2\pi r_o - l_g)/\mu S + l_g/\mu_0 S}$$

$$= \frac{\mathcal{V}_m}{\mathcal{R}_f + \mathcal{R}_g}$$

Reluctance:

$$\rightarrow \ \mathcal{R}_f = \frac{2\pi r_o - l_g}{\mu S} = \frac{l_f}{\mu S}$$

$$\rightarrow \ \mathcal{R}_g = \frac{l_g}{\mu_0 S}$$



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Analogous quantities:

Magnetic Circuits	Electric Circuits
mmf, $\mathcal{V}_m$ $(= NI)$	emf, $\mathcal{V}$
magnetic flux, Φ	electric current, I
reluctance, $\mathcal{R}$	resistance, R
permeability, μ	conductivity, σ

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Difficulties of magnetic circuits:

1. Leakage flux

2. Fringing effect

3. Nonlinearity: B/H

Magnetic Circuits (3)

Similarity to Kirchhoff's voltage law:

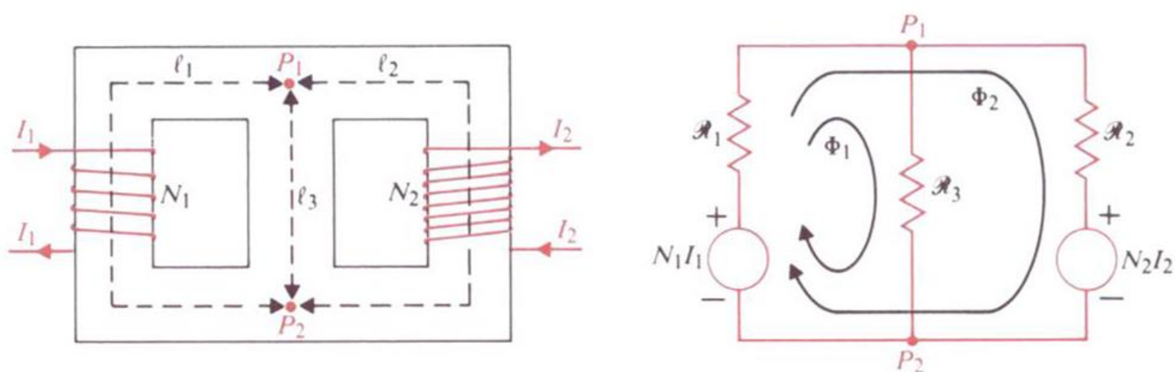
$$\Phi = \frac{V_m}{\mathcal{R}_f + \mathcal{R}_g} \rightarrow NI = \mathcal{R}_f \Phi + \mathcal{R}_g \Phi$$

$$\rightarrow \boxed{\sum_j N_j I_j = \sum_k \mathcal{R}_k \Phi_k}$$

Similarity to Kirchhoff's current law: $\leftarrow \nabla \cdot \mathbf{J} = 0$

$$\nabla \cdot \mathbf{B} = 0$$

$$\rightarrow \boxed{\sum_j \Phi_j = 0}$$



(a) Magnetic core with current-carrying windings

(b) Magnetic circuit for loop analysis.

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