

Electromagnetics:

Parallel-Plate Waveguides

(10-3)

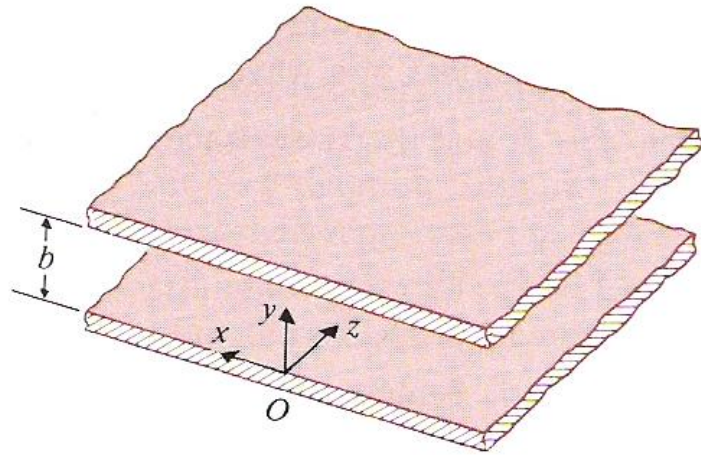
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Parallel-Plate Waveguides



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

An infinite parallel-plate waveguide

→ TEM, TM, & TE modes possible

Conditions given:

- Two perfectly conducting plates*
- Dielectric medium with ϵ, μ*
- Waves propagating in the z -direction*
- Infinitely wide in the x -direction: No variation in the x -direction*

TM Waves Between Parallel Plates (1)

Recall:

$$\gamma = \alpha + j\beta \quad \rightarrow \text{Propagation constant}$$

$$\mathbf{E}(x, y, z; t) = \text{Re}[\mathbf{E}^0(x, y)e^{(j\omega t - \gamma z)}]$$

$$H_z = 0 \quad \rightarrow \nabla_{xy}^2 E_z^0 + (\gamma^2 + k^2)E_z^0 = 0 \quad \rightarrow \nabla_{xy}^2 E_z^0 + h^2 E_z^0 = 0$$

$$\leftarrow h^2 = \gamma^2 + k^2$$

Characteristic equation:

$$\rightarrow \frac{\partial^2 E_z^0}{\partial y^2} + h^2 E_z^0 = 0 \quad (\because \frac{\partial}{\partial x} \rightarrow 0)$$

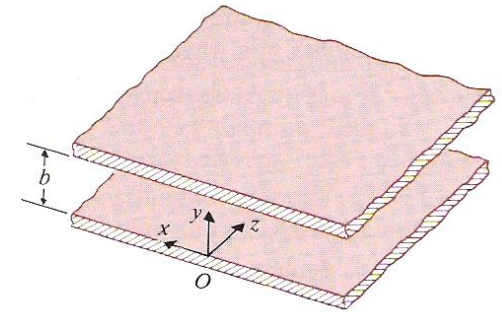
Boundary conditions:

$$E_z^0 = 0 \quad \leftarrow y = 0 \text{ \& } y = b$$

Solution:

$$E_z^0 = A_n \sin hy$$

$$\leftarrow h = \frac{n\pi}{b}, \quad n = 0, 1, 2, \dots$$



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Recall:

BCs for a D-PEC Interface

$$E_{1t} = 0, \quad E_{2t} = 0$$

$$\mathbf{a}_{n2} \times \mathbf{H}_1 = \mathbf{J}_s, \quad H_{2t} = 0$$

$$\mathbf{a}_{2n} \cdot \mathbf{D}_1 = \rho_s, \quad D_{2n} = 0$$

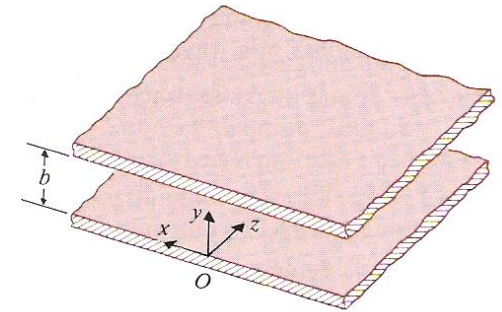
$$B_{1n} = 0, \quad B_{2n} = 0$$

TM Waves Between Parallel Plates (2)

