# Electromagnetics:
## Oblique Incidence at a Plane Dielectric Boundary
## (8-10)

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# Boundary Conditions for a Plane Dielectric Interface

Lossless dielectric media: $\sigma_1 = \sigma_2 = 0$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \rightarrow \quad \mathbf{a}_{n2} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \quad \rightarrow \quad \mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$$

$$\nabla \cdot \mathbf{D} = \rho \quad \rightarrow \quad \mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$$

$$\nabla \cdot \mathbf{B} = 0 \quad \rightarrow \quad \mathbf{a}_{n2} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$$

$\rightarrow$

$$E_{1t} = E_{2t}$$

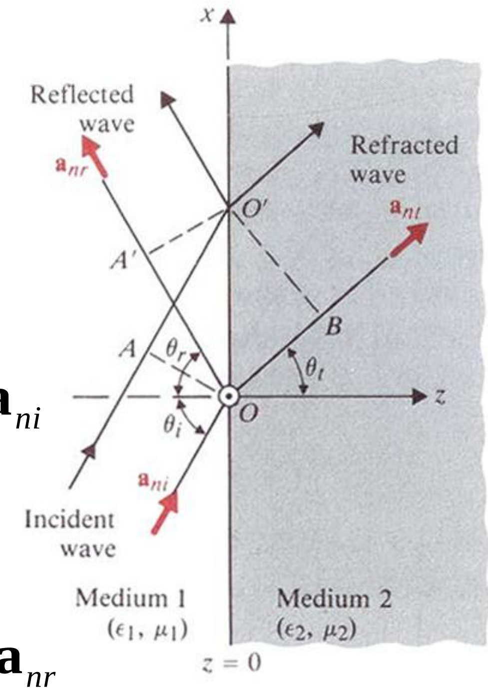
$$H_{1t} = H_{2t}$$

$$D_{1n} = D_{2n}$$

$$B_{1n} = B_{2n}$$

# Incidence at a Plane Dielectric Boundary

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.




For plane waves: $E, H \propto e^{-j\boldsymbol{\beta}\cdot\mathbf{r}}$

*(Isotropic media)*

In medium 1 ($z < 0$):

*Incident wave:*

$$\mathbf{E}_i = \mathbf{E}_{i0} e^{-j\boldsymbol{\beta}_i\cdot\mathbf{r}} \rightarrow \boldsymbol{\beta}_i = \beta_1(\mathbf{a}_x \sin\theta_i + \mathbf{a}_z \cos\theta_i) = \beta_1 \mathbf{a}_{ni}$$

$$\mathbf{H}_i = \mathbf{H}_{i0} e^{-j\boldsymbol{\beta}_i\cdot\mathbf{r}} = \frac{1}{\eta_1}\mathbf{a}_{ni} \times \mathbf{E}_{i0} e^{-j\boldsymbol{\beta}_i\cdot\mathbf{r}}$$

*Reflected wave:*

$$\mathbf{E}_r = \mathbf{E}_{r0} e^{-j\boldsymbol{\beta}_r\cdot\mathbf{r}} \rightarrow \boldsymbol{\beta}_r = \beta_1(\mathbf{a}_x \sin\theta_r - \mathbf{a}_z \cos\theta_r) = \beta_1 \mathbf{a}_{nr}$$

$$\mathbf{H}_r = \mathbf{H}_{r0} e^{-j\boldsymbol{\beta}_r\cdot\mathbf{r}} = \frac{1}{\eta_1}\mathbf{a}_{nr} \times \mathbf{E}_{r0} e^{-j\boldsymbol{\beta}_r\cdot\mathbf{r}}$$

In medium 2 ($z > 0$):

*Transmitted wave:*

$$\mathbf{E}_t = \mathbf{E}_{t0} e^{-j\boldsymbol{\beta}_t\cdot\mathbf{r}} \rightarrow \boldsymbol{\beta}_t = \beta_2(\mathbf{a}_x \sin\theta_t + \mathbf{a}_z \cos\theta_t) = \beta_2 \mathbf{a}_{nt}$$

$$\mathbf{H}_t = \mathbf{H}_{t0} e^{-j\boldsymbol{\beta}_t\cdot\mathbf{r}} = \frac{1}{\eta_2}\mathbf{a}_{nt} \times \mathbf{E}_{t0} e^{-j\boldsymbol{\beta}_t\cdot\mathbf{r}}$$

