Introduction to Electromagnetism
Static Magnetic Fields
(6-10, 6-11)

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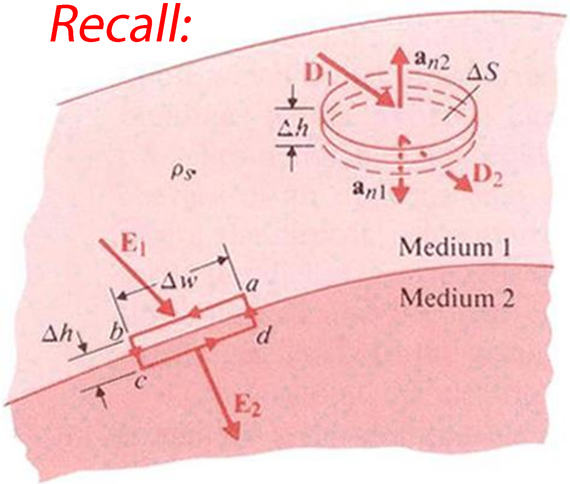
Boundary Conditions for Magnetostatic Fields

Two basic equations for magnetostatics:

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{H} = \mathbf{J}$$

Recall:



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Boundary conditions:

$$B_{1n} = B_{2n} \quad \text{(T)}$$

$$\mu_1 H_{1n} = \mu_2 H_{2n}$$

← For linear media

$$\rightarrow \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{s} = I$$

$$\rightarrow \oint_{abcda} \mathbf{H} \cdot d\mathbf{l} = \mathbf{H}_1 \cdot \Delta\mathbf{w} + \mathbf{H}_2 \cdot (-\Delta\mathbf{w}) = J_s \Delta w$$

$$\rightarrow H_{1t} - H_{2t} = J_s \quad \text{(A/m)}$$

$$\rightarrow \mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$$

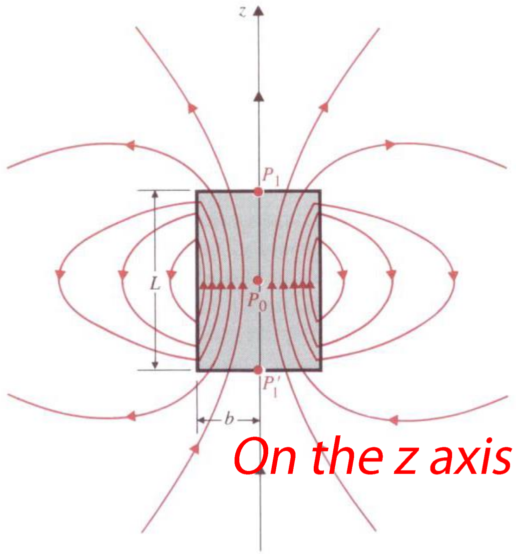
← $\mathbf{J}_s$ is non-zero only when an interface with an ideal perfect conductor is assumed.

Example 6-13

For a uniform axial magnetization: $\mathbf{M}=\mathbf{a}_z M_0$

$$\rightarrow \mathbf{J}_{ms}=\mathbf{M}\times\mathbf{a}_n'=(\mathbf{a}_z M_0)\times\mathbf{a}_r=\mathbf{a}_\phi M_0$$

$$\rightarrow \mathbf{J}_m=\nabla'\times\mathbf{M}=0$$



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Biot-Sarvart law:

$$d\mathbf{B}=\frac{\mu_0 I}{4\pi}\left(\frac{d\mathbf{l}'\times\mathbf{R}}{R^3}\right)=\frac{\mu_0 J_{ms}dz'}{4\pi}\left(\frac{d\mathbf{l}'\times\mathbf{R}}{R^3}\right)$$

$$\rightarrow d\mathbf{l}'=\mathbf{a}_\phi b\,d\phi'$$

Cylindrical symmetry

On the z axis

$$\rightarrow \mathbf{R}=\mathbf{a}_z(z-z')-\mathbf{a}_r b \quad \rightarrow R=[(z-z')^2+b^2]^{1/2}$$

$$\rightarrow d\mathbf{l}'\times\mathbf{R}=\mathbf{a}_\phi b\,d\phi'\times[\mathbf{a}_z(z-z')-\mathbf{a}_r b]$$