Solution:

$$E_z^0 = A_n \sin hy \quad \rightarrow H_x^0 = \frac{j\omega\epsilon}{h} A_n \cos hy$$

$$\rightarrow E_y^0 = -\frac{\gamma}{h} A_n \cos hy$$



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Recall:

$$H_x^0 = -\frac{1}{h^2} \left(\gamma \frac{\partial H_z^0}{\partial x} - j\omega\epsilon \frac{\partial E_z^0}{\partial y} \right)$$

$$H_y^0 = -\frac{1}{h^2} \left(\gamma \frac{\partial H_z^0}{\partial y} + j\omega\epsilon \frac{\partial E_z^0}{\partial x} \right)$$

$$E_x^0 = -\frac{1}{h^2} \left(\gamma \frac{\partial E_z^0}{\partial x} + j\omega\mu \frac{\partial H_z^0}{\partial y} \right)$$

$$E_y^0 = -\frac{1}{h^2} \left(\gamma \frac{\partial E_z^0}{\partial y} - j\omega\mu \frac{\partial H_z^0}{\partial x} \right)$$

$$\rightarrow \gamma = \sqrt{\left(\frac{n\pi}{b} \right)^2 - \omega^2 \mu\epsilon}$$

$$\rightarrow f_c = \frac{n}{2b\sqrt{\mu\epsilon}}$$

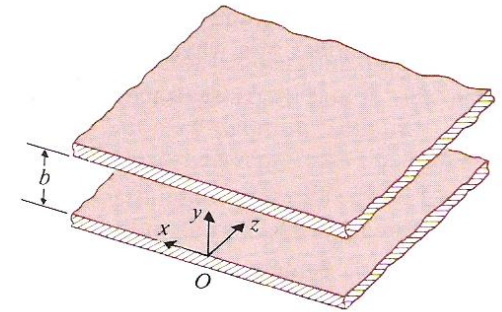
If $n = 0$:

$$\rightarrow E_z^0 = 0 \quad \text{TM}_0 \rightarrow \text{TEM mode}$$

$$\rightarrow \gamma = ik$$

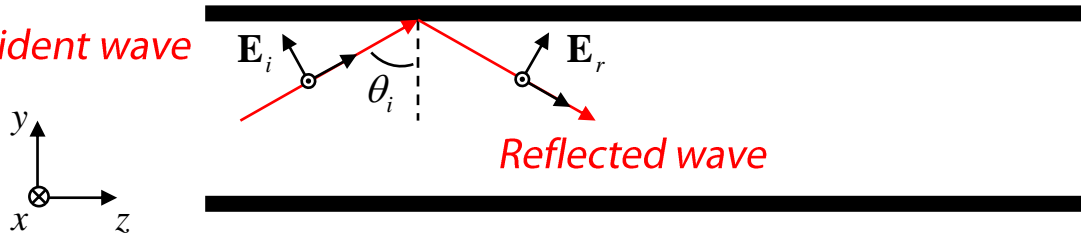
TM Waves: Alternative Approach (1)

