Introduction to Electromagnetism
Time-Varying Fields and Maxwell's Equations
(7-3, 7-4, 7-5)

Yoonchan Jeong
School of Electrical Engineering, Seoul National University
Tel: +82 (0)2 880 1623, Fax: +82 (0)2 873 9953
Email: yoonchan@snu.ac.kr

Electromagnetostatic Model vs. Maxwell's Equations

Electrostatic model:

$$\nabla \times \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\mathbf{D} = \varepsilon \mathbf{E}$$

Magnetostatic model:

$$\nabla \times \mathbf{H} = \mathbf{J} \quad = \sigma \mathbf{E}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{H} = \frac{1}{\mu} \mathbf{B}$$

Maxwell's equations (incl. time-varying):

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Faraday's law

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Ampère's law

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

Gauss's law

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{H} = \frac{1}{\mu} \mathbf{B} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}$$

Constitutive relations

Generalized electromagnetism!

Ampère's Law for Time-Varying EM Fields

What Ampère had discovered for *steady* currents:

$$\nabla \times \mathbf{H} = \mathbf{J}$$

However, the principle of conservation of charge must be satisfied at all times:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

Ampère's law for time-varying fields:

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 \overset{?}{=} \nabla \cdot \mathbf{J} \quad \leftarrow \text{Is this always true?}$$

This must be corrected as the following:

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = \nabla \cdot (\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t})$$

$$\rightarrow \boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}}$$

Displacement current density

"The major contribution of J. C. Maxwell"

Maxwell's Equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$ *Faraday's law*

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$ *Ampère's law*

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$ *Gauss's law*

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{H} = \frac{1}{\mu}\mathbf{B} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$$ *Constitutive relations*

The two divergence equations can be derived from the two curl equations! $\rightarrow$ *12 unknowns & 12 equations!*

Ready for doing anything with electromagnetism!

Integral Form of Maxwell's Equations

Differential form vs. integral form:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \rightarrow \quad \oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \rightarrow \quad \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{s}$$

$$\nabla \cdot \mathbf{D} = \rho \quad \rightarrow \quad \oint_S \mathbf{D} \cdot d\mathbf{s} = \int_V \rho dv$$

$$\nabla \cdot \mathbf{B} = 0 \quad \rightarrow \quad \oint_S \mathbf{B} \cdot d\mathbf{s} = 0$$

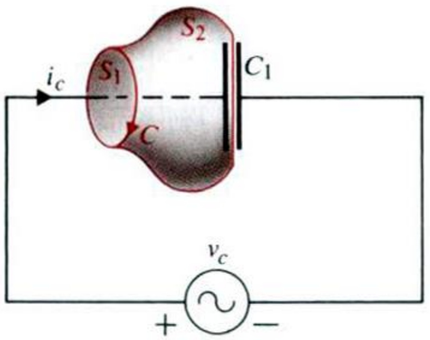
Example 7-5

For the conduction current in the connecting wire:

$$v_c = V_0 \sin \omega t$$

$$\rightarrow\ i_C = \frac{dQ_C}{dt} = C_1 \frac{dv_C}{dt} = C_1 V_0 \omega \cos \omega t$$

$$\leftarrow\ C_1 = \varepsilon \frac{A}{d}$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

For the displace current across the capacitor:

$$D = \varepsilon E = \varepsilon \frac{v_c}{d} = \varepsilon \frac{V_0}{d} \sin \omega t$$

$$\rightarrow\ i_D = \int_A \frac{d\mathbf{D}}{dt} \cdot d\mathbf{s} = \left(\varepsilon \frac{A}{d} \right) V_0 \omega \cos \omega t = C_1 V_0 \omega \cos \omega t = i_C$$

For the magnetic field intensity:

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = 2\pi r H_\phi = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{s}$$

$$\rightarrow\ \int_{S_1} \mathbf{J} \cdot d\mathbf{s} = i_C = C_1 V_0 \omega \cos \omega t \ \rightarrow\ \int_{S_2} \frac{d\mathbf{D}}{dt} \cdot d\mathbf{s} = i_D$$

$$\rightarrow\ H_\phi = \frac{C_1 V_0}{2\pi r} \omega \cos \omega t$$

Potential Functions

Vector magnetic potential:

$$\nabla \cdot \mathbf{B} = 0 \quad \rightarrow \boxed{\mathbf{B} = \nabla \times \mathbf{A} \quad \text{(T)}}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \rightarrow \nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A})$$

$$\rightarrow \nabla \times (\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}) = 0$$

Scalar electric potential:

$$\rightarrow \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V$$

$$\rightarrow \boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \quad \text{(V/m)}}$$

For static fields:

$$V = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho}{R} dv'$$

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\mathbf{J}}{R} dv'$$

→ Still valid for time-varying fields?

