Electromagnetics:

Plane Waves in Lossy Media
Group Velocity
(8-3, 8-4)

Yoonchan Jeong

School of Electrical Engineering, Seoul National University

Tel: +82 (0)2 880 1623, Fax: +82 (0)2 873 9953
Email: yoonchan@snu.ac.kr

EM Waves in Lossless Media

Source-free, linear, isotropic, homogeneous & lossless:

$$\rho = 0,\ \mathbf{J} = 0,\ \sigma = 0,\ \mu = const.,\ \&\ \varepsilon = const.$$

Maxwell's equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{B} = \mu \mathbf{H} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M}$$

$\rightarrow$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{H} = 0$$

$$\rightarrow \nabla^2 \mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\rightarrow \nabla^2 \mathbf{H} - \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$$

Homogeneous wave eqs.

EM Waves in "Lossy" Media (1)

Source-free, linear, isotropic, homogeneous & lossy:

$$\rho = ?,\ \mathbf{J} = ?,\ \mu = const.,\ \varepsilon = const.,\ \&\ \sigma \neq 0$$

Maxwell's equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{B} = \mu \mathbf{H} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M}$$

$$\mathbf{J} = \sigma \mathbf{E}$$ *← Conduction current*

$$\rightarrow \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

For a monochromatic wave:

$$\sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \rightarrow (\sigma + j\omega\varepsilon)\mathbf{E} = j\omega(\varepsilon + \frac{\sigma}{j\omega})\mathbf{E}$$

$$= j\omega\varepsilon_c \mathbf{E}$$

Complex permittivity

If we use the complex permittivity: $\mathbf{J} \rightarrow 0$*!*

For the charge density:

$$\nabla \cdot \mathbf{E} = 0 \quad \rightarrow \nabla \cdot \mathbf{D} = 0 \quad \rightarrow \rho = 0$$

EM Waves in "Lossy" Media (2)

From Faraday's law: $\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$

$$\rightarrow \nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$

$$\rightarrow -\mu \frac{\partial}{\partial t} \nabla \times \mathbf{H} = -\mu \frac{\partial}{\partial t}(\mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t})$$

$$= -\mu \frac{\partial}{\partial t}(\sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}) = -\mu\sigma \frac{\partial \mathbf{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Homogeneous wave equation for lossy media:

$$\rightarrow \nabla^2 \mathbf{E} - \mu\sigma \frac{\partial \mathbf{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

With the complex permittivity notation:

$$e^{j\omega t} \rightarrow \nabla^2 \mathbf{E} - \mu\varepsilon_c \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$ with $\varepsilon_c = \varepsilon + \frac{\sigma}{j\omega}$

Be careful with the complex permittivity and field notation!

$$e^{-j\omega t} \rightarrow \nabla^2 \mathbf{E}^* - \mu^* \varepsilon_c^* \frac{\partial^2 \mathbf{E}^*}{\partial t^2} = 0$$

Plane Waves in Lossy Media

For a monochromatic *"plane"* wave:

$$e^{j\omega t} \rightarrow \nabla^2 \mathbf{E} - \mu\varepsilon_c \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\rightarrow \mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{-j\mathbf{k}_c \cdot \mathbf{r}} = \mathbf{E}_0 e^{-j(k_{cx}x + k_{cy}y + k_{cz}z)}$$

$$\nabla \rightarrow -j\mathbf{k}_c, \quad \frac{\partial}{\partial t} \rightarrow j\omega$$

$$(-j\mathbf{k}_c)\cdot(-j\mathbf{k}_c) - \mu\varepsilon_c (j\omega)^2 = -k_c^2 + \mu\varepsilon_c\omega^2 = 0$$

$$\rightarrow k_c = \pm\omega(\mu\varepsilon_c)^{1/2}$$

*Propagating along +**r***

$$\rightarrow k_c = \omega\sqrt{\mu}\left(\varepsilon + \frac{\sigma}{j\omega}\right)^{1/2} = \omega\sqrt{\mu\varepsilon}\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)^{1/2}$$

$$k_c = \beta - j\alpha = \omega\sqrt{\mu\varepsilon}\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)^{1/2}$$

