

Introduction to Electromagnetism

Static Magnetic Fields

(6-13)

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Magnetic Forces (1)

Magnetic force: $\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$ (N) \rightarrow *See the Hall effect (H.W.)*

Magnetic force on the differential length element $d\mathbf{l}$:

$$d\mathbf{F}_m = NqS|d\mathbf{l}|\mathbf{u} \times \mathbf{B}$$

$$\rightarrow d\mathbf{F}_m = I d\mathbf{l} \times \mathbf{B} \quad (\text{N}) \quad \rightarrow d\mathbf{l} \text{ along the direction of } I$$

$$\rightarrow \mathbf{F}_m = I \oint_C d\mathbf{l} \times \mathbf{B} \quad (\text{N})$$

For two circuits carrying current I_1 and I_2 :

"Biot-Savart law"

$$\mathbf{F}_{21} = -\mathbf{F}_{12} \left\{ \begin{array}{l} \mathbf{F}_{21} = I_1 \oint_{C_1} d\mathbf{l}_1 \times \mathbf{B}_{21} \quad \leftarrow \quad \mathbf{B}_{21} = \frac{\mu_0 I_2}{4\pi} \oint_{C_2} \frac{d\mathbf{l}_2 \times \mathbf{a}_{R_{21}}}{R_{21}^2} \\ \\ = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1} \oint_{C_2} \frac{d\mathbf{l}_1 \times (d\mathbf{l}_2 \times \mathbf{a}_{R_{21}})}{R_{21}^2} \quad (\text{N}) \end{array} \right.$$

\rightarrow *Ampère's law of force*

*Newton's third law
of action and reaction*

$$\mathbf{F}_{12} = I_2 \oint_{C_2} d\mathbf{l}_2 \times \mathbf{B}_{12} \\ = \frac{\mu_0}{4\pi} I_2 I_1 \oint_{C_2} \oint_{C_1} \frac{d\mathbf{l}_2 \times (d\mathbf{l}_1 \times \mathbf{a}_{R_{12}})}{R_{12}^2} \quad (\text{N})$$

Magnetic Forces (2)

For two circuits carrying current I_1 and I_2 :

$$\mathbf{F}_{12} = \frac{\mu_0}{4\pi} I_2 I_1 \oint_{C_2} \oint_{C_1} \frac{d\mathbf{l}_2 \times (d\mathbf{l}_1 \times \mathbf{a}_{R_{12}})}{R_{12}^2} \quad (\text{N})$$

$$\mathbf{F}_{21} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1} \oint_{C_2} \frac{d\mathbf{l}_1 \times (d\mathbf{l}_2 \times \mathbf{a}_{R_{21}})}{R_{21}^2} \quad (\text{N})$$

Recall: "back-cab" rule

$$\rightarrow \frac{d\mathbf{l}_1 \times (d\mathbf{l}_2 \times \mathbf{a}_{R_{21}})}{R_{21}^2} = \frac{d\mathbf{l}_2 (d\mathbf{l}_1 \cdot \mathbf{a}_{R_{21}})}{R_{21}^2} - \frac{\mathbf{a}_{R_{21}} (d\mathbf{l}_1 \cdot d\mathbf{l}_2)}{R_{21}^2}$$

$$\begin{aligned} \rightarrow \oint_{C_1} \oint_{C_2} \frac{d\mathbf{l}_2 (d\mathbf{l}_1 \cdot \mathbf{a}_{R_{21}})}{R_{21}^2} &= \oint_{C_2} d\mathbf{l}_2 \oint_{C_1} \frac{(d\mathbf{l}_1 \cdot \mathbf{a}_{R_{21}})}{R_{21}^2} \\ &= \oint_{C_2} d\mathbf{l}_2 \oint_{C_1} d\mathbf{l}_1 \cdot \left[-\nabla_1 \left(\frac{1}{R_{21}} \right) \right] = 0 \end{aligned}$$

*Newton's third law
of action and reaction
verified!*

$$\rightarrow \mathbf{F}_{21} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1} \oint_{C_2} \frac{\mathbf{a}_{R_{21}} (d\mathbf{l}_1 \cdot d\mathbf{l}_2)}{R_{21}^2}$$

