Electromagnetics:

Time-Varying Fields
Maxwell's Equations
(7-6, 7-7)

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Farewell to Electromagnetostatic Model

Electrostatic model:

$$\nabla \times \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\mathbf{D} = \varepsilon \mathbf{E}$$

Magnetostatic model:

$$\nabla \times \mathbf{H} = \mathbf{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{B} = \mu \mathbf{H}$$

Maxwell's equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$ *Faraday's law*

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$ *Ampère's law*

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

Gauss's law

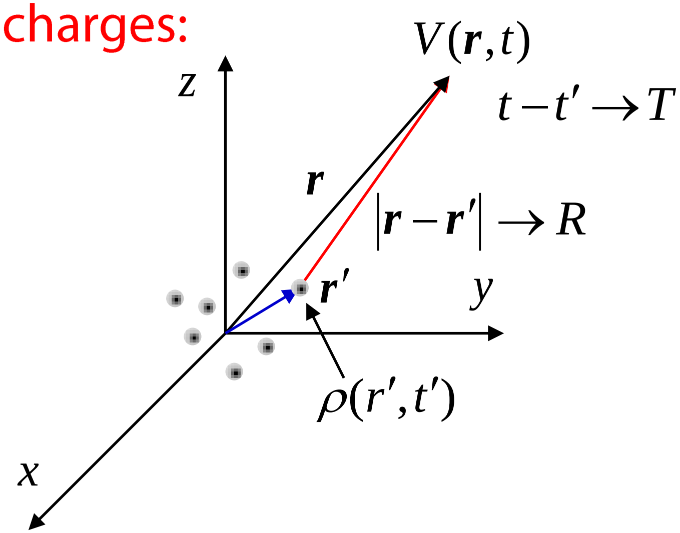
$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{H} = \frac{1}{\mu}\mathbf{B} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$$

Constitutive relations

Solution of Wave Equations for Potentials (1)

Wave equation for a time-varying distributed charges:



$$\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}$$

Homogeneous wave equation:

$$\nabla^2 V_h - \mu\varepsilon \frac{\partial^2 V_h}{\partial t^2} = 0$$

Spherical symmetry:

$$\rightarrow \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial V_{h0}}{\partial R}\right) - \mu\varepsilon \frac{\partial^2 V_{h0}}{\partial T^2} = 0$$

$$\rightarrow V_{h0}(R,T) \equiv \frac{1}{R} U(R,T) \quad \rightarrow \frac{\partial^2 U}{\partial R^2} - \mu\varepsilon \frac{\partial^2 U}{\partial T^2} = 0$$

$$\rightarrow U(R,T) = f(T - R/u) \;\; or \;\; f(T + R/u), \qquad \boxed{u = 1/\sqrt{\mu\varepsilon}}$$

Physically not useful for the moment!

Phase velocity

$$\rightarrow \boxed{V_{h0}(R,T) = \frac{1}{R} f(T - R/u)}$$

Solution of Wave Equations for Potentials (2)

Wave equation for a time-varying distributed charges:

$$\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}$$

Green's function:

Dirac delta functions

$$(\nabla^2 - \mu\varepsilon \frac{\partial^2}{\partial t^2})G = \underline{\delta(t-t')\delta(\mathbf{r}-\mathbf{r}')} \quad for \quad \mathbf{r} \to \mathbf{r}',\ t \to t'$$

$$(\nabla^2 - \mu\varepsilon \frac{\partial^2}{\partial t^2})G = 0 \quad for \quad \mathbf{r} \neq \mathbf{r}',\ t \neq t'$$

$$\to V = \iint G(\mathbf{r},\mathbf{r}',t,t')\,[-\frac{\rho(\mathbf{r}',t')}{\varepsilon}]\, d^3\mathbf{r}'dt'$$

For $\mathbf{r} \neq \mathbf{r}', t \neq t'$: $V_{h0}(R,T) = \frac{1}{R} f(T - R/u)$

$$\to G_{trial}(\mathbf{r},\mathbf{r}',t,t') = \frac{f(t-t'-|\mathbf{r}-\mathbf{r}'|/u)}{|\mathbf{r}-\mathbf{r}'|}$$

Valid for homogeneous eq.

