Electromagnetics:
Flow of EM Power and the Poynting Vector
Normal Incidence at a Plane Conducting Boundary
(8-5, 8-6)

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Energy Flow of EM Waves (1)

Maxwell's equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{B} = \mu \mathbf{H} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M}$$

For static constitutive parameters: $\varepsilon, \mu \,\&\, \sigma$

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H})$$

$$= \mathbf{H} \cdot (-\frac{\partial \mathbf{B}}{\partial t}) - \mathbf{E} \cdot (\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t})$$

$$\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = \mathbf{H} \cdot \frac{\partial (\mu \mathbf{H})}{\partial t} = \frac{\mu}{2} \frac{\partial (\mathbf{H} \cdot \mathbf{H})}{\partial t} = \frac{\partial}{\partial t} (\frac{1}{2} \mu H^2)$$

$$\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = \mathbf{E} \cdot \frac{\partial (\varepsilon \mathbf{E})}{\partial t} = \frac{\varepsilon}{2} \frac{\partial (\mathbf{E} \cdot \mathbf{E})}{\partial t} = \frac{\partial}{\partial t} (\frac{1}{2} \varepsilon E^2)$$

$$\mathbf{E} \cdot \mathbf{J} = \mathbf{E} \cdot (\sigma \mathbf{E}) = \sigma E^2$$

$$\rightarrow \nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\frac{\partial}{\partial t} (\frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2) - \sigma E^2$$

$$\rightarrow \oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s} = -\frac{\partial}{\partial t} \int_V (\frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2) dv - \int_V \sigma E^2 dv$$

Energy Flow of EM Waves (2)

Poynting theorem & Poynting vector:

Ohmic power dissipation

$$\rightarrow \oint_S (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{s} = -\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right)dv - \int_V \sigma E^2 dv$$

Negative rate of the temporal change

Stored energy in E-field

Stored energy in H-field

Poynting vector: Power flow per unit area

$$\mathbf{P} = \mathbf{E}\times\mathbf{H}$$

$$\oint_S \mathbf{P}\cdot d\mathbf{s} = -\frac{\partial}{\partial t}\int_V (w_e + w_m)dv - \int_V p_\sigma dv$$

Poynting's theorem

w_e : *Electric energy density*

w_m : *Magnetic energy density*

p_σ : *Ohmic power density*

Instantaneous and Average Power Densities

Complex notation for time-harmonic **E** & **H**:

$$\mathbf{E}(\mathbf{r},t)=\frac{1}{2}[\mathbf{E}(\mathbf{r})e^{j\omega t}+c.c.]=\mathrm{Re}[\mathbf{E}(\mathbf{r})e^{j\omega t}] \quad \rightarrow \mathrm{Re}(\mathbf{A})=\frac{1}{2}(\mathbf{A}+\mathbf{A}^{*})$$

$$\mathbf{H}(\mathbf{r},t)=\frac{1}{2}[\mathbf{H}(\mathbf{r})e^{j\omega t}+c.c.]=\mathrm{Re}[\mathbf{H}(\mathbf{r})e^{j\omega t}] \quad \rightarrow \mathrm{Re}(\mathbf{B})=\frac{1}{2}(\mathbf{B}+\mathbf{B}^{*})$$

For product operations:

$$\mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t),\ \mathbf{E}(\mathbf{r},t)\cdot\mathbf{D}(\mathbf{r},t),\ \cdots \rightarrow \mathrm{Re}(\mathbf{A})\times\mathrm{Re}(\mathbf{B}),\ \mathrm{Re}(\mathbf{A})\cdot\mathrm{Re}(\mathbf{D}),\ \cdots$$

$$\begin{aligned}\mathrm{Re}(\mathbf{A})\times\mathrm{Re}(\mathbf{B}) &= \frac{1}{2}(\mathbf{A}+\mathbf{A}^{*})\times\frac{1}{2}(\mathbf{B}+\mathbf{B}^{*}) = \frac{1}{4}(\mathbf{A}\times\mathbf{B}^{*}+\mathbf{A}^{*}\times\mathbf{B}+\mathbf{A}\times\mathbf{B}+\mathbf{A}^{*}\times\mathbf{B}^{*}) \\ &= \frac{1}{2}\mathrm{Re}(\mathbf{A}\times\mathbf{B}^{*}+\mathbf{A}\times\mathbf{B})\end{aligned}$$