Wave Equation for Vector Potential

For a homogeneous medium:

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \rightarrow \nabla \times \mu \mathbf{H} = \mu \mathbf{J} + \mu \frac{\partial \mathbf{D}}{\partial t}$$

$$\rightarrow \nabla \times \nabla \times \mathbf{A} = \mu \mathbf{J} + \mu\varepsilon \frac{\partial}{\partial t}(-\nabla V - \frac{\partial \mathbf{A}}{\partial t})$$

$$\rightarrow \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu \mathbf{J} - \nabla(\mu\varepsilon \frac{\partial V}{\partial t}) - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

$$\rightarrow \nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} + \nabla(\nabla \cdot \mathbf{A} + \mu\varepsilon \frac{\partial V}{\partial t})$$

Recall Helmholtz's theorem:

$$\rightarrow \nabla \times \mathbf{A} = \mathbf{B}$$

$$\rightarrow \nabla \cdot \mathbf{A} = ? \quad \rightarrow \boxed{\nabla \cdot \mathbf{A} + \mu\varepsilon \frac{\partial V}{\partial t} = 0}$$

→ Lorenz condition (or Lorenz gauge)

Wave equation in Lorenz condition:

$$\boxed{\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}}$$

Wave Equation for Scalar Potential

For a homogeneous medium:

$$\nabla \cdot \mathbf{D} = \rho \quad \rightarrow \nabla \cdot \varepsilon \mathbf{E} = \rho$$

$$\rightarrow -\nabla \cdot \varepsilon (\nabla V + \frac{\partial \mathbf{A}}{\partial t}) = \rho$$

$$\rightarrow \nabla^2 V + \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) = -\frac{\rho}{\varepsilon}$$

$\rightarrow$ *Recall the Lorenz condition*

$$\rightarrow \boxed{\nabla \cdot \mathbf{A} + \mu\varepsilon \frac{\partial V}{\partial t} = 0}$$

Wave equation in Lorenz condition:

$$\boxed{\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}}$$

Coulomb gauge (*Radiation or transverse gauge*):

$$\rightarrow \nabla \cdot \mathbf{A} = 0$$

$$\rightarrow \boxed{\nabla^2 V = -\frac{\rho}{\varepsilon}}$$

Electromagnetic Boundary Conditions

Continuity conditions:

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \rightarrow \oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

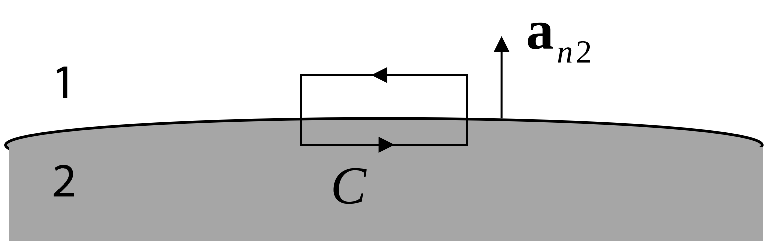
$$\rightarrow E_{1t} = E_{2t} \quad (\text{V/m})$$

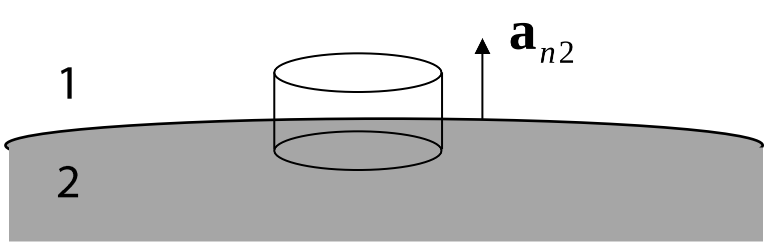
$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \rightarrow \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{s}$$

$$\rightarrow \mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \quad (\text{A/m})$$

$$\nabla \cdot \mathbf{D} = \rho \rightarrow \mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s \quad (\text{C/m}^2)$$

$$\nabla \cdot \mathbf{B} = 0 \rightarrow B_{1n} = B_{2n} \quad (\text{T})$$





Interface between Two Lossless Linear Media

Lossless non-conductive medium: $\sigma = 0$

$\rightarrow$ *No free charges & no surface currents*

General BCs

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \rightarrow E_{1t} = E_{2t}$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \rightarrow \mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$$

$$\nabla \cdot \mathbf{D} = \rho \rightarrow \mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$$

$$\nabla \cdot \mathbf{B} = 0 \rightarrow B_{1n} = B_{2n}$$

$\rightarrow$

BCs between two lossless media

$$E_{1t} = E_{2t} \rightarrow \frac{D_{1t}}{D_{2t}} = \frac{\varepsilon_1}{\varepsilon_2}$$

$$H_{1t} = H_{2t} \rightarrow \frac{B_{1t}}{B_{2t}} = \frac{\mu_1}{\mu_2}$$

$$D_{1n} = D_{2n} \rightarrow \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$$

$$B_{1n} = B_{2n} \rightarrow \mu_1 H_{1n} = \mu_2 H_{2n}$$

Interface between a Dielectric and a PEC

Perfect electric conductor: $\sigma = \infty$

$\rightarrow$ *No fields exist in the interior, i.e.,* $\mathbf{E} = \mathbf{D} = \mathbf{B} = \mathbf{H} = 0$*,*
"for time-varying fields"!

$\rightarrow$ *Any charges the perfect conductor may have will reside on the surface only!*

General BCs

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \rightarrow E_{1t} = E_{2t}$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \rightarrow \mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$$

$$\nabla \cdot \mathbf{D} = \rho \rightarrow \mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$$

$$\nabla \cdot \mathbf{B} = 0 \rightarrow B_{1n} = B_{2n}$$

$\rightarrow$

Medium 1 (Dielectric)	Medium 2 (PEC)
$E_{1t} = 0$	$E_{2t} = 0$
$\mathbf{a}_{n2} \times \mathbf{H}_1 = \mathbf{J}_s$	$H_{2t} = 0$
$\mathbf{a}_{n2} \cdot \mathbf{D}_1 = \rho_s$	$D_{2n} = 0$
$B_{1n} = 0$	$B_{2n} = 0$