# Snell's Law & Total Reflection

Boundary conditions at $z = 0$: $\mathbf{E}_{1t} = \mathbf{E}_{2t}$, $\mathbf{H}_{1t} = \mathbf{H}_{2t}$

$$\mathbf{E}_{i0t} e^{-j\beta_1 \sin\theta_i x} + \mathbf{E}_{r0t} e^{-j\beta_1 \sin\theta_r x} = \mathbf{E}_{t0t} e^{-j\beta_2 \sin\theta_t x}$$

$$\mathbf{H}_{i0t} e^{-j\beta_1 \sin\theta_i x} + \mathbf{H}_{r0t} e^{-j\beta_1 \sin\theta_r x} = \mathbf{H}_{t0t} e^{-j\beta_2 \sin\theta_t x}$$

*To hold for any x values:*

$$\rightarrow e^{-j\beta_1 \sin\theta_i x} = e^{-j\beta_1 \sin\theta_r x} = e^{-j\beta_2 \sin\theta_t x}$$

$$\rightarrow \beta_1 \sin\theta_i = \beta_1 \sin\theta_r = \beta_2 \sin\theta_t$$

$\rightarrow \theta_i = \theta_r$ *→ Snell's law of reflection*

$\rightarrow n_1 \sin\theta_i = n_2 \sin\theta_t$ *→ Snell's law of refraction*

For $n_1 > n_2$:

$\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i \rightarrow \sin\theta_t > 1 \; if \;\; \theta_i > \theta_c \leftarrow \theta_c = \sin^{-1}(\frac{n_2}{n_1})$ *Critical angle*

*→ Total internal reflection*

*Recall:* $\boldsymbol{\beta}_t = \beta_2(\mathbf{a}_x \sin\theta_t + \mathbf{a}_z \cos\theta_t)$

$\rightarrow \cos\theta_t = \sqrt{1-\sin^2\theta_t} \rightarrow -j\sqrt{\sin^2\theta_t - 1}$ *→ Evanescent wave in z-axis*

# Continuity of E- and H-fields

*Recall:*

Boundary conditions at $z = 0$: $\mathbf{E}_{1t} = \mathbf{E}_{2t}, \quad \mathbf{H}_{1t} = \mathbf{H}_{2t}$

$$\mathbf{E}_{i0t} e^{-j\beta_1 \sin\theta_i x} + \mathbf{E}_{r0t} e^{-j\beta_1 \sin\theta_r x} = \mathbf{E}_{t0t} e^{-j\beta_2 \sin\theta_t x} \rightarrow \mathbf{E}_{i0t} + \mathbf{E}_{r0t} = \mathbf{E}_{t0t}$$

$$\mathbf{H}_{i0t} e^{-j\beta_1 \sin\theta_i x} + \mathbf{H}_{r0t} e^{-j\beta_1 \sin\theta_r x} = \mathbf{H}_{t0t} e^{-j\beta_2 \sin\theta_t x} \rightarrow \mathbf{H}_{i0t} + \mathbf{H}_{r0t} = \mathbf{H}_{t0t}$$

Consider two orthogonal polarizations: $\perp$ & $//$

$$\mathbf{E}_{i0t} + \mathbf{E}_{r0t} = \mathbf{E}_{t0t} \rightarrow \mathbf{E}_{i0t}^{\perp} + \mathbf{E}_{i0t}^{//} + \mathbf{E}_{r0t}^{\perp} + \mathbf{E}_{r0t}^{//} = \mathbf{E}_{t0t}^{\perp} + \mathbf{E}_{t0t}^{//}$$

$$\mathbf{H}_{i0t} + \mathbf{H}_{r0t} = \mathbf{H}_{t0t} \rightarrow \mathbf{H}_{i0t}^{\perp} + \mathbf{H}_{i0t}^{//} + \mathbf{H}_{r0t}^{\perp} + \mathbf{H}_{r0t}^{//} = \mathbf{H}_{t0t}^{\perp} + \mathbf{H}_{t0t}^{//}$$