For a power density (Poynting vector):

Instantaneous: $$\rightarrow \mathbf{P}=\mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t)=\frac{1}{2}\mathrm{Re}[\mathbf{E}(\mathbf{r})\times\mathbf{H}^{*}(\mathbf{r})+\underline{\mathbf{E}(\mathbf{r})\times\mathbf{H}(\mathbf{r})e^{j2\omega t}}]$$

→ Zero if time-averaged!

Time-average: $$\rightarrow \mathbf{P}_{av}=\frac{1}{2}\mathrm{Re}[\mathbf{E}(\mathbf{r})\times\mathbf{H}^{*}(\mathbf{r})]$$

Electromagnetic Boundary Conditions

Continuity conditions:

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \rightarrow \oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} \rightarrow E_{1t} = E_{2t}$$

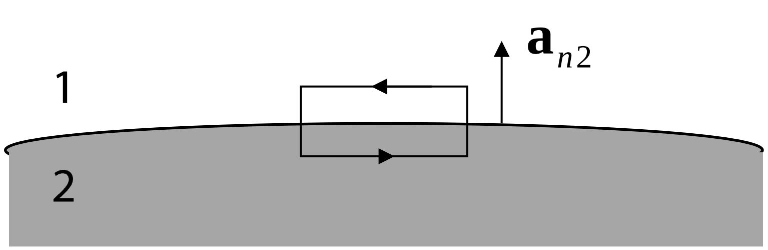
$$\rightarrow \boxed{\mathbf{a}_{n2} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0}$$

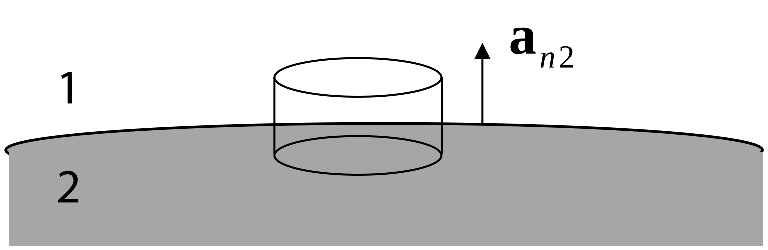
$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \rightarrow \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{s}$$

$$\rightarrow \boxed{\mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s}$$

$$\nabla \cdot \mathbf{D} = \rho \rightarrow \boxed{\mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s}$$

$$\nabla \cdot \mathbf{B} = 0 \rightarrow B_{1n} = B_{2n} \rightarrow \boxed{\mathbf{a}_{n2} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0}$$





Interface between a Dielectric and a Perfect Conductor

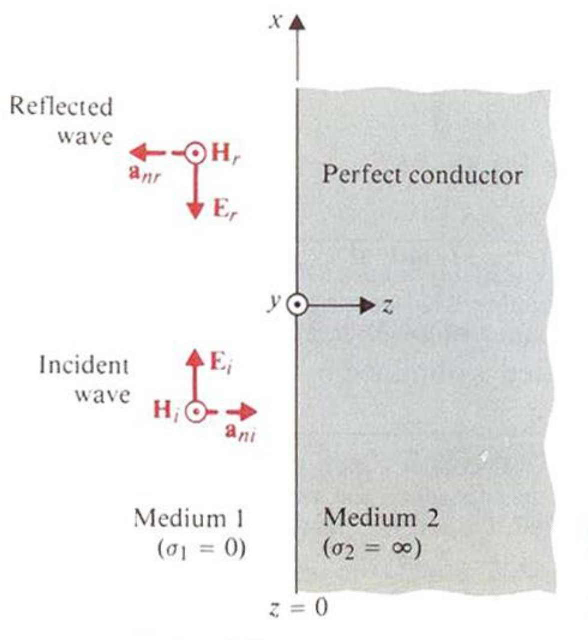
A perfect conductor: $\sigma = \infty$

→ *No fields exist in the interior, i.e.* $E = D = B = H = 0$*,*
for time-varying fields!