Phase constant (β) *Attenuation constant* (α)

Low-Loss Dielectrics

Nonzero but low conductivity: $\sigma \neq 0, \ \sigma / \omega\varepsilon << 1$

Taylor series expansion:

$$f = (1+x)^{1/2} = 1 + x \frac{df}{dx}\Big|_{x=0} + \frac{x^2}{2!}\frac{d^2 f}{dx^2}\Big|_{x=0} + \cdots$$

$$k_c = \beta - j\alpha = \omega\sqrt{\mu\varepsilon}(1 - j\frac{\sigma}{\omega\varepsilon})^{1/2}$$

$$\cong \omega\sqrt{\mu\varepsilon}[1 - j\frac{1}{2}\frac{\sigma}{\omega\varepsilon} + \frac{1}{8}(\frac{\sigma}{\omega\varepsilon})^2]$$

Phase constant $\rightarrow \beta \cong \omega\sqrt{\mu\varepsilon}[1 + \frac{1}{8}(\frac{\sigma}{\omega\varepsilon})^2]$

Attenuation constant $\rightarrow \alpha \cong \omega\sqrt{\mu\varepsilon}(\frac{1}{2}\frac{\sigma}{\omega\varepsilon}) = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$

Intrinsic impedance $\rightarrow \eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon}}(1 - j\frac{\sigma}{\omega\varepsilon})^{-1/2} \cong \sqrt{\frac{\mu}{\varepsilon}}(1 + j\frac{\sigma}{2\omega\varepsilon})$

Phase velocity $\rightarrow u_p = \frac{\omega}{\beta} \cong \frac{1}{\sqrt{\mu\varepsilon}}[1 - \frac{1}{8}(\frac{\sigma}{\omega\varepsilon})^2]$

Good Conductors

Nonzero and high conductivity: $\sigma \neq 0,\ \sigma / \omega\varepsilon >> 1$

Dominant

$$k_c = \beta - j\alpha = \omega\sqrt{\mu\varepsilon}\,(1 - j\frac{\sigma}{\omega\varepsilon})^{1/2}$$

$$(-j)^{1/2} \rightarrow (e^{-j\frac{\pi}{2}})^{1/2} = e^{-j\frac{\pi}{4}} = \frac{1-j}{\sqrt{2}}$$

$$\cong \omega\sqrt{\mu\varepsilon}\,(-j\frac{\sigma}{\omega\varepsilon})^{1/2} = \sqrt{\omega\mu\sigma}\,(-j)^{1/2} = \frac{1-j}{\sqrt{2}}\sqrt{\omega\mu\sigma}$$

Phase constant
Attenuation constant

$$\rightarrow \beta = \alpha \cong \sqrt{\frac{\omega\mu\sigma}{2}}$$

Intrinsic impedance

$$\rightarrow \eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} \cong \sqrt{\frac{\mu}{\varepsilon}}(-j\frac{\sigma}{\omega\varepsilon})^{-1/2} = \frac{1+j}{\sqrt{2}}\sqrt{\frac{\omega\mu}{\sigma}} = (1+j)\frac{\alpha}{\sigma}$$

Magnetic field lags behind electric field by $\pi/4$!

Phase velocity

$$\rightarrow u_p = \frac{\omega}{\beta} \cong \sqrt{\frac{2\omega}{\mu\sigma}}$$