*Perpendicular polarization (TE)*

$$E_{i0t}^{\perp} + E_{r0t}^{\perp} = E_{t0t}^{\perp}$$

$$H_{i0t}^{\perp} + H_{r0t}^{\perp} = H_{t0t}^{\perp}$$

*Parallel polarization (TM)*

$$E_{i0t}^{//} + E_{r0t}^{//} = E_{t0t}^{//}$$

$$H_{i0t}^{//} + H_{r0t}^{//} = H_{t0t}^{//}$$

# Perpendicular Polarization (TE) (1)

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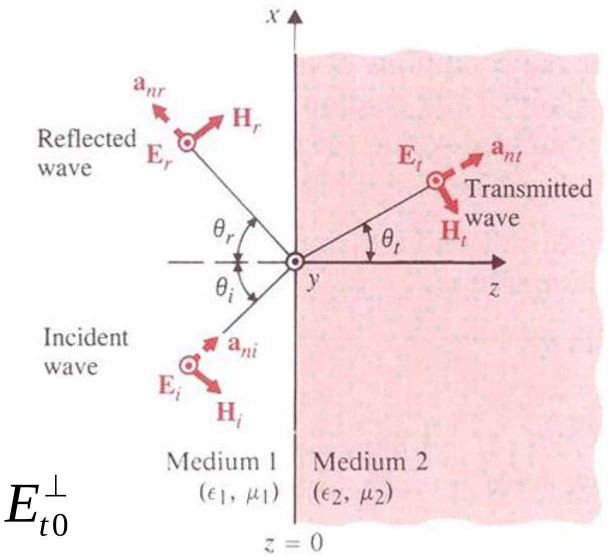


*Perpendicular polarization (TE)*

$$E_{i0t}^{\perp} + E_{r0t}^{\perp} = E_{t0t}^{\perp}$$

$$H_{i0t}^{\perp} + H_{r0t}^{\perp} = H_{t0t}^{\perp}$$

$$\mathbf{E}_{i0}^{\perp} = \mathbf{a}_y E_{i0}^{\perp}$$

$$\mathbf{H}_{i0}^{\perp} = \frac{1}{\eta_1}\mathbf{a}_{ni} \times \mathbf{E}_{i0}^{\perp} = \frac{E_{i0}^{\perp}}{\eta_1}(-\mathbf{a}_x \cos\theta_i + \mathbf{a}_z \sin\theta_i)$$

$$\mathbf{E}_{r0}^{\perp} = \mathbf{a}_y E_{r0}^{\perp}$$

$$\mathbf{H}_{r0}^{\perp} = \frac{1}{\eta_1}\mathbf{a}_{nr} \times \mathbf{E}_{r0}^{\perp} = \frac{E_{r0}^{\perp}}{\eta_1}(\mathbf{a}_x \cos\theta_r + \mathbf{a}_z \sin\theta_r)$$

$$\mathbf{E}_{t0}^{\perp} = \mathbf{a}_y E_{t0}^{\perp}$$

$$\mathbf{H}_{t0}^{\perp} = \frac{1}{\eta_2}\mathbf{a}_{nt} \times \mathbf{E}_{t0}^{\perp} = \frac{E_{t0}^{\perp}}{\eta_2}(-\mathbf{a}_x \cos\theta_t + \mathbf{a}_z \sin\theta_t)$$

$$E_{i0}^{\perp} + E_{r0}^{\perp} = E_{t0}^{\perp}$$

$$\frac{1}{\eta_1}(E_{i0}^{\perp} - E_{r0}^{\perp})\cos\theta_i = \frac{1}{\eta_2}E_{t0}^{\perp}\cos\theta_t$$

