Introduction to Electromagnetism
Static Magnetic Fields
(6-12)

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Magnetic Induction

Relationship between current and magnetic field:

$\rightarrow$ *Discovered in 1820 by Ørsted that an electric current produces a magnetic field.*

Ampère's circuital law:

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{s} = \mu_0 I$$

$\rightarrow$ *Discovered in 1826*

Faraday's law of induction:

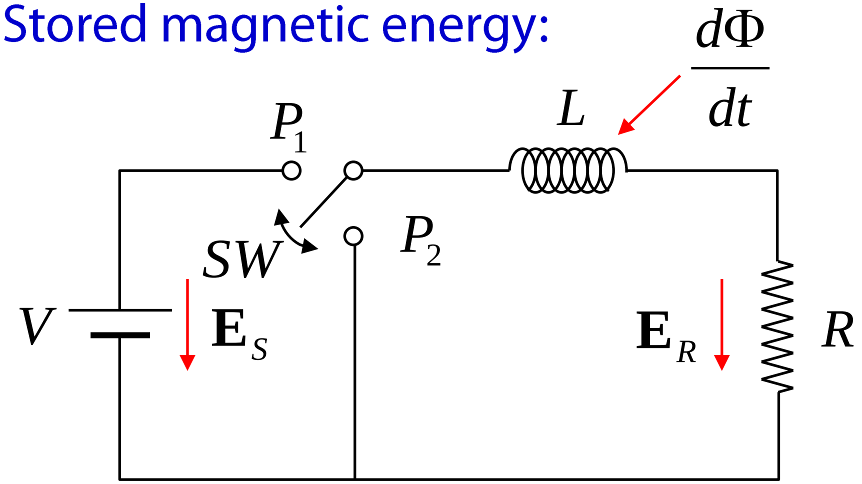
$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}$$

$\rightarrow$ *Discovered in 1831*

$\rightarrow$ *Lenz's law in 1833*

Magnetic Energy (1)

Stored magnetic energy:



Recall: $\oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d\Phi}{dt}$

→ *At $t=0$, switched to P_1:*

$$-V + Ri = -\frac{d\Phi}{dt} = -L\frac{di}{dt}$$

$$\to\ L\frac{di}{dt} + Ri = V$$

$$\to\ i = I(1-e^{-\frac{R}{L}t}),\quad I = \frac{V}{R}$$

→ *At $t = t_0$ ($t_0 >> 1$), switched to P_2:*

$$L\frac{di}{dt} + Ri = 0$$

$$\to\ i = I_0 e^{-\frac{R}{L}(t-t_0)},\quad I_0 \cong \frac{V}{R} = I$$

→ *Total energy stored in L = Total heat energy dissipated through R*

$$\to\ \int_{t_0}^{\infty} i^2 R\,dt = I^2 R\int_{t_0}^{\infty} e^{-\frac{2R}{L}(t-t_0)}dt = I^2 R\frac{L}{2R} = \frac{1}{2}LI^2 = \frac{1}{2}I\Phi$$ ← Stored magnetic energy

Magnetic Energy (2)

Work required for generating **B** field through *I*:

$$W_{11} = \int v_1 i_1 dt = \int \frac{d\phi_1}{dt} i_1 dt = \int L_1 \frac{di_1}{dt} i_1 dt = \int_0^{I_1} L_1 i_1 di_1 = \frac{1}{2} L_1 I_1^2$$

For two closed loops: $i_1 : 0 \rightarrow I_1 \;\; \rightarrow \;\; i_2 : 0 \rightarrow I_2$ ← Mutual coupling

$$W_{21} = \int v_{21} I_1 dt = \int_0^{I_2} L_{21} \frac{di_2}{dt} I_1 dt = L_{21} I_1 \int_0^{I_2} di_2 = L_{21} I_1 I_2$$

$$\rightarrow W_2 = W_{11} + W_{22} + W_{12} = \frac{1}{2} L_1 I_1^2 + L_{21} I_1 I_2 + \frac{1}{2} L_2 I_2^2 = \frac{1}{2} \sum_{j=1}^{2} \sum_{k=1}^{2} L_{jk} I_j I_k$$

Magnetic energy:

$$\rightarrow \boxed{W_m = \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} L_{jk} I_j I_k}$$

