

Introduction to Electromagnetism

Time-Varying Fields and Maxwell's Equations

(7-1, 7-2)

Yoonchan Jeong

School of Electrical Engineering, Seoul National University

Tel: +82 (0)2 880 1623, Fax: +82 (0)2 873 9953

Email: yunchan@snu.ac.kr

Electromagnetostatic Model vs. Maxwell's Equations

Electrostatic model:

$$\nabla \times \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\mathbf{D} = \varepsilon \mathbf{E}$$

Magnetostatic model:

$$\nabla \times \mathbf{H} = \mathbf{J} = \sigma \mathbf{E}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{H} = \frac{1}{\mu} \mathbf{B}$$

Maxwell's equations (incl. time-varying):

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Faraday's law

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Ampère's law

$$\nabla \cdot \mathbf{D} = \rho$$

Gauss's law

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

Constitutive relations

$$\mathbf{H} = \frac{1}{\mu} \mathbf{B} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}$$

Generalized electromagnetism!

Faraday's Law of Electromagnetic Induction

Faraday discovered in 1831:

→ *The electromotive force induced in a closed circuit is equal to the negative rate of increase of the magnetic flux linking the circuit.*

$$\mathcal{V} = \oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s}$$

← *Negative sign: "Lenz's law"*

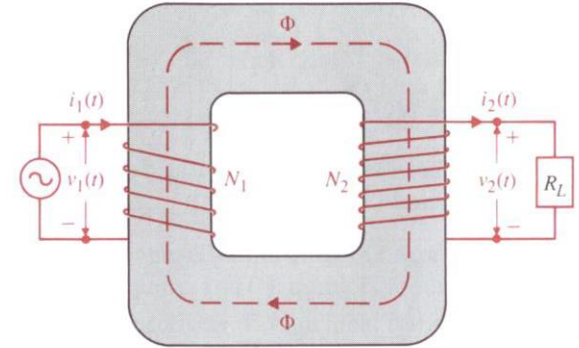
Transformers

Faraday's law of induction:

$$\mathcal{V} = \oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s}$$

For a magnetic circuit:

$$N_1 i_1 - N_2 i_2 = \mathcal{R} \Phi = \frac{l}{\mu S} \Phi$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

a) Ideal transformer: $\mu \rightarrow \infty$

$$\rightarrow N_1 i_1 - N_2 i_2 = 0$$

$$\rightarrow \frac{i_1}{i_2} = \frac{N_2}{N_1}$$

$$\rightarrow v_1 = \frac{d\Phi_1^L}{dt} = N_1 \frac{d\Phi}{dt}$$

$$\rightarrow v_2 = \frac{d\Phi_2^L}{dt} = N_2 \frac{d\Phi}{dt}$$

$$\rightarrow \frac{v_1}{v_2} = \frac{N_1}{N_2}$$

b) Real transformer:

$$\begin{aligned} \rightarrow v_1 &= \frac{d\Phi_1^L}{dt} = N_1 \frac{d\Phi}{dt} \\ &= \frac{\mu S}{l} \left(N_1^2 \frac{di_1}{dt} - N_1 N_2 \frac{di_2}{dt} \right) \end{aligned}$$

$$= L_1 \frac{di_1}{dt} - L_{12} \frac{di_2}{dt}$$

$$\begin{aligned} \rightarrow v_2 &= \frac{d\Phi_2^L}{dt} = N_2 \frac{d\Phi}{dt} \\ &= \frac{\mu S}{l} \left(N_1 N_2 \frac{di_1}{dt} - N_2^2 \frac{di_2}{dt} \right) \end{aligned}$$

$$= L_{12} \frac{di_1}{dt} - L_2 \frac{di_2}{dt}$$

Flux leakage $\rightarrow L_{12} = k \sqrt{L_1 L_2}, \quad k < 1$ ⁴

Faraday's Law: Fundamental Postulate

Faraday's law of induction:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s}$$

Fundamental postulate:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

→ *The electric field intensity in a region of time-varying magnetic flux is therefore nonconservative and cannot be expressed as the gradient of a scalar potential.*

A Stationary Circuit in a Time-Varying Magnetic Field

For a stationary circuit:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \leftarrow \text{Fundamental postulate}$$

$$\rightarrow \oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s}$$

$$\rightarrow \oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} = -\frac{d\Phi}{dt}$$

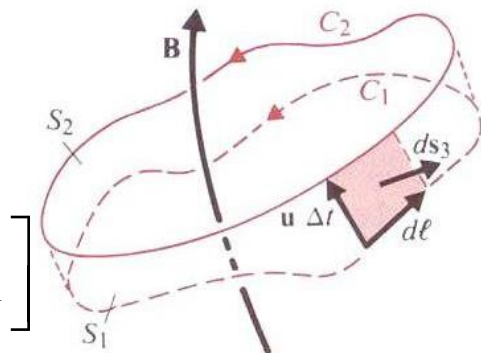
→ The postulate holds for a stationary circuit with Faraday's experimental law!

A Moving Circuit in a Time-Varying Magnetic Field (1)

For a moving circuit:

$$-\frac{d\Phi}{dt} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} \quad \leftarrow \text{Flux change rate}$$

$$= -\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left[\int_{S_2} \mathbf{B}(t + \Delta t) \cdot d\mathbf{s}_2 - \int_{S_1} \mathbf{B}(t) \cdot d\mathbf{s}_1 \right]$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

$$\rightarrow \mathbf{B}(t + \Delta t) = \mathbf{B}(t) + \frac{\partial \mathbf{B}(t)}{\partial t} \Delta t + H.O.T.$$

$$\rightarrow \frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} + \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left[\int_{S_2} \mathbf{B} \cdot d\mathbf{s}_2 - \int_{S_1} \mathbf{B} \cdot d\mathbf{s}_1 + H.O.T. \right]$$

$$\rightarrow \int_V \nabla \cdot \mathbf{B} \, dv = \int_{S_2} \mathbf{B} \cdot d\mathbf{s}_2 - \int_{S_1} \mathbf{B} \cdot d\mathbf{s}_1 + \int_{S_3} \mathbf{B} \cdot d\mathbf{s}_3 = 0$$

$$\leftarrow ds_3 = d\mathbf{l} \times \mathbf{u} \Delta t$$

$$\rightarrow \int_{S_2} \mathbf{B} \cdot d\mathbf{s}_2 - \int_{S_1} \mathbf{B} \cdot d\mathbf{s}_1 = -\Delta t \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

$$\rightarrow \frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} - \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

A Moving Circuit in a Time-Varying Magnetic Field (2)

For a moving circuit:

$$\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} - \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

$$\rightarrow -\frac{d\Phi}{dt} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} + \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = \oint_C \mathbf{E}' \cdot d\mathbf{l}$$

$$\rightarrow \oint_C (\mathbf{E}' - \mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

$$\rightarrow \mathbf{E}' = \mathbf{E} + \mathbf{u} \times \mathbf{B} \quad \leftarrow \mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

← Is this really true?

$$\rightarrow \oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

(Only valid for a nonrelativistic charge, i.e., $u \ll c$)

$$\rightarrow \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\mathbf{E}' = \gamma(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \frac{\gamma^2}{c^2(\gamma + 1)} (\mathbf{E} \cdot \mathbf{u})\mathbf{u}$$

$$\leftarrow \gamma = (1 - u^2/c^2)^{-1/2}$$

→ The postulate still holds for a moving circuit with Faraday's experimental law!