# Perpendicular Polarization (TE) (2)

$$E_{i0}^{\perp} + E_{r0}^{\perp} = E_{t0}^{\perp}$$

$$\frac{1}{\eta_1}(E_{i0}^{\perp} - E_{r0}^{\perp})\cos\theta_i = \frac{1}{\eta_2} E_{t0}^{\perp} \cos\theta_t$$

$$\Gamma_{\perp} = \frac{E_{r0}^{\perp}}{E_{i0}^{\perp}} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

*→ Fresnel's equations for S-pol.*

$$\tau_{\perp} = \frac{E_{t0}^{\perp}}{E_{i0}^{\perp}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

$$1 + \Gamma_{\perp} = \tau_{\perp}$$

*Note:* $\eta_2 \cos\theta_i - \eta_1 \cos\theta_t = 0 \quad \rightarrow \Gamma_{\perp} = 0$

*→ Brewster angle*

# Brewster Angle for S-Pol. (TE) Waves

In case: $\Gamma_\perp = 0 \quad \rightarrow \boxed{\eta_2 \cos\theta_{B\perp} = \eta_1 \cos\theta_t}$

$$\rightarrow \cos^2\theta_{B\perp} = \frac{\eta_1^2}{\eta_2^2}\cos^2\theta_t \rightarrow 1-\sin^2\theta_{B\perp} = \frac{\eta_1^2}{\eta_2^2}\left(1-\frac{n_1^2}{n_2^2}\sin^2\theta_{B\perp}\right)$$

$$\rightarrow 1-\frac{\eta_1^2}{\eta_2^2} = \left(1-\frac{\eta_1^2}{\eta_2^2}\frac{n_1^2}{n_2^2}\right)\sin^2\theta_{B\perp}$$

$$\rightarrow \sin^2\theta_{B\perp} = \frac{1-\eta_1^2/\eta_2^2}{1-\eta_1^2 n_1^2/\eta_2^2 n_2^2} = \frac{1-\dfrac{\mu_1}{\varepsilon_1}\dfrac{\varepsilon_2}{\mu_2}}{1-\dfrac{\mu_1}{\varepsilon_1}\dfrac{\varepsilon_2}{\mu_2}\dfrac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}} = \frac{1-\mu_1\varepsilon_2/\mu_2\varepsilon_1}{1-(\mu_1/\mu_2)^2}$$

*No Brewster angle for* $\mu_1 = \mu_2$

In case: $\varepsilon_1 = \varepsilon_2, \mu_1 \neq \mu_2$

$$\rightarrow \sin\theta_{B\perp} = \frac{1}{\sqrt{1+\mu_1/\mu_2}}$$

$\rightarrow$ *For non-magnetic media, no Brewster angle exists!*

# Parallel Polarization (TM) (1)

*Parallel polarization (TM)*

$$E^{//}_{i0t} + E^{//}_{r0t} = E^{//}_{t0t}$$

$$H^{//}_{i0t} + H^{//}_{r0t} = H^{//}_{t0t}$$

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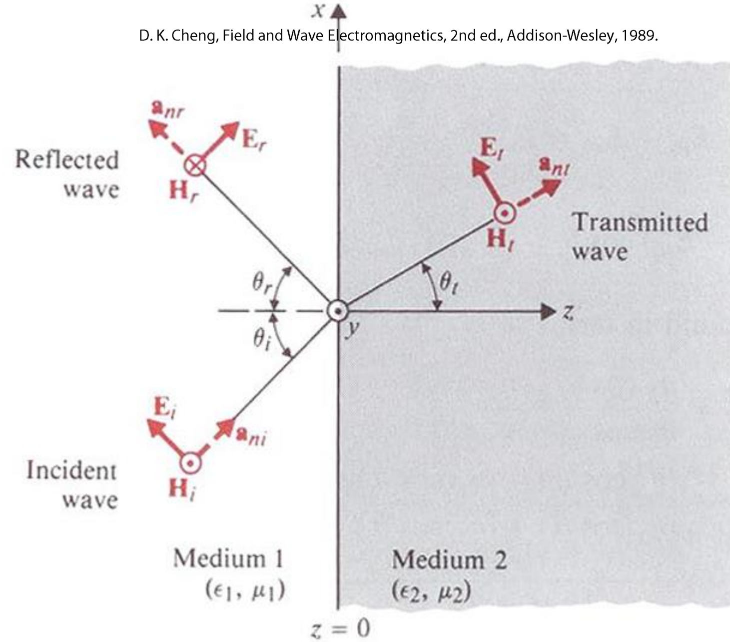