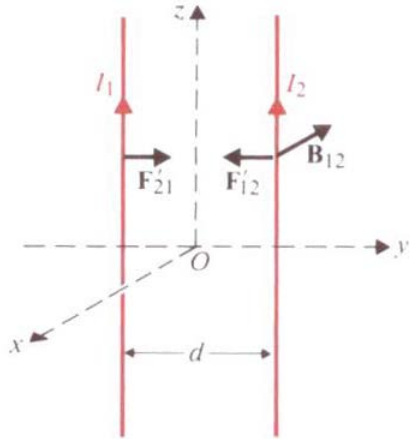
Recall:

$$\leftarrow \nabla \times (\nabla V) = 0$$

$$\rightarrow \mathbf{F}_{12} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1} \oint_{C_2} \frac{\mathbf{a}_{R_{12}} (d\mathbf{l}_1 \cdot d\mathbf{l}_2)}{R_{12}^2} = -\mathbf{F}_{21}$$

Example 6-21

For two parallel current-carrying wires:



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

$$\frac{\mathbf{F}_{12}}{l_2} = \mathbf{F}'_{12} = I_2 (\mathbf{a}_z \times \mathbf{B}_{12})$$

$$\rightarrow \mathbf{B}_{12} = -\mathbf{a}_x \frac{\mu_0 I_1}{2\pi d} \quad \leftarrow \text{Ampère's circuital law}$$

$$\rightarrow \mathbf{F}'_{12} = -\mathbf{a}_y \frac{\mu_0 I_1 I_2}{2\pi d} \quad (\text{N/m})$$

→ Attraction for currents in the same direction

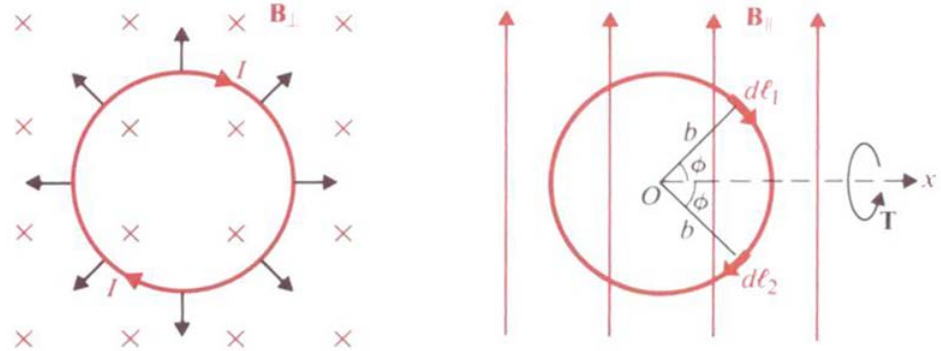
→ Repulsion for currents in opposite directions

Magnetic Forces and Torques

Magnetic flux density:

$$\mathbf{B} = \mathbf{B}_{\perp} + \mathbf{B}_{\parallel}$$

$$(\mathbf{B}_{\perp} = \mathbf{a}_z B_{\perp})$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Forces on the loop:

→ $\mathbf{F}_{\mathbf{B}_{\perp}} \propto \mathbf{a}_r$ → *Radial symmetry* → *No net contribution*

→ $d\mathbf{F}_{1,\mathbf{B}_{\parallel}} \propto -\mathbf{a}_z$ (upward)

→ $d\mathbf{F}_{2,\mathbf{B}_{\parallel}} \propto +\mathbf{a}_z$ (downward)

} → *Rotational motion*
→ *Torque*

Torque on the loop:

$$d\mathbf{T} = \mathbf{a}_x 2(dF)b \sin \phi$$

$$= \mathbf{a}_x 2(I dl B_{\parallel} \sin \phi) b \sin \phi$$

$$= \mathbf{a}_x 2I b^2 B_{\parallel} \sin^2 \phi d\phi$$

$$\rightarrow \mathbf{T} = \int d\mathbf{T} = \mathbf{a}_x 2I b^2 B_{\parallel} \int_0^{\pi} \sin^2 \phi d\phi = \mathbf{a}_x I (\pi b^2) B_{\parallel}$$

$$= \mathbf{m} \times \mathbf{B} \quad (\text{N} \cdot \text{m})$$

Forces and Torques - Stored Magnetic Energy (1)

Principle of virtual displacement:

Work done by the source

$$dW_s = dW + dW_m$$

*Mechanical work
done by the system*

*Increase in the stored
magnetic energy*

System of circuits with constant flux linkage:

Virtual displacement $d\mathbf{l}$ with no changes in flux linkages

→ No emf → No energy from the source to the system

$$\rightarrow dW_s = dW + dW_m = 0$$

Magnetic forces:

$$dW = \mathbf{F}_\Phi \cdot d\mathbf{l} = -dW_m = -(\nabla W_m) \cdot d\mathbf{l}$$

$$\rightarrow \mathbf{F}_\Phi = -\nabla W_m \quad (\text{N})$$

Magnetic torques:

$$\rightarrow (T_\Phi)_z = -\frac{\partial W_m}{\partial \phi} \quad (\text{N} \cdot \text{m})$$

Forces and Torques - Stored Magnetic Energy (2)

System of circuits with constant currents:

Virtual displacement $d\mathbf{l}$ with constant currents \rightarrow Flux linkage changed

\rightarrow Emf induced \rightarrow Energy from the source to the system

$$\rightarrow dW_s = dW + dW_m$$

Magnetic forces:

$$\begin{aligned} \rightarrow dW_s &= \sum_k I_k V_k dt = \sum_k I_k d\Phi_k \\ \rightarrow dW_m &= \frac{1}{2} \sum_k I_k d\Phi_k \end{aligned} \quad \left. \vphantom{\begin{aligned} \rightarrow dW_s \\ \rightarrow dW_m \end{aligned}} \right\} \rightarrow dW = dW_m$$

$$\rightarrow dW = \mathbf{F}_I \cdot d\mathbf{l} = dW_m = \nabla(W_m) \cdot d\mathbf{l}$$

$$\rightarrow \boxed{\mathbf{F}_I = \nabla W_m \quad (\text{N})}$$

Magnetic torques:

$$\rightarrow \boxed{(T_I)_z = \frac{\partial W_m}{\partial \phi} \quad (\text{N} \cdot \text{m})}$$

Force and Torque in Terms of Mutual Inductance

For two circuits with currents I_1 and I_2 :

$$W_m = \frac{1}{2} \sum_{k=1}^N I_k \Phi_k = \frac{1}{2} \sum_{j=1}^N \sum_{k=1}^N L_{jk} I_j I_k$$
$$\rightarrow W_m = \frac{1}{2} L_1 I_1^2 + L_{12} I_1 I_2 + \frac{1}{2} L_2 I_2^2$$

Magnetic force between two circuits:

$$\rightarrow \mathbf{F}_I = I_1 I_2 (\nabla L_{12})$$

Magnetic torque between two circuits:

$$\rightarrow (T_I)_z = I_1 I_2 \frac{\partial L_{12}}{\partial \phi} \quad (\text{N} \cdot \text{m})$$

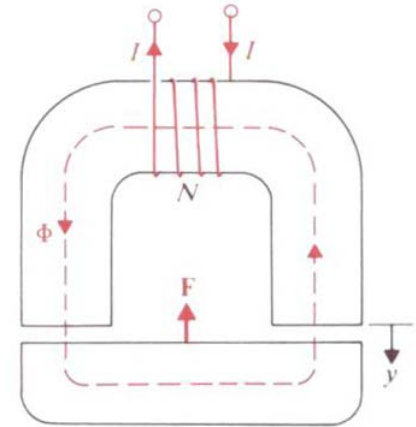
Example 6-23

System of circuits with constant flux linkage:

$$\begin{aligned} \rightarrow dW_m &= d(W_m)_{air\ gap} \\ &= 2 \left(\frac{B^2}{2\mu_0} S dy \right) \end{aligned}$$

$$\rightarrow \mathbf{F}_\Phi = -\mathbf{a}_y \frac{\partial W_m}{\partial y} = -\mathbf{a}_y \frac{B^2 S}{\mu_0} = -\mathbf{a}_y \frac{\Phi^2}{\mu_0 S} \quad (\text{N})$$

Magnetic flux



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

System of circuits with constant currents:

$$\rightarrow \Phi = \frac{\mathcal{V}_m}{\mathcal{R}_c + 2\mathcal{R}_g} = \frac{NI}{\mathcal{R}_c + 2y / \mu_0 S}$$