$$=\mathbf{a}_r b(z-z')d\phi'+\mathbf{a}_z b^2 d\phi'$$

$$\rightarrow \mathbf{B}=\frac{\mu_0}{4\pi}\int_0^L\int_0^{2\pi}\frac{M_0 d\mathbf{l}'\times\mathbf{R}}{R^3}dz'=\mathbf{a}_z\frac{\mu_0}{2}\int_0^L\frac{M_0 b^2 dz'}{[(z-z')^2+b^2]^{3/2}}=\mathbf{a}_z\frac{\mu_0 M_0}{2}\left[\frac{-(z-z')}{[(z-z')^2+b^2]^{1/2}}\right]_0^L$$

$$=\mathbf{a}_z\frac{\mu_0 M_0}{2}\left[\frac{z}{\sqrt{z^2+b^2}}-\frac{z-L}{\sqrt{(z-L)^2+b^2}}\right]$$

$R >> b, L$

$$\leftarrow \boxed{m=IS=M_0 L\pi b^2}$$

$$\rightarrow \mathbf{B}=\frac{\mu_0 m}{4\pi R^3}(\mathbf{a}_R 2\cos\theta+\mathbf{a}_\theta\sin\theta)\quad \text{(T)}$$

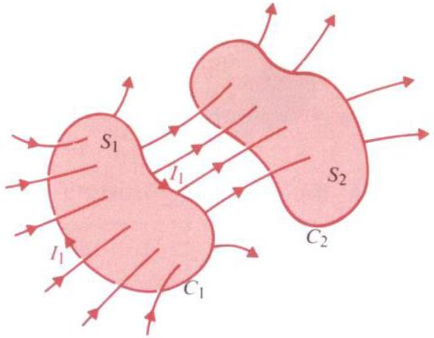
Inside

$$\rightarrow \boxed{\mathbf{H}=\frac{\mathbf{B}}{\mu_0}-\mathbf{M}}$$

Outside

$$\rightarrow \boxed{\mathbf{H}=\frac{\mathbf{B}}{\mu_0}}$$

Inductance and Inductors



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Magnetic flux:

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} \quad (\text{Wb})$$

Any surface

Inductance (Self inductance):

$$\rightarrow \Phi_{11}^L = \int_{S_1} \mathbf{B}_1 \cdot d\mathbf{s}_1 \rightarrow B_1 \propto I_1 \rightarrow \Phi_{11}^L \propto I_1$$

Entire surface enclosed by I_1

Total magnetic flux linkage

$$= L_{11} I_1$$

$$\rightarrow \boxed{L_{11} = \frac{\Phi_{11}^L}{I_1} \quad (\text{H})} \rightarrow \boxed{L_{11} = \frac{d\Phi_{11}^L}{dI_1} \quad (\text{H})}$$

Mutual inductance:

$$\rightarrow \Phi_{12}^L = \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s}_2$$

Linked to S_2

By I_1

$$= L_{12} I_1$$

$$\rightarrow \boxed{L_{12} = \frac{\Phi_{12}^L}{I_1} \quad (\text{H})} \rightarrow \boxed{L_{12} = \frac{d\Phi_{12}^L}{dI_1} \quad (\text{H})}$$

Question: $L_{21} = \frac{\Phi_{21}^L}{I_2} \overset{?}{=} L_{12}$

Mutual Inductance

Mutual magnetic flux linkage:

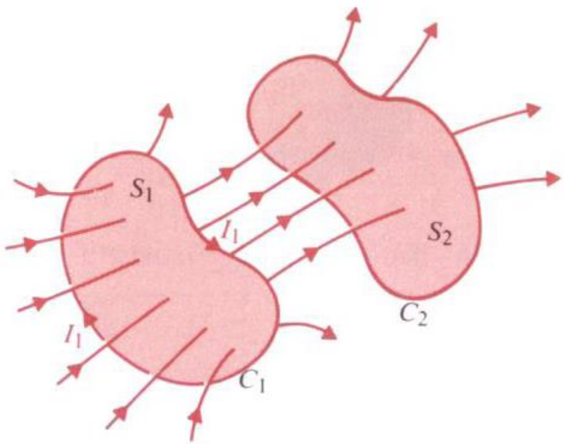
$$\rightarrow \Phi_{12}^{L} = \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s}_2 = L_{12} I_1$$