Solution of Wave Equations for Potentials (3)

For $\mathbf{r} \to \mathbf{r}'$, $t \neq t'$:

$$(\nabla^2 - \mu\varepsilon \frac{\partial^2}{\partial t^2}) G_{trial} = \lim_{\mathbf{r}\to\mathbf{r}'} \left[(\nabla^2 - \mu\varepsilon \frac{\partial^2}{\partial t^2}) \frac{f(t - t' - |\mathbf{r} - \mathbf{r}'|/u)}{|\mathbf{r} - \mathbf{r}'|} \right]$$

$$= \lim_{\mathbf{r}\to\mathbf{r}'} \left[(\nabla^2 - \mu\varepsilon \frac{\partial^2}{\partial t^2}) \frac{f(t-t')}{|\mathbf{r} - \mathbf{r}'|} \right]$$

$$= \lim_{\mathbf{r}\to\mathbf{r}'} \left\{ [f(t-t') \nabla^2 (\frac{1}{|\mathbf{r} - \mathbf{r}'|}) - \frac{\mu\varepsilon}{|\mathbf{r} - \mathbf{r}'|} \frac{d^2 f(t-t')}{dt^2}] \right\}$$

$\rightarrow$ *Negligible compared to the 1st term*

$$= \lim_{\mathbf{r}\to\mathbf{r}'} \left[f(t-t') \nabla^2 (\frac{1}{|\mathbf{r} - \mathbf{r}'|}) \right] = -f(t-t') 4\pi \delta(\mathbf{r} - \mathbf{r}')$$

Recall:

$$\leftarrow \boxed{\nabla^2 V = -\frac{q}{\varepsilon_0} \delta(\mathbf{r} - \mathbf{r}')}$$

In result:

$$(\nabla^2 - \mu\varepsilon \frac{\partial^2}{\partial t^2}) G_{trial} = -4\pi f(t-t') \delta(\mathbf{r} - \mathbf{r}')$$

Solution of Wave Equations for Potentials (4)

For $\mathbf{r} \to \mathbf{r}'$, $t \neq t'$:

$$(\nabla^2 - \mu\varepsilon \frac{\partial^2}{\partial t^2}) G_{trial} = -4\pi f(t-t')\delta(\mathbf{r}-\mathbf{r}')$$

For $\mathbf{r} \to \mathbf{r}'$, $t \to t'$:

$$(\nabla^2 - \mu\varepsilon \frac{\partial^2}{\partial t^2}) G = \delta(t-t')\delta(\mathbf{r}-\mathbf{r}')$$

$$\to G_{trial} = G$$

$$if \quad f(\xi) = -\frac{1}{4\pi}\delta(\xi)$$

$$\to G_{trial}(\mathbf{r},\mathbf{r}',t,t') = \frac{f(t-t'-|\mathbf{r}-\mathbf{r}'|/u)}{|\mathbf{r}-\mathbf{r}'|} = -\frac{1}{4\pi}\frac{\delta(t-t'-|\mathbf{r}-\mathbf{r}'|/u)}{|\mathbf{r}-\mathbf{r}'|} = G$$

Green's function

Finally:

$$V = \iint G(\mathbf{r},\mathbf{r}',t,t')\,[-\frac{\rho(\mathbf{r}',t')}{\varepsilon}]\,d^3\mathbf{r}'dt'$$

$$= \iint -\frac{1}{4\pi}\frac{\delta(t-t'-|\mathbf{r}-\mathbf{r}'|/u)}{|\mathbf{r}-\mathbf{r}'|}[-\frac{\rho(\mathbf{r}',t')}{\varepsilon}]\,d^3\mathbf{r}'dt'$$

$$= \frac{1}{4\pi\varepsilon}\int \frac{\rho(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/u)}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}'$$

→ Retarded scalar potential

→ Is this a general solution?

Retarded Vector Potential

Similarly, for a vector magnetic potential:

$$\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}$$

$$\mathbf{A}(r,t) = \frac{\mu}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/u)}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'$$

"It takes time for electromagnetic waves to travel in order that the effects of the time-varying charges or currents are to appear at distant points."