Skin Depth

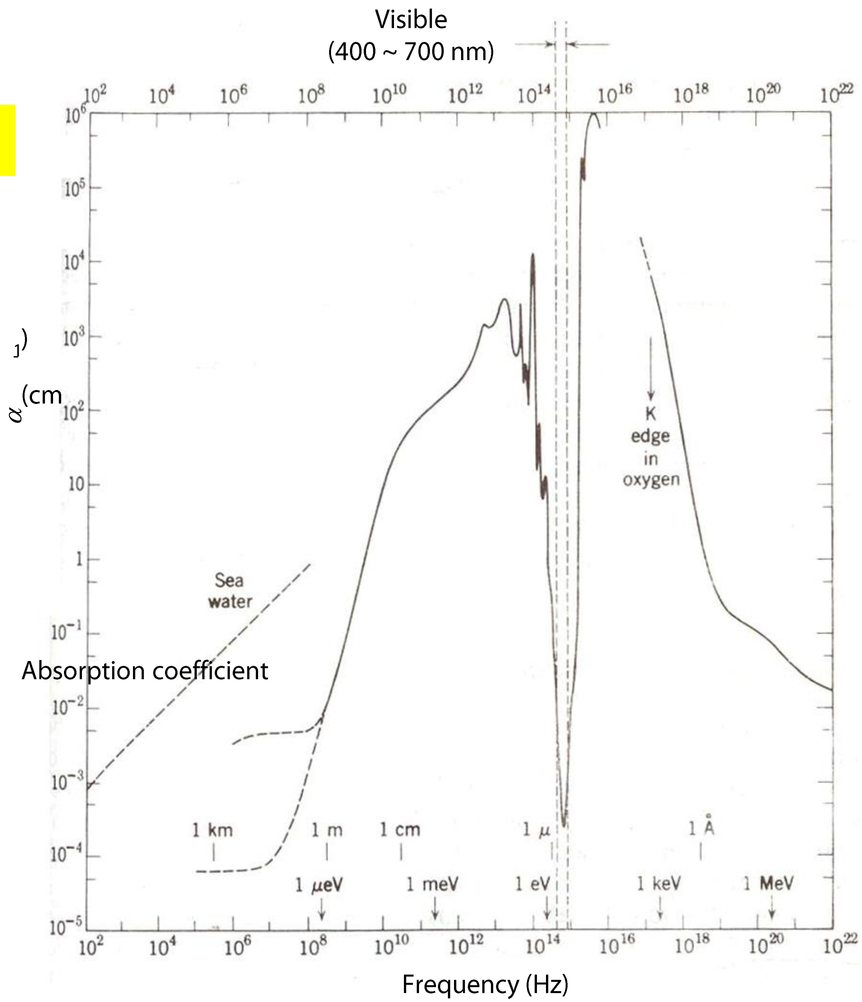
Propagation distance for *1/e* attenuation for good conductors:

$$\rightarrow \delta = \frac{1}{\alpha} \cong \sqrt{\frac{2}{\omega\mu\sigma}}$$

Strongly frequency-dependent!

However, σ is also frequency dependent!

cf. Absorption coefficient of water:



J. D. Jackson, Classical Electrodynamics, 3rd ed. Wiley, 1999.

Ionized Gases (1)

Plasmas: Ionised gases with equal electron and ion densities

Electrons much lighters than ions → as free electron gases

Recall Newton's law of motion II: $\mathbf{F} = m\mathbf{a}$

$$\rightarrow -e\mathbf{E} = m\frac{d^2\mathbf{x}}{dt^2} = -m\omega^2\mathbf{x}$$ *for a monochromatic wave at* ω

Displacement: $$\rightarrow \mathbf{x} = \frac{e}{m\omega^2}\mathbf{E}$$

Electric dipole moment: $$\rightarrow \mathbf{p} = -e\mathbf{x}$$ *N: Density of number of electrons*

Polarization vector: $$\rightarrow \mathbf{P} = N\mathbf{p} = -\frac{Ne^2}{m\omega^2}\mathbf{E}$$

Ionized Gases (2)

Polarization vector: $$\mathbf{P} = N\mathbf{p} = -\frac{Ne^2}{m\omega^2}\mathbf{E}$$

Electric displacement vector:

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$$ ← *Constitutive relation*

$$= \varepsilon_0(1 - \frac{Ne^2}{m\omega^2\varepsilon_0})\mathbf{E} = \varepsilon_0(1 - \frac{\omega_p^2}{\omega^2})\mathbf{E}$$ *with* $$\omega_p = \sqrt{\frac{Ne^2}{m\varepsilon_0}}$$

Equivalent permittivity:

$$\varepsilon_p = \varepsilon_0(1 - \frac{\omega_p^2}{\omega^2}) = \varepsilon_0(1 - \frac{f_p^2}{f^2})$$ $$\leftarrow f_p \cong 9\sqrt{N}$$

