

# Introduction to Electromagnetism

## Static Magnetic Fields

(6-12)

Yoonchan Jeong

School of Electrical Engineering, Seoul National University

Tel: +82 (0)2 880 1623, Fax: +82 (0)2 873 9953

Email: [yunchan@snu.ac.kr](mailto:yunchan@snu.ac.kr)

# Magnetic Induction

Relationship between current and magnetic field:

→ *Discovered in 1820 by Ørsted that an electric current produces a magnetic field.*

Ampère's circuital law:

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{s} = \mu_0 I$$

→ *Discovered in 1826*

Faraday's law of induction:

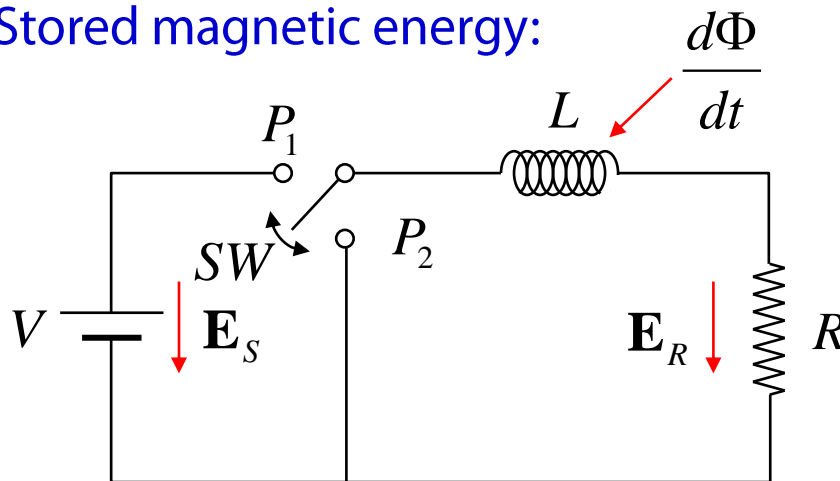
$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}$$

→ *Discovered in 1831*

→ *Lenz's law in 1833*

# Magnetic Energy (1)

Stored magnetic energy:



Recall:  $\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}$

→ At  $t = 0$ , switched to  $P_1$ :

$$-V + Ri = -\frac{d\Phi}{dt} = -L \frac{di}{dt}$$

$$\rightarrow L \frac{di}{dt} + Ri = V$$

$$\rightarrow i = I(1 - e^{-\frac{R}{L}t}), \quad I = \frac{V}{R}$$

→ At  $t = t_0$  ( $t_0 \gg 1$ ), switched to  $P_2$ :

$$L \frac{di}{dt} + Ri = 0$$

$$\rightarrow i = I_0 e^{-\frac{R}{L}(t-t_0)}, \quad I_0 \cong \frac{V}{R} = I$$

→ Total heat energy dissipated through  $R$  = Total energy stored in  $L$

$$\rightarrow \int_{t_0}^{\infty} i^2 R dt = I^2 R \int_{t_0}^{\infty} e^{-\frac{2R}{L}(t-t_0)} dt = I^2 R \frac{L}{2R} = \frac{1}{2} LI^2 = \frac{1}{2} I\Phi \leftarrow \text{Stored magnetic energy}$$

# Magnetic Energy (2)

Work required for generating **B** field through  $I$ :

$$W_{11} = \int v_1 i_1 dt = \int \frac{d\phi_1}{dt} i_1 dt = \int L_1 \frac{di_1}{dt} i_1 dt = \int_0^{I_1} L_1 i_1 di_1 = \frac{1}{2} L_1 I_1^2$$

For two closed loops:  $i_1 : 0 \rightarrow I_1 \rightarrow i_2 : 0 \rightarrow I_2$  ← Mutual coupling

$$W_{21} = \int v_{21} I_1 dt = \int_0^{I_2} L_{21} \frac{di_2}{dt} I_1 dt = L_{21} I_1 \int_0^{I_2} di_2 = L_{21} I_1 I_2$$

$$\rightarrow W_{total} = W_{11} + W_{22} + W_{21} = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + L_{21} I_1 I_2 = \frac{1}{2} \sum_{j=1}^2 \sum_{k=1}^2 L_{jk} I_j I_k$$

