Electromagnetics:
Plane Waves in Lossless Media
(8-1, 8-2)

Yoonchan Jeong

School of Electrical Engineering, Seoul National University

Tel: +82 (0)2 880 1623, Fax: +82 (0)2 873 9953
Email: yoonchan@snu.ac.kr

Scalar and Vector Potentials

Wave equations for a homogeneous medium:

$$\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}$$

$$\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}$$

$\rightarrow$ *Recall the Lorenz condition*

$$\leftarrow \nabla \cdot \mathbf{A} + \mu\varepsilon \frac{\partial V}{\partial t} = 0$$

Fields from potentials:

$$\rightarrow \ \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

$$\rightarrow \ \mathbf{B} = \nabla \times \mathbf{A}$$

$\leftarrow$ *Formalism by Maxwell*

Plane Waves in Lossless Media (1)

Source-free, lossless, & uniform isotropic:

$$\rho = 0,\ \mathbf{J} = 0,\ \sigma = 0,\ \mu = const.,\ \&\ \varepsilon = const.$$

Maxwell's equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{B} = \mu \mathbf{H} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M}$$

$\rightarrow$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{H} = 0$$

$$\rightarrow \nabla^2 \mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\rightarrow \nabla^2 \mathbf{H} - \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$$

Homogeneous wave eqs.

Plane Waves in Lossless Media (2)

Homogeneous wave equation:

$$\nabla^2 \mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

For a monochromatic wave: $\leftarrow$ *ω: fixed to one value*

$$\rightarrow \mathbf{E} = \frac{1}{2}[\mathbf{E}(\mathbf{r})e^{j\omega t} + c.c] = \mathrm{Re}[\mathbf{E}(\mathbf{r})e^{j\omega t}]$$

For a monochromatic *"plane"* wave: $\leftarrow$ *$\mathbf{k}$: fixed to one value*

$$\rightarrow \mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} = \mathbf{E}_0 e^{-j(k_x x + k_y y + k_z z)}$$

$$\rightarrow \nabla^2[\mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}] = \mathbf{E}_0 \nabla^2 (e^{-j\mathbf{k}\cdot\mathbf{r}}) = -k^2 \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}$$

Note: $\nabla \rightarrow -j\mathbf{k}$

$$\leftarrow k^2 = |\mathbf{k}|^2 = k_x^2 + k_y^2 + k_z^2$$

$$\rightarrow \frac{\partial^2}{\partial t^2}[\mathbf{E}(\mathbf{r})e^{j\omega t}] = -\omega^2 \mathbf{E}(\mathbf{r})e^{j\omega t}$$

Note: $\frac{\partial}{\partial t} \rightarrow j\omega$

In result:

$$\nabla^2 \mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \;\rightarrow\; -k^2 + \omega^2 \mu\varepsilon = 0 \;\rightarrow\; k = \pm\omega\sqrt{\mu\varepsilon}$$

Note that $\mathbf{E}_0$ and $\mathbf{k}$ are still arbitrary!

$$\rightarrow \mathbf{E} = \frac{1}{2}[\mathbf{E}_0 e^{j(\omega t - \mathbf{k}\cdot\mathbf{r})} + c.c] = \mathrm{Re}[\mathbf{E}_0 e^{j(\omega t - \mathbf{k}\cdot\mathbf{r})}]$$

Plane Waves in Lossless Media (3)

Recall source-free, lossless, & uniform isotropic media:

$$\rho = 0,\ \mathbf{J} = 0,\ \sigma = 0,\ \mu = const.,\ \&\ \varepsilon = const.$$

Relation between $\mathbf{E}$ & $\mathbf{k}$ or $\mathbf{H}$ & $\mathbf{k}$:

For an electric field: $\rightarrow \nabla \cdot \mathbf{E} = 0$

$$\nabla \cdot (f\mathbf{A}) = \nabla f \cdot \mathbf{A} + f(\nabla \cdot \mathbf{A})$$

$$\rightarrow \nabla \cdot [\mathbf{E}_0 e^{j(\omega t - \mathbf{k}\cdot\mathbf{r})}] = (\nabla \cdot \mathbf{E}_0) e^{j(\omega t - \mathbf{k}\cdot\mathbf{r})} + \mathbf{E}_0 \cdot \nabla [e^{j(\omega t - \mathbf{k}\cdot\mathbf{r})}]$$

$$\rightarrow \nabla \cdot [\mathbf{E}_0 e^{j(\omega t - \mathbf{k}\cdot\mathbf{r})}] = \mathbf{E}_0 \cdot [-j\mathbf{k}\, e^{j(\omega t - \mathbf{k}\cdot\mathbf{r})}] = 0$$

Note: $\nabla \rightarrow -j\mathbf{k}$

$$\rightarrow \mathbf{E}_0 \cdot \mathbf{k} = 0$$ $\leftarrow$ ***E** is perpendicular to **k**!*

For a magnetic field:

Similarly:

$$\nabla^2 \mathbf{H} - \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \quad \rightarrow \mathbf{H} = \frac{1}{2}[\mathbf{H}_0 e^{j(\omega t - \mathbf{k}\cdot\mathbf{r})} + c.c] = \mathrm{Re}[\mathbf{H}_0 e^{j(\omega t - \mathbf{k}\cdot\mathbf{r})}]$$

$$\rightarrow \nabla \cdot \mathbf{H} = 0 \rightarrow \mathbf{H}_0 \cdot \mathbf{k} = 0$$ $\leftarrow$ ***H** is also perpendicular to **k**!*

Plane Waves in Lossless Media (4)

Relation between $\mathbf{E}$ & $\mathbf{H}$:

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad \rightarrow \mathbf{H}(\mathbf{r}) = -\frac{1}{j\omega\mu} \nabla \times \mathbf{E}(\mathbf{r})$$

$$\rightarrow \nabla \times \mathbf{E}(\mathbf{r}) = \nabla \times (\mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}})$$

$$\boxed{\nabla \times (f\mathbf{A}) = \nabla f \times \mathbf{A} + f(\nabla \times \mathbf{A})}$$

$$= (\nabla \times \mathbf{E}_0) e^{-j\mathbf{k}\cdot\mathbf{r}} + \nabla(e^{-j\mathbf{k}\cdot\mathbf{r}}) \times \mathbf{E}_0$$

$$= [-j\mathbf{k}\, e^{-j\mathbf{k}\cdot\mathbf{r}}] \times \mathbf{E}_0$$

Note: $\nabla \rightarrow -j\mathbf{k}$

$$= -je^{-j\mathbf{k}\cdot\mathbf{r}} (\mathbf{k} \times \mathbf{E}_0)$$

$$\rightarrow \mathbf{H}(\mathbf{r}) = \mathbf{H}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} = -\frac{1}{j\omega\mu} \nabla \times \mathbf{E}(\mathbf{r})$$

$$= -\frac{1}{j\omega\mu}(-je^{-j\mathbf{k}\cdot\mathbf{r}})(\mathbf{k} \times \mathbf{E}_0) = \frac{1}{\omega\mu}(\mathbf{k} \times \mathbf{E}_0) e^{-j\mathbf{k}\cdot\mathbf{r}}$$

$$\rightarrow \mathbf{H}_0 = \frac{1}{\omega\mu}(\mathbf{k} \times \mathbf{E}_0)$$

← $\mathbf{H}$ *is perpendicular to* $\mathbf{E}_0$ *as well as* $\mathbf{k}$*!*

Plane Waves in Lossless Media (4)

Relations among **E**, **H** & **k**:

$$\nabla \cdot \mathbf{E} = 0 \quad \rightarrow \quad -j\mathbf{k} \cdot \mathbf{E} = 0 \quad \rightarrow \quad \mathbf{k} \perp \mathbf{E}$$

$$\nabla \cdot \mathbf{H} = 0 \quad \rightarrow \quad -j\mathbf{k} \cdot \mathbf{H} = 0 \quad \rightarrow \quad \mathbf{k} \perp \mathbf{H}$$

$$\mathbf{H} = -\frac{1}{j\omega\mu}(\nabla \times \mathbf{E}) \quad \rightarrow \quad \mathbf{H} = -\frac{1}{j\omega\mu}(-j\mathbf{k} \times \mathbf{E}) \quad \rightarrow \quad \mathbf{k} \perp \mathbf{H} \ \ \& \ \ \mathbf{H} \perp \mathbf{E}$$

$$\rightarrow \quad \mathbf{k} \propto \mathbf{E} \times \mathbf{H}$$ ← *Transverse electromagnetic (TEM) wave*

Plane Waves in Lossless Media (5)

Travelling wave:

$$\mathbf{E} = \mathrm{Re}[\mathbf{E}_0 e^{j(\omega t - \mathbf{k}\cdot\mathbf{r})}], \quad \mathbf{H} = \mathrm{Re}[\mathbf{H}_0 e^{j(\omega t - \mathbf{k}\cdot\mathbf{r})}]$$

Phase velocity:

$$\omega t - \mathbf{k}\cdot\mathbf{r} = \mathrm{const.}$$

$$u_p = \frac{dr}{dt} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}}$$

For free space:

$$\rightarrow \frac{1}{\sqrt{\mu_0\varepsilon_0}} \equiv c \cong 3\times 10^8 \text{ (m/s)}$$

Intrinsic impedance:

$$\left|\frac{\mathbf{E}}{\mathbf{H}}\right| = \left|\frac{\omega\mu}{\mathbf{k}}\right| = \left|\frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}}\right| = \sqrt{\frac{\mu}{\varepsilon}} \rightarrow \sqrt{\frac{\mu_0}{\varepsilon_0}} \cong 377 \cong 120\pi \ (\Omega)$$

Significance of parameters:

Wavelength λ: Minimum travelling distance to lead to the same position of the wave

