Introduction to Electromagnetism
Time-Varying Fields and Maxwell's Equations
(7-1, 7-2)

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Electromagnetostatic Model vs. Maxwell's Equations

Electrostatic model:

$$\nabla \times \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{D} = \rho$$
$$\mathbf{D} = \varepsilon \mathbf{E}$$

Magnetostatic model:

$$\nabla \times \mathbf{H} = \mathbf{J} \quad = \sigma \mathbf{E}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\mathbf{H} = \frac{1}{\mu}\mathbf{B}$$

Maxwell's equations (incl. time-varying):

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$ *Faraday's law*

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$ *Ampère's law*

$$\nabla \cdot \mathbf{D} = \rho$$
$$\nabla \cdot \mathbf{B} = 0$$
Gauss's law

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$
$$\mathbf{H} = \frac{1}{\mu}\mathbf{B} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$$
Constitutive relations

Generalized electromagnetism!

Faraday's Law of Electromagnetic Induction

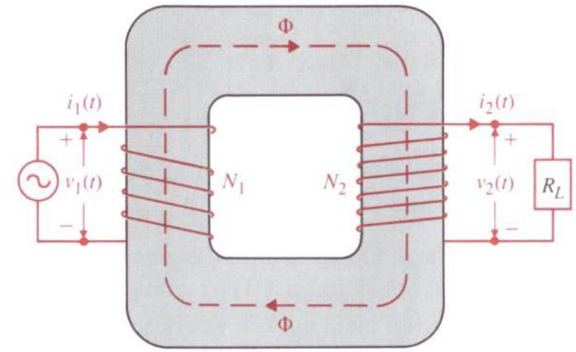
Faraday discovered in 1831:

→ *The electromotive force induced in a closed circuit is equal to the negative rate of increase of the magnetic flux linking the circuit.*

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{s}$$

← *Negative sign: "Lenz's law"*

Transformers



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Faraday's law of induction:

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s}$$

For a magnetic circuit:

$$N_1 i_1 - N_2 i_2 = \mathcal{R}\Phi = \frac{l}{\mu S}\Phi$$

a) Ideal transformer: $\mu \to \infty$

$$\to N_1 i_1 - N_2 i_2 = 0$$

$$\to \frac{i_1}{i_2} = \frac{N_2}{N_1}$$

$$\to v_1 = \frac{d\Phi_1^L}{dt} = N_1\frac{d\Phi}{dt}$$

$$\to v_2 = \frac{d\Phi_2^L}{dt} = N_2\frac{d\Phi}{dt}$$

$$\to \frac{v_1}{v_2} = \frac{N_1}{N_2}$$

b) Real transformer:

$$\to v_1 = \frac{d\Phi_1^L}{dt} = N_1\frac{d\Phi}{dt} = \frac{\mu S}{l}N_1\left(N_1\frac{di_1}{dt} - N_2\frac{di_2}{dt}\right) = L_1\frac{di_1}{dt} - L_{12}\frac{di_2}{dt}$$

$$\to v_2 = \frac{d\Phi_2^L}{dt} = N_2\frac{d\Phi}{dt} = \frac{\mu S}{l}N_2\left(N_1\frac{di_1}{dt} - N_2\frac{di_2}{dt}\right) = L_{12}\frac{di_1}{dt} - L_2\frac{di_2}{dt}$$

Flux leakage $\to L_{12} = k\sqrt{L_1 L_2}, \quad k < 1$

Faraday's Law: Fundamental Postulate

Faraday's law of induction:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{s}$$

Fundamental postulate:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$\rightarrow$ *The electric field intensity in a region of time-varying magnetic flux is therefore nonconservative and cannot be expressed as the gradient of a scalar potential any more.*

A Stationary Circuit in a Time-Varying Magnetic Field

For a stationary circuit:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \leftarrow \textit{Fundamental postulate}$$

$$\rightarrow \oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{s}$$

$$\rightarrow \oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{s} = -\frac{d\Phi}{dt}$$

