Electromagnetics:

Time-Varying Fields
Maxwell's Equations
(7-1, 7-2, 7-3, 7-4, 7-5)

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Electromagnetostatic Model vs. Maxwell's Equations

Electrostatic model:

$$\nabla \times \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\mathbf{D} = \varepsilon \mathbf{E}$$

Magnetostatic model:

$$\nabla \times \mathbf{H} = \mathbf{J} \quad \boxed{= \sigma \mathbf{E}}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{H} = \frac{1}{\mu} \mathbf{B}$$

Maxwell's equations (incl. time-varying):

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$ *Faraday's law*

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$ *Ampère's law*

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

Gauss's law

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{H} = \frac{1}{\mu} \mathbf{B} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}$$

Constitutive relations

Generalized electromagnetism!

Faraday's Law of Electromagnetic Induction

Faraday discovered in 1831:

$\rightarrow$ *The electromotive force induced in a closed circuit is equal to the negative rate of increase of the magnetic flux linking the circuit.*

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{s}$$

$\leftarrow$ *Negative sign: "Lenz's law"*

Faraday's Law: Fundamental Postulate

Faraday's law of induction:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{s}$$

Fundamental postulate:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$\rightarrow$ *The electric field intensity in a region of time-varying magnetic flux is therefore nonconservative and cannot be expressed as the gradient of a scalar potential any more.*

A Stationary Circuit in a Time-Varying Magnetic Field

For a stationary circuit:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \leftarrow \textit{Fundamental postulate}$$

$$\rightarrow \oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{s}$$

$$\rightarrow \oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{s} = -\frac{d\Phi}{dt}$$

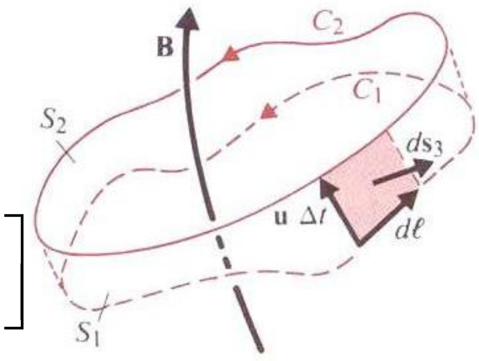
→ *The postulate holds for a stationary circuit with Faraday's experimental law!*

A Moving Circuit in a Time-Varying Magnetic Field (1)

For a moving circuit:

$$-\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} \qquad \leftarrow \text{Flux change rate}$$

$$= -\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{S_2}\mathbf{B}(t+\Delta t)\cdot d\mathbf{s}_2 - \int_{S_1}\mathbf{B}(t)\cdot d\mathbf{s}_1\right]$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

$$\rightarrow \mathbf{B}(t+\Delta t) = \mathbf{B}(t) + \frac{\partial \mathbf{B}(t)}{\partial t}\Delta t + H.O.T.$$

$$\rightarrow \frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s} + \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{S_2}\mathbf{B}\cdot d\mathbf{s}_2 - \int_{S_1}\mathbf{B}\cdot d\mathbf{s}_1 + H.O.T.\right]$$

$$\rightarrow \int_V \nabla\cdot\mathbf{B}\, dv = \int_{S_2}\mathbf{B}\cdot d\mathbf{s}_2 - \int_{S_1}\mathbf{B}\cdot d\mathbf{s}_1 + \int_{S_3}\mathbf{B}\cdot d\mathbf{s}_3 = 0$$

$$\leftarrow d\mathbf{s}_3 = d\mathbf{l}\times\mathbf{u}\Delta t$$

$$\rightarrow \int_{S_2}\mathbf{B}\cdot d\mathbf{s}_2 - \int_{S_1}\mathbf{B}\cdot d\mathbf{s}_1 = -\Delta t\oint_C (\mathbf{u}\times\mathbf{B})\cdot d\mathbf{l}$$

$$\rightarrow \frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s} - \oint_C (\mathbf{u}\times\mathbf{B})\cdot d\mathbf{l}$$

A Moving Circuit in a Time-Varying Magnetic Field (2)

For a moving circuit:

$$\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s} - \oint_C (\mathbf{u}\times\mathbf{B})\cdot d\mathbf{l}$$

$$\rightarrow\ -\frac{d\Phi}{dt} = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s} + \oint_C (\mathbf{u}\times\mathbf{B})\cdot d\mathbf{l} = \oint_C \mathbf{E}'\cdot d\mathbf{l}$$

$$\rightarrow\ \oint_C (\mathbf{E}' - \mathbf{u}\times\mathbf{B})\cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s}$$

$$\rightarrow\ \mathbf{E}' = \mathbf{E} + \mathbf{u}\times\mathbf{B} \ \leftarrow\ \mathbf{F} = q(\mathbf{E} + \mathbf{u}\times\mathbf{B})$$