Wavenumber k: $2\pi \times$ number of wavelength per unit distance

$$\rightarrow k = \frac{2\pi}{\lambda} \qquad \rightarrow u_p = \frac{\omega}{k} = \frac{2\pi f}{2\pi/\lambda} = f\lambda$$

Polarization of Plane Waves (1)

For a fixed value of $\mathbf{k}$:

$$\mathbf{E} = \mathrm{Re}[\mathbf{E}_0 e^{j(\omega t - \mathbf{k}\cdot\mathbf{r})}], \quad \mathbf{H} = \mathrm{Re}[\mathbf{H}_0 e^{j(\omega t - \mathbf{k}\cdot\mathbf{r})}]$$

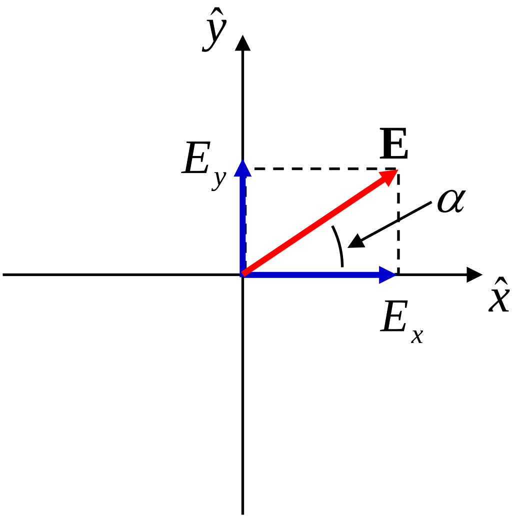
Note that the direction of the E-field vector (polarization) and its phase are still arbitrary.

Let $\mathbf{k}$ // z-axis:

$$\rightarrow \mathbf{E} = \mathrm{Re}(E_x\hat{x} + E_y\hat{y}) = \mathrm{Re}[E_{0,x}e^{j(\omega t - kz)}\hat{x} + E_{0,y}e^{j(\omega t - kz)}\hat{y}]$$

$$\rightarrow E_{0,x} = |E_{0,x}|e^{j\phi_{0,x}}, \quad E_{0,y} = |E_{0,y}|e^{j\phi_{0,y}}$$

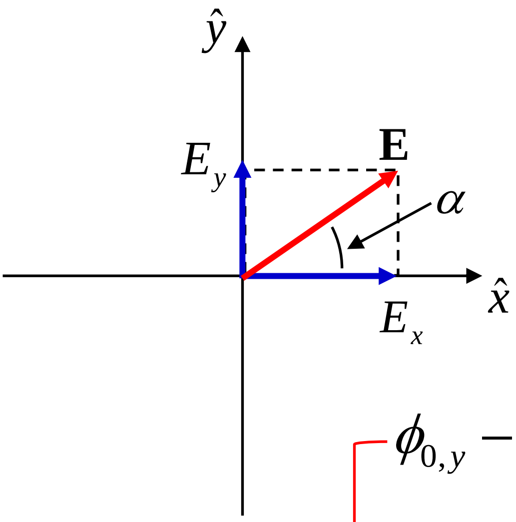
$$\rightarrow \mathbf{E} = \mathrm{Re}[|E_{0,x}|e^{j(\omega t - kz + \phi_{0,x})}\hat{x} + |E_{0,y}|e^{j(\omega t - kz + \phi_{0,y})}\hat{y}]$$



Instantaneous angle of $\mathbf{E}$ against x-axis:

$$\alpha = \tan^{-1}\frac{\mathrm{Re}[|E_{0,y}|e^{j(\omega t - kz + \phi_{0,y})}]}{\mathrm{Re}[|E_{0,x}|e^{j(\omega t - kz + \phi_{0,x})}]}$$

Polarization of Plane Waves (2)



Instantaneous angle of $\mathbf{E}$ against x-axis:

$$\alpha = \tan^{-1}\frac{\mathrm{Re}[|E_{0,y}|e^{j(\omega t - kz + \phi_{0,y})}]}{\mathrm{Re}[|E_{0,x}|e^{j(\omega t - kz + \phi_{0,x})}]}$$

$$\phi_{0,y} - \phi_{0,x} = 0, \pi : \rightarrow \alpha = \pm\tan^{-1}\left|\frac{E_{0,y}}{E_{0,x}}\right|$$ *← Linearly polarized*

$$\phi_{0,y} - \phi_{0,x} = \pm\frac{\pi}{2},\ |E_{0,x}| = |E_{0,y}| :$$

$$\rightarrow \mathbf{E} = |E_{0,x}|[\cos(\omega t - kz + \phi_{0,x})\hat{x} \mp \sin(\omega t - kz + \phi_{0,x})\hat{y}]$$

$$\rightarrow \alpha = \mp(\omega t - kz + \phi_{0,x})$$ *←Circularly polarized (Left-hand / Right-hand)*

Otherwise: *←Elliptically polarized*

Proof? *→ More suitable for homework!*