Alternatively:

$$\rightarrow dW_m = \sum_{k=1}^{N} dW_k = \sum_{k=1}^{N} v_k i_k dt = \sum_{k=1}^{N} \frac{d\phi_k}{dt} i_k dt = \sum_{k=1}^{N} i_k d\phi_k$$

$$\rightarrow W_m = \int dW_m = \sum_{k=1}^{N} I_k \Phi_k \int_0^1 \alpha d\alpha$$

$$\rightarrow \boxed{W_m = \frac{1}{2} \sum_{k=1}^{N} I_k \Phi_k} \quad \leftarrow \Phi_k = \sum_{j=1}^{N} L_{jk} I_j$$

Magnetic Energy in Terms of Field Quantities (1)

For a current-carrying loop:

$$\Phi_k = \int_{S_k} \mathbf{B}\cdot d\mathbf{s}'_k = \int_{S_k} \nabla\times\mathbf{A}\cdot d\mathbf{s}'_k = \oint_{C_k} \mathbf{A}\cdot d\mathbf{l}'_k$$

Magnetic energy:

$$W_m = \lim_{N\to\infty} \frac{1}{2}\sum_{k=1}^{N} I_k \oint_{C_k} \mathbf{A}\cdot d\mathbf{l}'_k \quad \to \quad I_k d\mathbf{l}'_k = J_k(a'_k)d\mathbf{l}'_k = \mathbf{J}_k dv'_k$$

$$= \lim_{N\to\infty} \frac{1}{2}\sum_{k=1}^{N} \oint_{C_k} \mathbf{A}\cdot\mathbf{J}_k dv'_k = \frac{1}{2}\int_{V'} \mathbf{A}\cdot\mathbf{J}dv' \quad \text{(J)}$$

Magnetic energy in terms of field quantities:

$$\nabla\cdot(\mathbf{A}\times\mathbf{H}) = \mathbf{H}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{H})$$

$$\to \mathbf{A}\cdot(\nabla\times\mathbf{H}) = \mathbf{H}\cdot(\nabla\times\mathbf{A}) - \nabla\cdot(\mathbf{A}\times\mathbf{H})$$

$$\to \mathbf{A}\cdot\mathbf{J} = \mathbf{H}\cdot\mathbf{B} - \nabla\cdot(\mathbf{A}\times\mathbf{H})$$

$$\to W_m = \frac{1}{2}\int_{V'} \mathbf{H}\cdot\mathbf{B}dv' - \frac{1}{2}\oint_{S'} (\mathbf{A}\times\mathbf{H})\cdot d\mathbf{s}' = \frac{1}{2}\int_{V'} \mathbf{H}\cdot\mathbf{B}dv' \quad \text{(J)}$$

$$\to \mathbf{H} = \mathbf{B}/\mu$$

$$\to W_m = \frac{1}{2}\int_{V'} \frac{B^2}{\mu} dv' = \frac{1}{2}\int_{V'} \mu H^2 dv' \quad \text{(J)}$$

Magnetic Energy in Terms of Field Quantities (2)

Magnetic energy density:

$$W_m = \int_{V'} w_m dv'$$

$$\rightarrow \ w_m = \frac{1}{2}\mathbf{H}\cdot\mathbf{B} \quad (\text{J/m}^3)$$

$$\rightarrow \ w_m = \frac{B^2}{2\mu} \quad (\text{J/m}^3)$$

$$\rightarrow \ w_m = \frac{1}{2}\mu H^2 \quad (\text{J/m}^3)$$

Example 6-20:

For a coaxial transmission line:

$$W'_{m1} = \frac{1}{2\mu_0}\int_0^a B_{\phi 1}^2 (2\pi r) dr = \frac{\mu_0 I^2}{4\pi a^4}\int_0^a r^3 dr = \frac{\mu_0 I^2}{16\pi} \quad (\text{J/m})$$

$$W'_{m2} = \frac{1}{2\mu_0}\int_a^b B_{\phi 2}^2 (2\pi r) dr = \frac{\mu_0 I^2}{4\pi}\int_a^b \frac{1}{r} dr = \frac{\mu_0 I^2}{4\pi}\ln\frac{b}{a} \quad (\text{J/m})$$

$$\rightarrow \ W'_m = W'_{m1} + W'_{m2} = \frac{1}{2}LI^2 \ \rightarrow \ L = \frac{\mu_0}{8\pi} + \frac{\mu_0}{2\pi}\ln\frac{b}{a} \quad (\text{H/m})$$