Source-Free Wave Equations

Source free, simple medium: $\rho = 0$ & $\mathbf{J} = 0$ $(\sigma = 0)$

→ *Linear, isotropic and homogeneous*

Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{B} = \mu \mathbf{H} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M}$$

→

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{H} = 0$$

$$\nabla \times \nabla \times \mathbf{E} = -\mu \frac{\partial}{\partial t}(\nabla \times \mathbf{H}) = -\mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$= \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$

$$\rightarrow \nabla^2 \mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$u = 1/\sqrt{\mu\varepsilon}$$

→ *Phase velocity*

$$\rightarrow \nabla^2 \mathbf{E} - \frac{1}{u^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\rightarrow \nabla^2 \mathbf{H} - \frac{1}{u^2} \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$$

→ *Homogeneous vector wave eqs.*

Time-Harmonic Fields (1)

Maxwell's equations: *Linear differential equations*

$\rightarrow$ *Principle of superposition*

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

Wave equation for a scalar potential:

$$\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}$$

For a monochromatic charge density:

$$\rho(\mathbf{r}, t) = \frac{1}{2}[\rho(\mathbf{r})e^{j\omega t} + c.c.]$$

(c.c.: complex conjugate)

$$= \mathrm{Re}[\rho(\mathbf{r})e^{j\omega t}]$$

Let a trial solution be:

$$V(\mathbf{r}, t) = \frac{1}{2}[V(\mathbf{r})e^{j\omega t} + c.c.]$$

$$= \mathrm{Re}[V(\mathbf{r})e^{j\omega t}]$$

Time-Harmonic Fields (2)

Wave equation:

$$\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}$$

Recall the principle of superposition!

$$\rightarrow (\nabla^2 - \mu\varepsilon \frac{\partial^2}{\partial t^2})\left\{\frac{1}{2}[V(\mathbf{r})e^{j\omega t} + c.c.]\right\} = -\frac{1}{\varepsilon}\frac{1}{2}[\rho(\mathbf{r})e^{j\omega t} + c.c.]$$

$$\rightarrow (\nabla^2 - \mu\varepsilon \frac{\partial^2}{\partial t^2})[V(\mathbf{r})e^{j\omega t}] = -\frac{1}{\varepsilon}[\rho(\mathbf{r})e^{j\omega t}]$$

Note: $\frac{\partial}{\partial t} \rightarrow j\omega$

$$\rightarrow \nabla^2 V(\mathbf{r}) + k^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon}, \quad k = \omega\sqrt{\mu\varepsilon} = \omega / u$$

Recall Green's function:

Phase retardation

H.W.

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon}\int \frac{\rho(\mathbf{r}')e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}'$$

Similarly:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}'$$

Time-Harmonic Fields (3)

Maxwell's equations: *Linear differential equations*

$\rightarrow$ *Principle of superposition*

$\rightarrow$ *Phasor notation for monochromatic fields:*

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$
$$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}$$
$$\nabla\cdot\mathbf{D} = \rho$$
$$\nabla\cdot\mathbf{B} = 0$$

$\rightarrow$

$$\nabla\times\mathbf{E} = -j\omega\mathbf{B}$$
$$\nabla\times\mathbf{H} = \mathbf{J} + j\omega\mathbf{D}$$
$$\nabla\cdot\mathbf{D} = \rho$$
$$\nabla\cdot\mathbf{B} = 0$$

or

$$\nabla\times\mathbf{E} = -j\omega\mu\mathbf{H}$$
$$\nabla\times\mathbf{H} = j\omega\varepsilon\mathbf{E}$$
$$\nabla\cdot\mathbf{E} = 0$$
$$\nabla\cdot\mathbf{H} = 0$$

Source-free, simple medium

$\downarrow$

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = 0$$
$$\nabla^2\mathbf{H} + k^2\mathbf{H} = 0$$

$\rightarrow$ *Homogeneous Helmholtz's eqs.*

For non-monochromatic fields:

$$\mathbf{E}(r,t) = \mathrm{Re}\sum_i \boldsymbol{\psi}_i(\mathbf{r})e^{j\omega_i t}$$

or

$$\mathbf{E}(r,t) = \mathrm{Re}\int \boldsymbol{\psi}(\mathbf{r},\omega)e^{j\omega t}d\omega$$