Entire surface enclosed by I_2

$$\rightarrow L_{12} = \frac{\Phi_{12}^{L}}{I_1} = \frac{1}{I_1}\int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s}_2 = \frac{1}{I_1}\int_{S_2} (\nabla \times \mathbf{A}_1) \cdot d\mathbf{s}_2$$

$$= \frac{1}{I_1}\oint_{C_2} \mathbf{A}_1 \cdot d\mathbf{l}_2 \quad \leftarrow \quad \mathbf{A}_1 = \frac{\mu_0 I_1}{4\pi}\oint_{C_1} \frac{d\mathbf{l}_1}{R}$$

$$= \frac{\mu_0}{4\pi}\oint_{C_2}\oint_{C_1} \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{R}$$



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Similarly:

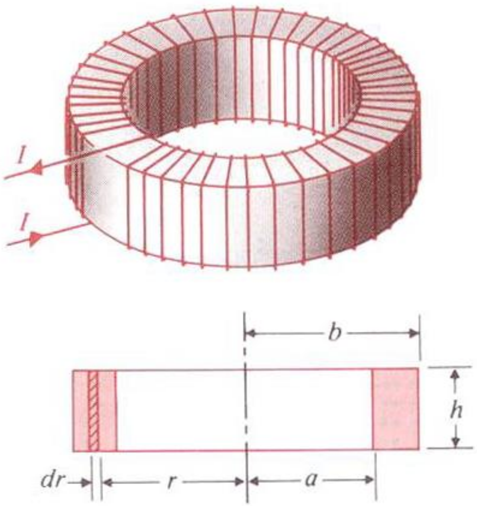
$$\rightarrow L_{21} = \frac{\mu_0}{4\pi}\oint_{C_1}\oint_{C_2} \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{R} \quad \leftarrow \text{Neumann formula}$$

Consequently:

$$\rightarrow L_{21} = L_{12}$$

Example 6-14

For a toroid:



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$$\oint_C \mathbf{B}\cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J}\cdot d\mathbf{s}$$

$$\rightarrow \mathbf{B} = \mathbf{a}_\phi B_\phi$$

$$\rightarrow \oint_C \mathbf{B}\cdot d\mathbf{l} = 2\pi r B_\phi = \mu_0 N I$$

$$\rightarrow \mathbf{B} = \mathbf{a}_\phi \frac{\mu_0 N I}{2\pi r}$$

Entire surface enclosed by I

$$\rightarrow L = \frac{\Phi^L}{I} = \frac{1}{I}\int_S \mathbf{B}\cdot d\mathbf{s}$$

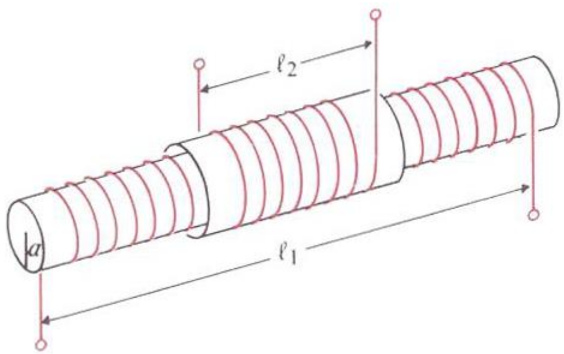
$$= \frac{1}{I}\int_S \left(\mathbf{a}_\phi \frac{\mu_0 N I}{2\pi r}\right)\cdot(\mathbf{a}_\phi N h\, dr)$$

$$= \frac{\mu_0 N^2 h}{2\pi}\int_a^b \frac{1}{r} dr$$

$$= \frac{\mu_0 N^2 h}{2\pi} \ln\frac{b}{a} \quad \text{(H)}$$

Example 6-18

For a solenoid with two windings:



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Entire surface enclosed by I_2

$$\rightarrow \Phi_{12}^{L} = \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s}_2 = L_{12} I_1$$

$$= \mu \left(\frac{N_1}{l_1} \right) I_1 \cdot N_2 \pi a^2 \quad \leftarrow B_1 l_1 = \mu N_1 I_1$$

$$\rightarrow L_{12} = \frac{\mu}{l_1} N_1 N_2 \pi a^2 \quad \text{(H)}$$

Alternatively:

$$\rightarrow \Phi_{21}^{L} = \int_{S_1} \mathbf{B}_2 \cdot d\mathbf{s}_1 = L_{21} I_2$$

$$= \mu \left(\frac{N_2}{l_2} \right) I_2 \cdot N_1 \left(\frac{l_2}{l_1} \right) \pi a^2$$

$$\rightarrow L_{21} = \frac{\mu}{l_1} N_1 N_2 \pi a^2 \quad \text{(H)}$$

As expected:

$$\rightarrow L_{12} = L_{21}$$