$\nabla \times \mathbf{E} + \dfrac{\partial \mathbf{B}}{\partial t} = 0 \quad \rightarrow \quad \mathbf{a}_{n2} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$		$E_{1t} = 0, \quad E_{2t} = 0$
$\nabla \times \mathbf{H} - \dfrac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \quad \rightarrow \quad \mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$	→	$\mathbf{a}_{n2} \times \mathbf{H}_1 = \mathbf{J}_s, \quad H_{2t} = 0$
$\nabla \cdot \mathbf{D} = \rho \quad \rightarrow \quad \mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$		$\mathbf{a}_{2n} \cdot \mathbf{D}_1 = \rho_s, \quad D_{2n} = 0$
$\nabla \cdot \mathbf{B} = 0 \quad \rightarrow \quad \mathbf{a}_{n2} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$		$B_{1n} = 0, \quad B_{2n} = 0$

→ *Any charges or currents the perfect conductor may have*
exist on the "surface" only!

Normal Incidence at a Plane Conducting Boundary (1)



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

In medium 1 ($z < 0$):

$$\mathbf{E}_i(z) = \mathbf{a}_x E_{i0} e^{-j\beta_1 z}$$

$$\mathbf{H}_i(z) = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z}$$

← Intrinsic impedance

$$\mathbf{E}_r(z) = \mathbf{a}_x E_{r0} e^{+j\beta_1 z}$$

$$\mathbf{H}_r(z) = \mathbf{a}_y H_{r0} e^{+j\beta_1 z}$$

$$\rightarrow \mathbf{E}_1(z) = \mathbf{E}_i(z) + \mathbf{E}_r(z)$$

$$\rightarrow \mathbf{H}_1(z) = \mathbf{H}_i(z) + \mathbf{H}_r(z)$$

In medium 2 ($z > 0$):

$$\rightarrow \mathbf{E}_2(z) = 0$$

$$\rightarrow \mathbf{H}_2(z) = 0$$

for a perfect conductor ($\sigma = \infty$)

Normal Incidence at a Plane Conducting Boundary (2)

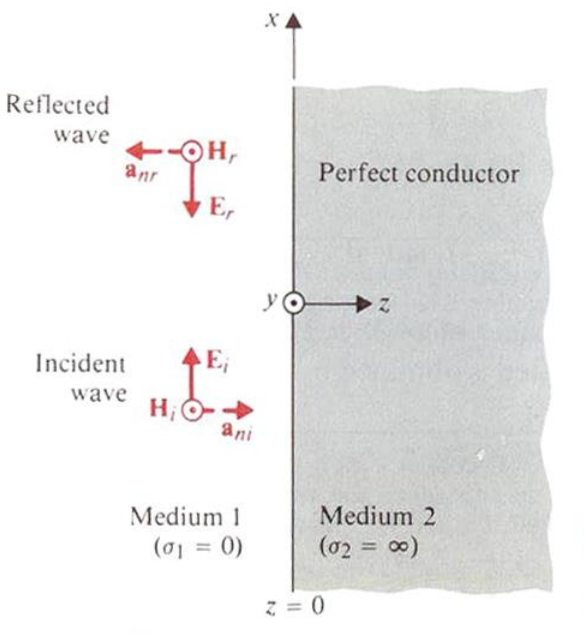
For tangential components of E-fields:

$$\mathbf{a}_{n2} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0 \qquad \rightarrow \mathbf{E}_1(z=0) = \mathbf{a}_x (E_{i0} + E_{r0}) = \mathbf{E}_2(z=0) = 0$$

$$\rightarrow E_{r0} = -E_{i0}$$

$$\rightarrow \mathbf{E}_1(z) = \mathbf{a}_x E_{i0}(e^{-j\beta_1 z} - e^{+j\beta_1 z})$$