$$\mathbf{E}^{//}_{i0} = E^{//}_{i0}(\mathbf{a}_x \cos\theta_i - \mathbf{a}_z \sin\theta_i)$$

$$\mathbf{H}^{//}_{i0} = \mathbf{a}_y \frac{1}{\eta_1} E^{//}_{i0}$$

$$\mathbf{E}^{//}_{t0} = E^{//}_{t0}(\mathbf{a}_x \cos\theta_t - \mathbf{a}_z \sin\theta_t)$$

$$\mathbf{H}^{//}_{t0} = \mathbf{a}_y \frac{1}{\eta_2} E^{//}_{t0}$$

$$\mathbf{E}^{//}_{r0} = E^{//}_{r0}(\mathbf{a}_x \cos\theta_r + \mathbf{a}_z \sin\theta_r)$$

$$\mathbf{H}^{//}_{r0} = -\mathbf{a}_y \frac{1}{\eta_1} E^{//}_{r0}$$

$$(E^{//}_{i0} + E^{//}_{r0})\cos\theta_i = E^{//}_{t0}\cos\theta_t$$

$$\frac{1}{\eta_1}(E^{//}_{i0} - E^{//}_{r0}) = \frac{1}{\eta_2} E^{//}_{t0}$$

# Parallel Polarization (TM) (2)

$$(E_{i0}^{//} + E_{r0}^{//})\cos\theta_i = E_{t0}^{//}\cos\theta_t$$

$$\frac{1}{\eta_1}(E_{i0}^{//} - E_{r0}^{//}) = \frac{1}{\eta_2}E_{t0}^{//}$$

$$\Gamma_{//} = \frac{E_{r0}^{//}}{E_{i0}^{//}} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

*→ Fresnel's equations for P-pol.*

$$\tau_{//} = \frac{E_{t0}^{//}}{E_{i0}^{//}} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

$$(1+\Gamma_{//})\cos\theta_i = \tau_{//}\cos\theta_t$$

*Note:* $\eta_2\cos\theta_t - \eta_1\cos\theta_i = 0 \quad \rightarrow \Gamma_{//} = 0$

*→ Brewster angle*

# Brewster Angle for P-Pol. (TM) Waves

In case: $\Gamma_{//} = 0 \quad \rightarrow \boxed{\eta_2 \cos\theta_t = \eta_1 \cos\theta_{B//}}$

$$\rightarrow \cos^2\theta_{B//} = \frac{\eta_2^2}{\eta_1^2}\cos^2\theta_t \quad \rightarrow 1-\sin^2\theta_{B//} = \frac{\eta_2^2}{\eta_1^2}(1-\frac{n_1^2}{n_2^2}\sin^2\theta_{B//})$$

$$\rightarrow 1-\frac{\eta_2^2}{\eta_1^2} = (1-\frac{\eta_2^2}{\eta_1^2}\frac{n_1^2}{n_2^2})\sin^2\theta_{B//}$$

$$\rightarrow \sin^2\theta_{B//} = \frac{1-\eta_2^2/\eta_1^2}{1-\eta_2^2 n_1^2/\eta_1^2 n_2^2} = \frac{1-\frac{\mu_2}{\varepsilon_2}\frac{\varepsilon_1}{\mu_1}}{1-\frac{\mu_2}{\varepsilon_2}\frac{\varepsilon_1}{\mu_1}\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}} = \frac{1-\mu_2\varepsilon_1/\mu_1\varepsilon_2}{1-(\varepsilon_1/\varepsilon_2)^2}$$