$\leftarrow$ Is this really true?

$$\rightarrow\ \oint_C \mathbf{E}\cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s}$$

(Only valid for a nonrelativistic charge, i.e., $u << c$)

$$\rightarrow\ \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\mathbf{E}' = \gamma(\mathbf{E} + \mathbf{u}\times\mathbf{B}) - \frac{\gamma^2}{c^2(\gamma+1)}(\mathbf{E}\cdot\mathbf{u})\mathbf{u}$$

$$\leftarrow\ \gamma = (1 - u^2/c^2)^{-1/2}$$

$\rightarrow$ The postulate still holds for a moving circuit with Faraday's experimental law!

$\rightarrow$ Special theory of relativity

Ampère’s Law for Time-Varying EM Fields

What Ampère had discovered for *steady* currents:

$$\nabla \times \mathbf{H} = \mathbf{J}$$

However, the principle of conservation of charge must be satisfied at all times:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

Ampère’s law for time-varying fields:

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 \overset{?}{=} \nabla \cdot \mathbf{J} \quad \leftarrow \textit{Is this always true?}$$

This must be corrected as the following:

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = \nabla \cdot (\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t})$$

$$\rightarrow \boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}}$$

Displacement current density

“The major contribution of J. C. Maxwell”

Maxwell's Equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$ *Faraday's law*

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$ *Ampère's law*

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

Gauss's law

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{H} = \frac{1}{\mu}\mathbf{B} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$$

Constitutive relations

The two divergence equations can be derived from the two curl equations!

$\rightarrow$ 12 unknowns & 12 equations!

Ready for electromagnetics!

Differential/Integral Form of Maxwell's Equations

Differential form vs. integral form:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \rightarrow \oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \rightarrow \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{s}$$

$$\nabla \cdot \mathbf{D} = \rho \rightarrow \oint_S \mathbf{D} \cdot d\mathbf{s} = \int_V \rho dv$$

$$\nabla \cdot \mathbf{B} = 0 \rightarrow \oint_S \mathbf{B} \cdot d\mathbf{s} = 0$$

Potential Functions

Vector magnetic potential:

$$\nabla \cdot \mathbf{B} = 0 \quad \rightarrow \boxed{\mathbf{B} = \nabla \times \mathbf{A} \quad \text{(T)}}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \rightarrow \nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A})$$

$$\rightarrow \nabla \times (\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}) = 0$$

Scalar electric potential:

$$\rightarrow \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V$$

$$\rightarrow \boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \quad \text{(V/m)}}$$

For static fields:

$$V = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho}{R} dv'$$

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J}}{R} dv'$$

→ *Still valid for time-varying fields?*

Wave Equation for Vector Potential

For a homogeneous medium:

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \rightarrow \nabla \times \mu \mathbf{H} = \mu \mathbf{J} + \mu \frac{\partial \mathbf{D}}{\partial t}$$

$$\rightarrow \nabla \times \nabla \times \mathbf{A} = \mu \mathbf{J} + \mu\varepsilon \frac{\partial}{\partial t}(-\nabla V - \frac{\partial \mathbf{A}}{\partial t})$$

$$\rightarrow \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu \mathbf{J} - \nabla(\mu\varepsilon \frac{\partial V}{\partial t}) - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

$$\rightarrow \nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} + \nabla(\nabla \cdot \mathbf{A} + \mu\varepsilon \frac{\partial V}{\partial t})$$

Recall Helmholtz's theorem:

$$\rightarrow \nabla \times \mathbf{A} = \mathbf{B}$$

$$\rightarrow \nabla \cdot \mathbf{A} = ? \quad \leftarrow \boxed{\nabla \cdot \mathbf{A} = -\mu\varepsilon \frac{\partial V}{\partial t}}$$

$\rightarrow$ *Lorenz condition (or Lorenz gauge)*

Wave equation in Lorenz condition:

$$\boxed{\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}}$$

Wave Equation for Scalar Potential

For a homogeneous medium:

$$\nabla \cdot \mathbf{D} = \rho \quad \rightarrow \nabla \cdot \varepsilon \mathbf{E} = \rho$$

$$\rightarrow -\nabla \cdot \varepsilon (\nabla V + \frac{\partial \mathbf{A}}{\partial t}) = \rho$$

$$\rightarrow \nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\varepsilon}$$