An infinite parallel-plate waveguide



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Incident wave



Reflected wave

Plane-wave representation:

$$\rightarrow \mathbf{E}_i = E_i (\mathbf{a}_y \sin \theta_i - \mathbf{a}_z \cos \theta_i) e^{-jk(\cos \theta_i y + \sin \theta_i z)}$$

$$\rightarrow \mathbf{E}_r = E_r (\mathbf{a}_y \sin \theta_i + \mathbf{a}_z \cos \theta_i) e^{-jk(-\cos \theta_i y + \sin \theta_i z)}$$

$$\rightarrow \mathbf{E}_{total} = \mathbf{E}_i + \mathbf{E}_r \quad \rightarrow E_{total,t} = E_{i,z} + E_{r,z}$$

$$= (-E_i \cos \theta_i e^{-jk \cos \theta_i y} + E_r \cos \theta_i e^{+jk \cos \theta_i y}) e^{-jk \sin \theta_i z}$$

$$\rightarrow E_{total,t}(y = 0 \& b) = 0$$

$$\rightarrow E_i = E_r \quad \rightarrow \sin(k \cos \theta_i b) = 0 \quad \rightarrow k \cos \theta_i = \frac{n\pi}{b}$$

$$\rightarrow \mathbf{E}_{total} = E_i [\mathbf{a}_y 2 \sin \theta_i \cos(k \cos \theta_i y) + \mathbf{a}_z 2j \cos \theta_i \sin(k \cos \theta_i y)] e^{-jk \sin \theta_i z}$$

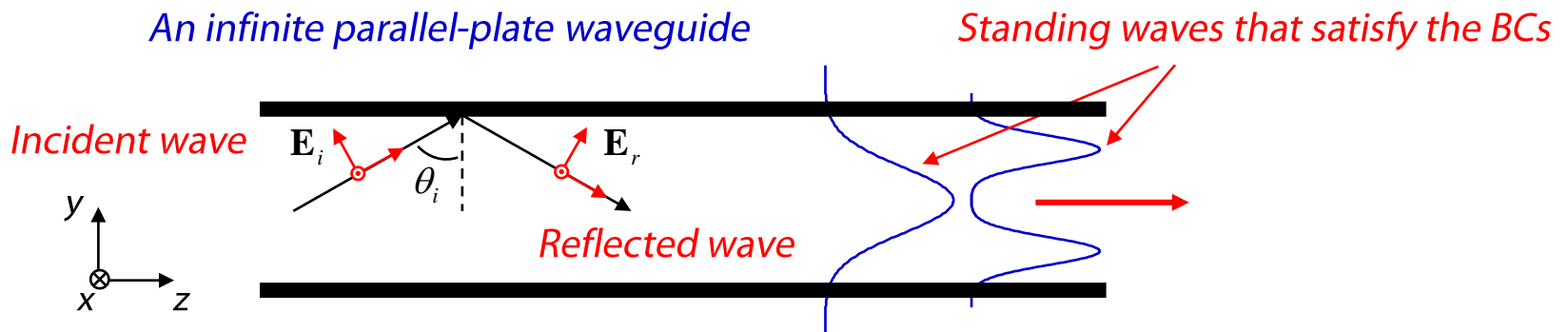
TM Waves: Alternative Approach (2)

$$\begin{aligned}
 \rightarrow \mathbf{E}_{total} &= E_i [\mathbf{a}_y 2 \sin \theta_i \overbrace{\cos(k \cos \theta_i y)}^{\rightarrow h} + \mathbf{a}_z 2 j \cos \theta_i \sin(k \cos \theta_i y)] e^{-\overbrace{jk \sin \theta_i z}^{\rightarrow \gamma}} \\
 &= A_n [\mathbf{a}_y \frac{2 \sin \theta_i}{2 j \cos \theta_i} \cos hy + \mathbf{a}_z \sin hy] e^{-\gamma z} && \leftarrow h^2 = \gamma^2 + k^2 \\
 &= A_n [\mathbf{a}_y \frac{-jk \sin \theta_i}{k \cos \theta_i} \cos hy + \mathbf{a}_z \sin hy] e^{-\gamma z} \\
 &= A_n [\mathbf{a}_y \frac{-\gamma}{h} \cos hy + \mathbf{a}_z \sin hy] e^{-\gamma z}
 \end{aligned}$$

Exactly the same outcome!