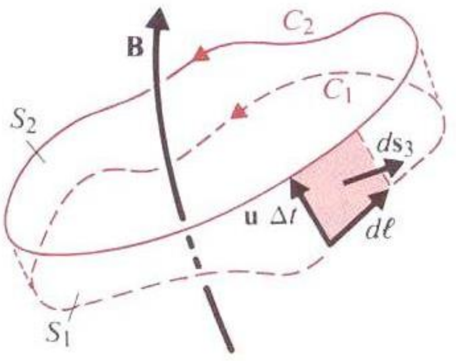
$\rightarrow$ *The postulate holds for a stationary circuit with Faraday's experimental law!*

A Moving Circuit in a Time-Varying Magnetic Field (1)

For a moving circuit:

$$-\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} \quad \leftarrow \text{Flux change rate}$$

$$= -\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{S_2}\mathbf{B}(t+\Delta t)\cdot d\mathbf{s}_2 - \int_{S_1}\mathbf{B}(t)\cdot d\mathbf{s}_1\right]$$



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$$\rightarrow \mathbf{B}(t+\Delta t) = \mathbf{B}(t) + \frac{\partial \mathbf{B}(t)}{\partial t}\Delta t + H.O.T.$$

$$\rightarrow \frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s} + \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\boxed{\int_{S_2}\mathbf{B}\cdot d\mathbf{s}_2 - \int_{S_1}\mathbf{B}\cdot d\mathbf{s}_1} + \cancel{H.O.T.}\right]$$

$$\rightarrow \int_V \nabla\cdot\mathbf{B}\,dv = \int_{S_2}\mathbf{B}\cdot d\mathbf{s}_2 - \int_{S_1}\mathbf{B}\cdot d\mathbf{s}_1 + \int_{S_3}\mathbf{B}\cdot d\mathbf{s}_3 = 0$$

$$\leftarrow d\mathbf{s}_3 = d\mathbf{l}\times\mathbf{u}\Delta t$$

$$\rightarrow \int_{S_2}\mathbf{B}\cdot d\mathbf{s}_2 - \int_{S_1}\mathbf{B}\cdot d\mathbf{s}_1 = \boxed{-\Delta t\oint_C (\mathbf{u}\times\mathbf{B})\cdot d\mathbf{l}}$$

$$\rightarrow \frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s} - \oint_C (\mathbf{u}\times\mathbf{B})\cdot d\mathbf{l}$$

A Moving Circuit in a Time-Varying Magnetic Field (2)

For a moving circuit:

$$\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s} - \oint_C (\mathbf{u}\times\mathbf{B})\cdot d\mathbf{l}$$

$$\rightarrow\ -\frac{d\Phi}{dt} = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s} + \oint_C (\mathbf{u}\times\mathbf{B})\cdot d\mathbf{l} = \oint_C \mathbf{E}'\cdot d\mathbf{l}$$

$$\rightarrow\ \oint_C (\mathbf{E}' - \mathbf{u}\times\mathbf{B})\cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s}$$

$$\rightarrow\ \mathbf{E}' = \mathbf{E} + \mathbf{u}\times\mathbf{B} \ \leftarrow\ \mathbf{F} = q(\mathbf{E} + \mathbf{u}\times\mathbf{B})$$

$\leftarrow$ Is this really true?

(Only valid for a nonrelativistic case, i.e., $u << c$)

$$\rightarrow\ \oint_C \mathbf{E}\cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s}$$

$$\rightarrow\ \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\mathbf{E}' = \gamma(\mathbf{E} + \mathbf{u}\times\mathbf{B}) - \frac{\gamma^2}{c^2(\gamma+1)}(\mathbf{E}\cdot\mathbf{u})\mathbf{u}$$

$$\leftarrow\ \gamma = (1 - u^2/c^2)^{-1/2}$$

$\rightarrow$ The postulate still holds for a moving circuit with Faraday's experimental law!