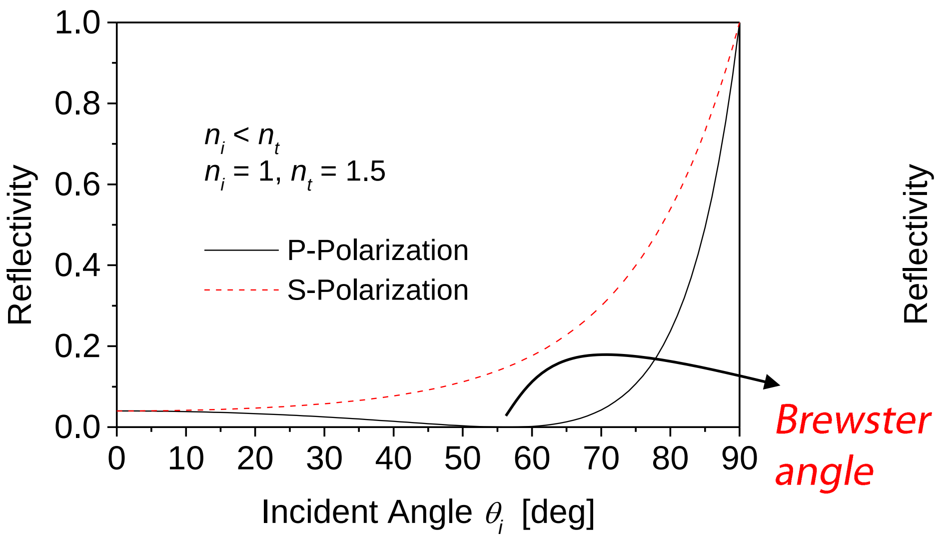
*No Brewster angle for* $\varepsilon_1 = \varepsilon_2$

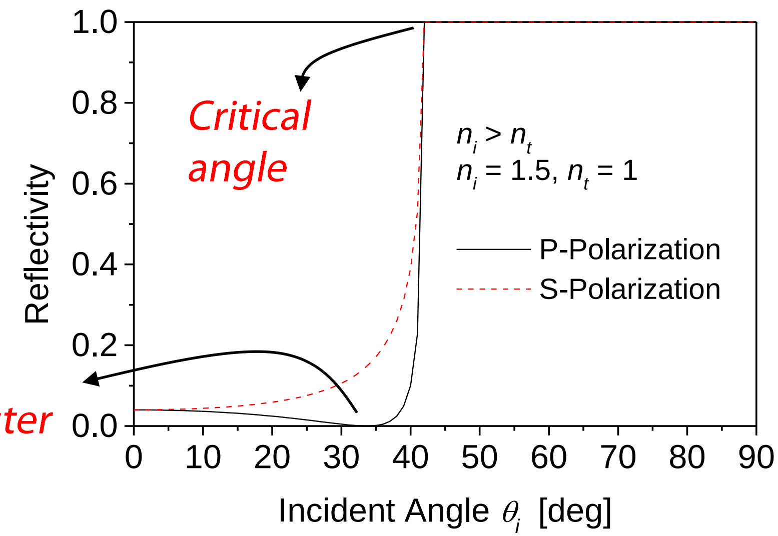
In case: $\varepsilon_1 \neq \varepsilon_2,\ \mu_1 = \mu_2 = \mu_0$

$$\rightarrow \sin\theta_{B//} = \frac{1}{\sqrt{1+\varepsilon_1/\varepsilon_2}} \quad \rightarrow \tan\theta_{B//} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{n_2}{n_1} \quad \rightarrow \theta_i + \theta_t = \frac{\pi}{2}$$

# Brewster Angle & Critical Angle

## Numerical examples:




*Brewster angle (Non-magnetic):*

*For P-polarization*

$$\theta_B = \tan^{-1}\left(\frac{n_t}{n_i}\right)$$

$$\rightarrow \theta_i + \theta_t = \frac{\pi}{2}$$

*Critical angle:* *Total internal reflection*

$$\theta_c = \sin^{-1}\left(\frac{n_t}{n_i}\right)$$

$$\rightarrow n_i > n_t, \ \theta_t = \frac{\pi}{2}$$