→ *Recall the Lorenz condition*

$$\rightarrow \boxed{\nabla \cdot \mathbf{A} + \mu\varepsilon \frac{\partial V}{\partial t} = 0}$$

Wave equation in Lorenz condition:

$$\boxed{\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}}$$

cf. Coulomb gauge (*Radiation or transverse gauge*):

$$\rightarrow \nabla \cdot \mathbf{A} = 0$$

$$\rightarrow \boxed{\nabla^2 V = -\frac{\rho}{\varepsilon}}$$

Electromagnetic Boundary Conditions

Continuity conditions:

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \rightarrow \oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

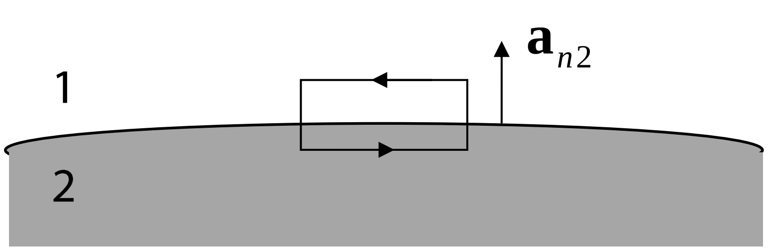
$$\rightarrow E_{1t} = E_{2t} \quad (\text{V/m})$$

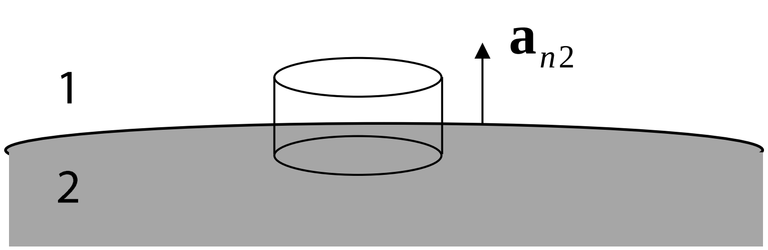
$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \rightarrow \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{s}$$

$$\rightarrow \mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \quad (\text{A/m})$$

$$\nabla \cdot \mathbf{D} = \rho \rightarrow \mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s \quad (\text{C/m}^2)$$

$$\nabla \cdot \mathbf{B} = 0 \rightarrow B_{1n} = B_{2n} \quad (\text{T})$$





Interface between Two Lossless Linear Media

Lossless non-conductive medium: $\sigma = 0$

$\rightarrow$ *No free charges & no surface currents*

General BCs

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \rightarrow E_{1t} = E_{2t}$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \rightarrow \mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \cancel{\mathbf{J}_s}$$

$$\nabla \cdot \mathbf{D} = \rho \rightarrow \mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \cancel{\rho_s}$$

$$\nabla \cdot \mathbf{B} = 0 \rightarrow B_{1n} = B_{2n}$$

$\rightarrow$

BCs between two lossless media

$$E_{1t} = E_{2t} \rightarrow \frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$

$$H_{1t} = H_{2t} \rightarrow \frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2}$$

$$D_{1n} = D_{2n} \rightarrow \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$$

$$B_{1n} = B_{2n} \rightarrow \mu_1 H_{1n} = \mu_2 H_{2n}$$

Interface between a Dielectric and a PEC

Perfect electric conductor: $\sigma = \infty$

→ *No fields exist in the interior, i.e.,* $\mathbf{E} = \mathbf{D} = \mathbf{B} = \mathbf{H} = 0$, *"for time-varying fields"!*

→ *Any charges the perfect conductor may have will reside on the surface only!*

General BCs

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \rightarrow E_{1t} = E_{2t}$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \rightarrow \mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$$

$$\nabla \cdot \mathbf{D} = \rho \rightarrow \mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$$

$$\nabla \cdot \mathbf{B} = 0 \rightarrow B_{1n} = B_{2n}$$

→

Medium 1 (Dielectric)	Medium 2 (PEC)
$E_{1t} = 0$	$E_{2t} = 0$
$\mathbf{a}_{n2} \times \mathbf{H}_1 = \mathbf{J}_s$	$H_{2t} = 0$
$\mathbf{a}_{n2} \cdot \mathbf{D}_1 = \rho_s$	$D_{2n} = 0$
$B_{1n} = 0$	$B_{2n} = 0$