$$= -\mathbf{a}_x j2E_{i0} \sin \beta_1 z$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Recall: $$\mathbf{H} = -\frac{1}{j\omega\mu}(\nabla \times \mathbf{E})$$

$$\rightarrow \mathbf{H}_r(z) = \frac{1}{\eta_1}\mathbf{a}_{nr} \times \mathbf{E}_r = \frac{1}{\eta_1}(-\mathbf{a}_z) \times \mathbf{E}_r$$

$$= -\mathbf{a}_y \frac{1}{\eta_1} E_{r0} e^{+\beta_1 z} = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{+\beta_1 z}$$

$$\rightarrow \mathbf{H}_1(z) = \mathbf{H}_i(z) + \mathbf{H}_r(z) = \mathbf{a}_y \frac{E_{i0}}{\eta_1}(e^{-i\beta_1 z} + e^{+i\beta_1 z})$$

$$= \mathbf{a}_y 2\frac{E_{i0}}{\eta_1} \cos \beta_1 z$$

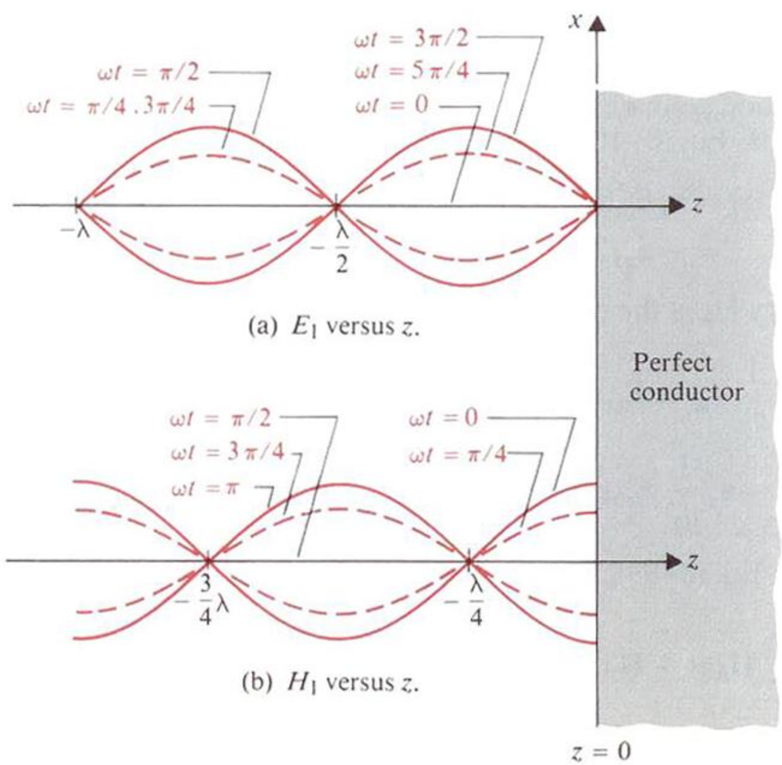
Normal Incidence at a Plane Conducting Boundary (3)

$$\rightarrow \mathbf{E}_1(z) = -\mathbf{a}_x j2E_{i0} \sin \beta_1 z$$

$$\rightarrow \mathbf{H}_1(z) = \mathbf{a}_y 2\frac{E_{i0}}{\eta_1} \cos \beta_1 z$$

$$\rightarrow \mathbf{E}_1(z,t) = \mathrm{Re}[\mathbf{E}_1(z)e^{j\omega t}] = \mathbf{a}_x 2E_{i0} \sin \beta_1 z \sin \omega t$$

$$\rightarrow \mathbf{H}_1(z,t) = \mathrm{Re}[\mathbf{H}_1(z)e^{j\omega t}] = \mathbf{a}_y 2\frac{E_{i0}}{\eta_1} \cos \beta_1 z \cos \omega t$$



→ Standing wave

Poynting vector: $\mathbf{E}_1 \times \mathbf{H}_1 = ?$

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