↗
↖

Normal at the boundary
Tangential at the boundary



TE Waves Between Parallel Plates

Recall:

$$\gamma = \alpha + j\beta \quad \rightarrow \text{Propagation constant}$$

$$\mathbf{E}(x, y, z; t) = \text{Re}[\mathbf{E}^0(x, y)e^{(j\omega t - \gamma z)}]$$

$$E_z = 0 \quad \rightarrow \nabla_{xy}^2 H_z^0 + (\gamma^2 + k^2)H_z^0 = 0 \quad \rightarrow \nabla_{xy}^2 H_z^0 + h^2 H_z^0 = 0$$

Recall:

Characteristic equation:

$$\rightarrow \frac{\partial^2 H_z^0}{\partial y^2} + h^2 H_z^0 = 0 \quad (\because \frac{\partial}{\partial x} \rightarrow 0)$$

Boundary conditions:

$$E_x^0 = 0 \quad \leftarrow y = 0 \text{ \& } y = b$$

Solution:

$$H_z^0 = B_n \cosh y$$

$$\leftarrow h = \frac{n\pi}{b}, \quad n = 0, 1, 2, \dots$$

$$\begin{aligned} H_x^0 &= -\frac{1}{h^2} \left(\gamma \frac{\partial H_z^0}{\partial x} - j\omega\epsilon \frac{\partial E_z^0}{\partial y} \right) \\ H_y^0 &= -\frac{1}{h^2} \left(\gamma \frac{\partial H_z^0}{\partial y} + j\omega\epsilon \frac{\partial E_z^0}{\partial x} \right) \\ E_x^0 &= -\frac{1}{h^2} \left(\gamma \frac{\partial E_z^0}{\partial x} + j\omega\mu \frac{\partial H_z^0}{\partial y} \right) \\ E_y^0 &= -\frac{1}{h^2} \left(\gamma \frac{\partial E_z^0}{\partial y} - j\omega\mu \frac{\partial H_z^0}{\partial x} \right) \end{aligned}$$

If $n = 0$:

$TE_0 \rightarrow$ Null mode

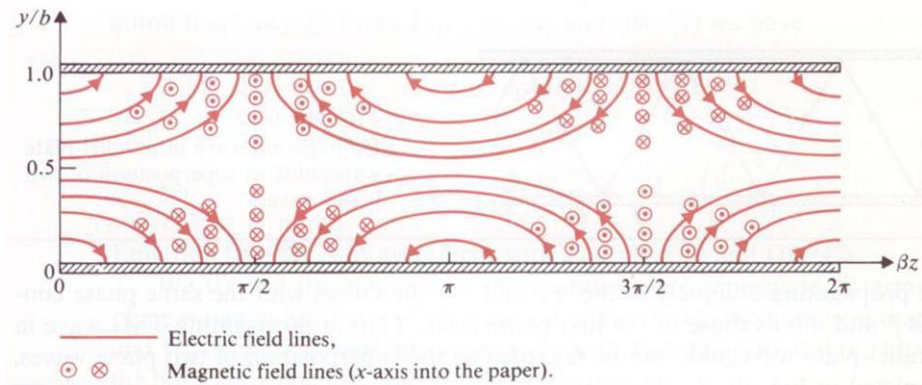
TM/TE Waves in Parallel-Plate Waveguides

TM_1 mode:

$$E_z^0 = A_n \sin\left(\frac{\pi}{b} y\right)$$

$$E_y^0 = -\frac{\gamma b}{\pi} A_n \cos\left(\frac{\pi}{b} y\right)$$

$$H_x^0 = \frac{j\omega\epsilon b}{\pi} A_n \cos\left(\frac{\pi}{b} y\right)$$



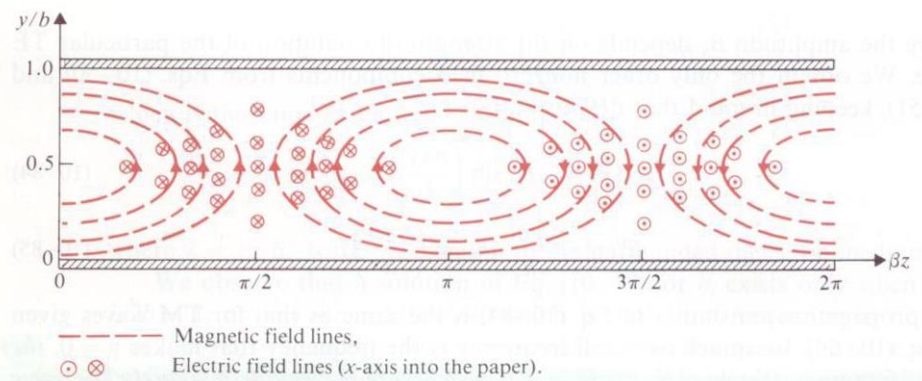
D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

TE_1 mode:

$$H_z^0 = B_n \cos\left(\frac{\pi}{b} y\right)$$

$$H_y^0 = \frac{\gamma b}{\pi} B_n \sin\left(\frac{\pi}{b} y\right)$$

$$E_x^0 = \frac{j\omega\mu b}{\pi} B_n \sin\left(\frac{\pi}{b} y\right)$$



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Attenuation in Parallel-Plate Waveguides

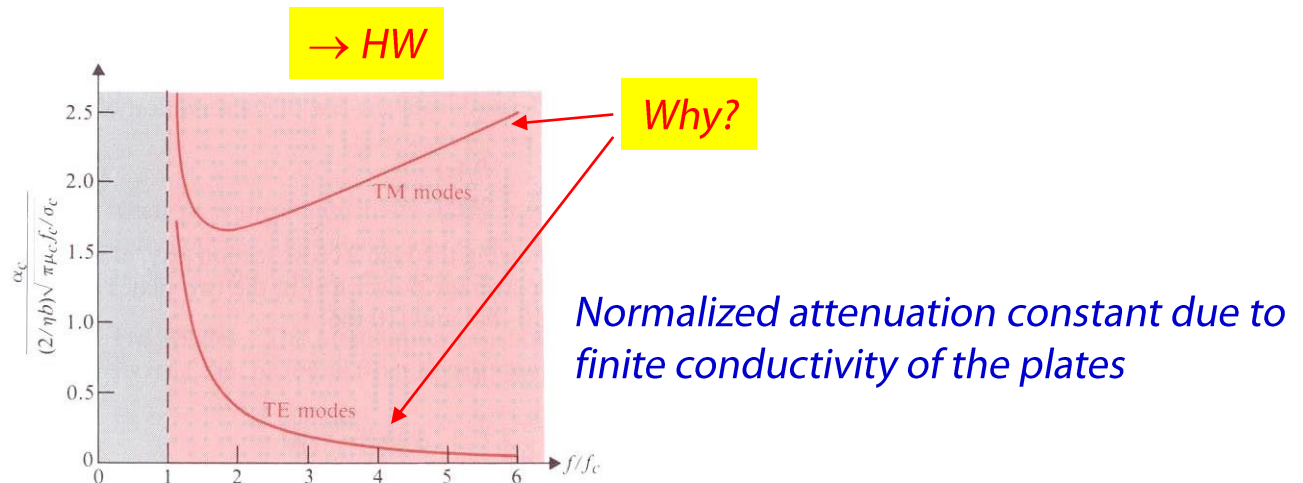
Attenuation constant:

$$\rightarrow \alpha = \alpha_d + \alpha_c$$

\swarrow Loss in the dielectric
 \searrow Loss in the conducting wall

$$\leftarrow \frac{dP}{dz} = -2\alpha P$$

$$\rightarrow \alpha_c = \frac{P_L}{2P} = \frac{\text{Time-averaged power lost in the conducting wall per unit length}}{\text{Time-averaged power flow through the waveguide}}$$