Principle of Duality

For a source-free, simple medium: $(\mathbf{E}, \mathbf{H})$ $\leftarrow$ *Solution of Maxwell's eqs.*

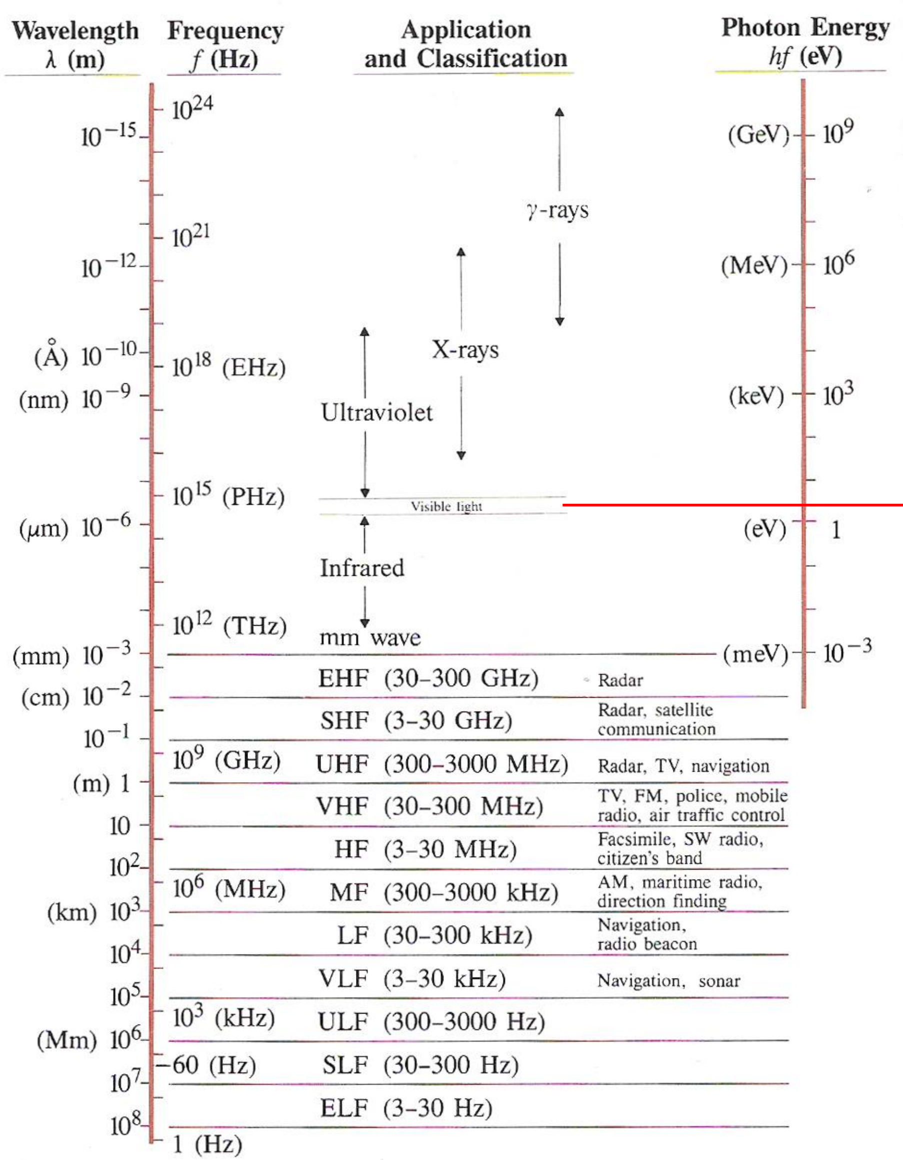
$$\rightarrow \mathbf{E}' = \eta \mathbf{H}$$

$$\leftarrow \eta = \sqrt{\mu / \varepsilon}$$ $\leftarrow$ *Intrinsic impedance*

$$\rightarrow \mathbf{H}' = -\frac{1}{\eta} \mathbf{E}$$

Principle of duality: *H.W.*

Electromagnetic Spectrum



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Visible light
(400 nm ~ 700 nm)