Magnetic energy:

$$\rightarrow W_m = \frac{1}{2} \sum_{j=1}^N \sum_{k=1}^N L_{jk} I_j I_k$$

Alternatively:

$$\rightarrow dW_m = \sum_{k=1}^N dW_k = \sum_{k=1}^N v_k i_k dt = \sum_{k=1}^N \frac{d\phi_k}{dt} i_k dt = \sum_{k=1}^N i_k d\phi_k$$

$$\rightarrow W_m = \int dW_m = \sum_{k=1}^N I_k \Phi_k \int_0^1 \alpha d\alpha$$

$$\rightarrow W_m = \frac{1}{2} \sum_{k=1}^N I_k \Phi_k \quad \leftarrow \quad \Phi_k = \sum_{j=1}^N L_{jk} I_j$$

# Magnetic Energy in Terms of Field Quantities (1)

For a current-carrying loop:

$$\Phi_k = \int_{S_k} \mathbf{B} \cdot d\mathbf{s}'_k = \int_{S_k} \nabla \times \mathbf{A} \cdot d\mathbf{s}'_k = \oint_{C_k} \mathbf{A} \cdot d\mathbf{l}'_k$$

Magnetic energy:

$$\begin{aligned} W_m &= \lim_{N \rightarrow \infty} \frac{1}{2} \sum_{k=1}^N I_k \oint_{C_k} \mathbf{A} \cdot d\mathbf{l}'_k \rightarrow I_k d\mathbf{l}'_k = J_k(a'_k) d\mathbf{l}'_k = \mathbf{J}_k dv'_k \\ &= \lim_{N \rightarrow \infty} \frac{1}{2} \sum_{k=1}^N \oint_{C_k} \mathbf{A} \cdot \mathbf{J}_k dv'_k = \frac{1}{2} \int_{V'} \mathbf{A} \cdot \mathbf{J} dv' \quad (\text{J}) \end{aligned}$$

Magnetic energy in terms of field quantities:

$$\nabla \cdot (\mathbf{A} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{H})$$

$$\rightarrow \mathbf{A} \cdot (\nabla \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{A}) - \nabla \cdot (\mathbf{A} \times \mathbf{H})$$

$$\rightarrow \mathbf{A} \cdot \mathbf{J} = \mathbf{H} \cdot \mathbf{B} - \nabla \cdot (\mathbf{A} \times \mathbf{H})$$

$$\rightarrow W_m = \frac{1}{2} \int_{V'} \mathbf{H} \cdot \mathbf{B} dv' - \frac{1}{2} \oint_{S'} (\mathbf{A} \times \mathbf{H}) \cdot d\mathbf{s}' = \frac{1}{2} \int_{V'} \mathbf{H} \cdot \mathbf{B} dv' \quad (\text{J})$$

$$\rightarrow \mathbf{H} = \mathbf{B} / \mu$$

$$\rightarrow W_m = \frac{1}{2} \int_{V'} \frac{B^2}{\mu} dv' = \frac{1}{2} \int_{V'} \mu H^2 dv' \quad (\text{J})$$

# Magnetic Energy in Terms of Field Quantities (2)

Magnetic energy density:

$$W_m = \int_{V'} w_m dv'$$

$$\rightarrow w_m = \frac{1}{2} \mathbf{H} \cdot \mathbf{B} \quad (\text{J/m}^3)$$

$$\rightarrow w_m = \frac{B^2}{2\mu} \quad (\text{J/m}^3)$$

$$\rightarrow w_m = \frac{1}{2} \mu H^2 \quad (\text{J/m}^3)$$

*Recall:*

$$\mathbf{B}_1 = \mathbf{a}_\phi B_{\phi 1} = \mathbf{a}_\phi \frac{\mu_0 r I}{2\pi a^2}, \quad r \leq a$$

$$\mathbf{B}_2 = \mathbf{a}_\phi B_{\phi 2} = \mathbf{a}_\phi \frac{\mu_0 I}{2\pi r}, \quad a < r \leq b$$

Example 6-20:

*For a coaxial transmission line:*

$$W'_{m1} = \frac{1}{2\mu_0} \int_0^a B_{\phi 1}^2 (2\pi r) dr = \frac{\mu_0 I^2}{4\pi a^4} \int_0^a r^3 dr = \frac{\mu_0 I^2}{16\pi} \quad (\text{J/m})$$

$$W'_{m2} = \frac{1}{2\mu_0} \int_a^b B_{\phi 2}^2 (2\pi r) dr = \frac{\mu_0 I^2}{4\pi} \int_a^b \frac{1}{r} dr = \frac{\mu_0 I^2}{4\pi} \ln \frac{b}{a} \quad (\text{J/m})$$

$$\rightarrow W'_m = W'_{m1} + W'_{m2} = \frac{1}{2} LI^2 \rightarrow L = \frac{\mu_0}{8\pi} + \frac{\mu_0}{2\pi} \ln \frac{b}{a} \quad (\text{H/m}) \quad 6$$