Energy-Transport Velocity (1)

Energy-transport velocity:

$$\rightarrow u_{en} = \frac{P_{z,av}}{W'_{av}} = \frac{\text{Time-averaged propagated power}}{\text{Time-averaged stored energy per unit guide length}}$$

$$\leftarrow P_{z,av} = \int_S \mathbf{P}_{av} \cdot d\mathbf{s} = \frac{1}{2} \text{Re} \int_S \mathbf{E} \times \mathbf{H}^* \cdot d\mathbf{s}$$

$$\leftarrow W'_{av} = \int_S (w_{e,av} + w_{m,av}) ds = \frac{1}{4} \text{Re} \int_S (\mathbf{E} \cdot \epsilon \mathbf{E}^* + \mathbf{H} \cdot \mu \mathbf{H}^*) ds$$

$$\rightarrow u_{en} = \frac{P_{z,av}}{W'_{av}} = u_g \rightarrow \text{Group velocity}$$

Derivation:

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \rightarrow \nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \rightarrow (\nabla_t + \nabla_z) \times \mathbf{E} = -j\omega\mu\mathbf{H}$$

$$\rightarrow \nabla_t \times \mathbf{E} - j\beta(\mathbf{a}_z \times \mathbf{E}) = -j\omega\mu\mathbf{H}$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \rightarrow \nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E} \rightarrow (\nabla_t + \nabla_z) \times \mathbf{H} = j\omega\epsilon\mathbf{E}$$

$$\rightarrow \nabla_t \times \mathbf{H} - j\beta(\mathbf{a}_z \times \mathbf{H}) = j\omega\epsilon\mathbf{E}$$

Energy-Transport Velocity (2)

First-order perturbation: $\rightarrow \delta\beta \rightarrow \delta\omega, \delta\mathbf{E}, \& \delta\mathbf{H}$

Recall:

$$\rightarrow \nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\rightarrow \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

$$\rightarrow \nabla_t \times \mathbf{E} - j\beta(\mathbf{a}_z \times \mathbf{E}) = -j\omega\mu\mathbf{H}$$

$$\rightarrow \nabla_t \times (\mathbf{E} + \delta\mathbf{E}) - j(\beta + \delta\beta)[\mathbf{a}_z \times (\mathbf{E} + \delta\mathbf{E})] = -j(\omega + \delta\omega)\mu(\mathbf{H} + \delta\mathbf{H})$$

$$\rightarrow \nabla_t \times \delta\mathbf{E} - j\beta(\mathbf{a}_z \times \delta\mathbf{E}) - j\delta\beta(\mathbf{a}_z \times \mathbf{E}) = -j\omega\mu\delta\mathbf{H} - j\delta\omega\mu\mathbf{H} \quad \leftarrow \cdot \mathbf{H}^*$$

$$\begin{aligned} \rightarrow \delta\mathbf{E} \cdot (\nabla_t \times \mathbf{H}^*) + \nabla_t \cdot (\delta\mathbf{E} \times \mathbf{H}^*) + j\beta\delta\mathbf{E} \cdot (\mathbf{a}_z \times \mathbf{H}^*) - j\delta\beta(\mathbf{E} \times \mathbf{H}^*) \cdot \mathbf{a}_z \\ = -j\omega\mathbf{H}^* \cdot \mu\delta\mathbf{H} - j\delta\omega\mathbf{H}^* \cdot \mu\mathbf{H} \end{aligned}$$

$$\rightarrow \delta\mathbf{E} \cdot (\nabla \times \mathbf{H}^*) + \nabla_t \cdot (\delta\mathbf{E} \times \mathbf{H}^*) - j\delta\beta(\mathbf{E} \times \mathbf{H}^*) \cdot \mathbf{a}_z = -j\omega\mathbf{H}^* \cdot \mu\delta\mathbf{H} - j\delta\omega\mathbf{H}^* \cdot \mu\mathbf{H}$$