Plasma frequency or cutoff frequency

$$k_p = \beta - j\alpha = \omega\sqrt{\mu\varepsilon_p}$$

Freq. response: $$\begin{cases} f < f_p \\ f > f_p \end{cases}$$

- $f < f_p$ → *No transmission* → *Back reflection by ionospheric layers*
- $f > f_p$ → *Un-attenuated transmission*

Group velocity

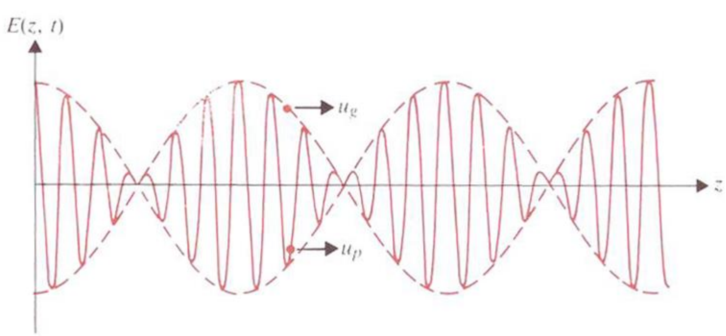
Phase velocity: $u_p = \dfrac{\omega}{\beta}$ Material dispersion: $\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu(\omega)\varepsilon(\omega)}$

Consider a dual-tone wave packet: $\omega_0 + \Delta\omega$ & $\omega_0 - \Delta\omega$ $(\Delta\omega << \omega_0)$

$$E(z,t) = E_0\cos[(\omega_0 + \Delta\omega)t - (\beta_0 + \Delta\beta)z] + E_0\cos[(\omega_0 - \Delta\omega)t - (\beta_0 - \Delta\beta)z]$$
$$= 2E_0 \underbrace{\cos(t\Delta\omega - z\Delta\beta)}_{\text{Slowly oscillating} \to \text{Envelope}} \underbrace{\cos(\omega_0 t - \beta_0 z)}_{\text{Rapidly oscillating} \to \text{Carrier}}$$

Slowly oscillating
→ Envelope

Rapidly oscillating
→ Carrier



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Phase: $\omega_0 t - \beta_0 z = const.$

Phase velocity: $\rightarrow u_p = \dfrac{dz}{dt} = \dfrac{\omega_0}{\beta_0}$

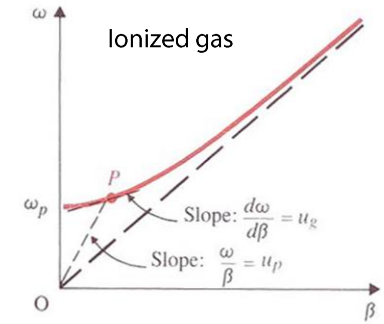
Envelope: $\Delta\omega t - \Delta\beta z = const.$

Group velocity: $\rightarrow u_g = \dfrac{dz}{dt} = \dfrac{\Delta\omega}{\Delta\beta} \rightarrow \dfrac{d\omega}{d\beta}$

Dispersion

Group velocity: $u_g = \frac{d\omega}{d\beta}$

$$u_p = \frac{\omega}{\beta}$$



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$$\frac{d\beta}{d\omega} = \frac{d}{d\omega}\left(\frac{\omega}{u_p}\right) = \frac{1}{u_p} - \frac{\omega}{u_p^2}\frac{du_p}{d\omega}$$

$$\rightarrow u_g = \frac{1}{\frac{1}{u_p} - \frac{\omega}{u_p^2}\frac{du_p}{d\omega}} = \frac{u_p}{1 - \frac{\omega}{u_p}\frac{du_p}{d\omega}}$$

Dispersion:

$$\begin{cases} \frac{du_p}{d\omega} = 0 \rightarrow u_g = u_p & \rightarrow \textit{No dispersion} \\ \frac{du_p}{d\omega} < 0 \rightarrow u_g < u_p & \rightarrow \textit{Normal dispersion} \\ \frac{du_p}{d\omega} > 0 \rightarrow u_g > u_p & \rightarrow \textit{Abnormal dispersion} \end{cases}$$