Magnetic flux linkage

$$\rightarrow W_m = \frac{1}{2} \sum_k I_k \Phi_k = \frac{1}{2} IN\Phi$$

Magnetic flux

$$\rightarrow \mathbf{F}_I = \mathbf{a}_y \frac{\partial W_m}{\partial y} = \mathbf{a}_y \frac{1}{2} \left(-\Phi^2 \frac{2}{\mu_0 S} \right) = -\mathbf{a}_y \frac{\Phi^2}{\mu_0 S} \quad (\text{N})$$

Example 6-24

Force between two coaxial circular coils:

Recall:

$$\rightarrow \mathbf{A} = \mathbf{a}_\phi \frac{\mu_0 I b^2}{4R^2} \sin \theta \quad (6-43)$$

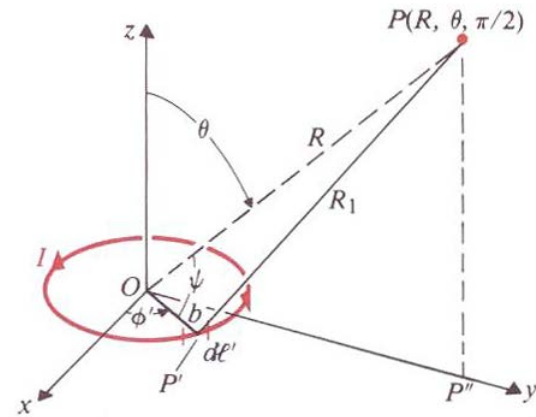
$$\begin{aligned} \rightarrow \mathbf{A}_{12} &= \mathbf{a}_\phi \frac{\mu_0 N_1 I_1 b_1^2}{4R^2} \sin \theta \\ &= \mathbf{a}_\phi \frac{\mu_0 N_1 I_1 b_1^2}{4R^2} \left(\frac{b_2}{R} \right) = \mathbf{a}_\phi \frac{\mu_0 N_1 I_1 b_1^2 b_2}{4(z^2 + b_2^2)^{3/2}} \end{aligned}$$

$$\begin{aligned} \rightarrow \Phi_{12}^L &= \oint_{C_2} \mathbf{A}_{12} \cdot d\mathbf{l}_2 = \frac{\mu_0 N_1 I_1 b_1^2 b_2}{4(z^2 + b_2^2)^{3/2}} \oint_{C_2} b_2 d\phi \\ &= \frac{\mu_0 N_1 N_2 \pi I_1 b_1^2 b_2^2}{2(z^2 + b_2^2)^{3/2}} \end{aligned}$$

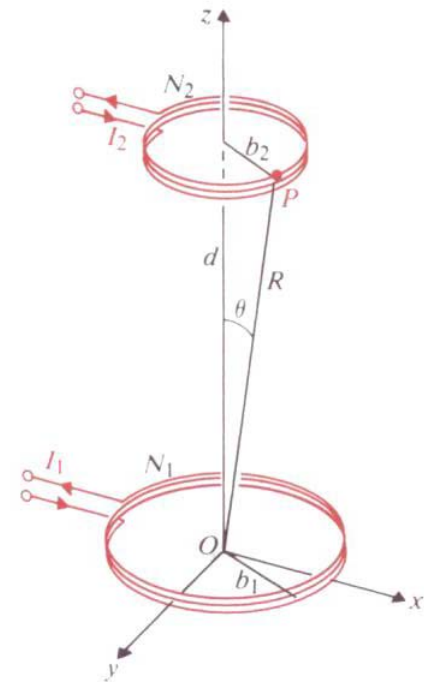
$$\rightarrow L_{12} = \frac{\Phi_{12}^L}{I_1} = \frac{\mu_0 N_1 N_2 \pi b_1^2 b_2^2}{2(z^2 + b_2^2)^{3/2}}$$

$$\rightarrow \mathbf{F}_{12} = \mathbf{a}_z I_1 I_2 \left. \frac{dL_{12}}{dz} \right|_{z=d} = -\mathbf{a}_z I_1 I_2 \frac{3\mu_0 N_1 N_2 \pi b_1^2 b_2^2 d}{2(d^2 + b_2^2)^{5/2}}$$

$$\cong -\mathbf{a}_z \frac{3\mu_0 m_1 m_2}{2\pi d^4} \quad (\text{N}) \quad \leftarrow m_1 = N_1 I_1 \pi b_1^2, \quad m_2 = N_2 I_2 \pi b_2^2$$



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