$$\rightarrow \nabla_t \times \mathbf{H} - j\beta(\mathbf{a}_z \times \mathbf{H}) = j\omega\epsilon\mathbf{E}$$

$$\rightarrow \delta\mathbf{H} \cdot (\nabla \times \mathbf{E}^*) + \nabla_t \cdot (\delta\mathbf{H} \times \mathbf{E}^*) + j\delta\beta(\mathbf{E}^* \times \mathbf{H}) \cdot \mathbf{a}_z = j\omega\mathbf{E}^* \cdot \epsilon\delta\mathbf{E} + j\omega\mathbf{E}^* \cdot \epsilon\mathbf{E}$$

$$\begin{aligned} \rightarrow \delta\mathbf{E} \cdot (\nabla \times \mathbf{H}^* + j\omega\epsilon\mathbf{E}^*) - \delta\mathbf{H} \cdot (\nabla \times \mathbf{E}^* - j\omega\mu\mathbf{H}^*) + \nabla_t \cdot (\delta\mathbf{E} \times \mathbf{H}^* - \delta\mathbf{H} \times \mathbf{E}^*) - j\delta\beta \cdot 2\text{Re}(\mathbf{E} \times \mathbf{H}^*) \cdot \mathbf{a}_z \\ = -j\delta\omega(\mathbf{E}^* \cdot \epsilon\mathbf{E} + \mathbf{H}^* \cdot \mu\mathbf{H}) \end{aligned}$$

$$\rightarrow \nabla_t \cdot \mathbf{F} - j\delta\beta \cdot 4\text{Re}(\mathbf{E} \times \mathbf{H}^*) \cdot \mathbf{a}_z = -j\delta\omega \cdot 2\text{Re}(\mathbf{E} \cdot \epsilon\mathbf{E}^* + \mathbf{H} \cdot \mu\mathbf{H}^*)$$

$$\leftarrow \mathbf{F} = \delta\mathbf{E} \times \mathbf{H}^* - \delta\mathbf{H} \times \mathbf{E}^* - \delta\mathbf{E}^* \times \mathbf{H} + \delta\mathbf{H}^* \times \mathbf{E}$$

Energy-Transport Velocity (3)

Continued:

$$\rightarrow \nabla_t \cdot \mathbf{F} - j\delta\beta \cdot 4 \operatorname{Re}(\mathbf{E} \times \mathbf{H}^*) \cdot \mathbf{a}_z = -j\delta\omega \cdot 2 \operatorname{Re}(\mathbf{E} \cdot \epsilon \mathbf{E}^* + \mathbf{H} \cdot \mu \mathbf{H}^*)$$

$$\leftarrow \mathbf{F} = \delta \mathbf{E} \times \mathbf{H}^* - \delta \mathbf{H} \times \mathbf{E}^* - \delta \mathbf{E}^* \times \mathbf{H} + \delta \mathbf{H}^* \times \mathbf{E}$$

$$\rightarrow \int_S \nabla_t \cdot \mathbf{F} ds = \oint_C \mathbf{F} \cdot \mathbf{n} dl \rightarrow 0$$

$$\leftarrow \frac{1}{2} \operatorname{Re} \int_S \mathbf{E} \times \mathbf{H}^* \cdot \mathbf{a}_z ds = P_{z,av}$$

$$\leftarrow \frac{1}{4} \operatorname{Re} \int_S (\mathbf{E} \cdot \epsilon \mathbf{E}^* + \mathbf{H} \cdot \mu \mathbf{H}^*) ds = W'_{av}$$

$$\rightarrow \delta\beta P_{z,av} = \delta\omega W'_{av}$$

$$\rightarrow \frac{P_{z,av}}{W'_{av}} = \frac{\delta\omega}{\delta\beta} \rightarrow u_{en} = u_g$$

\rightarrow Energy-transport velocity = Group velocity