

Electromagnetics: Rectangular Waveguides (10-4)

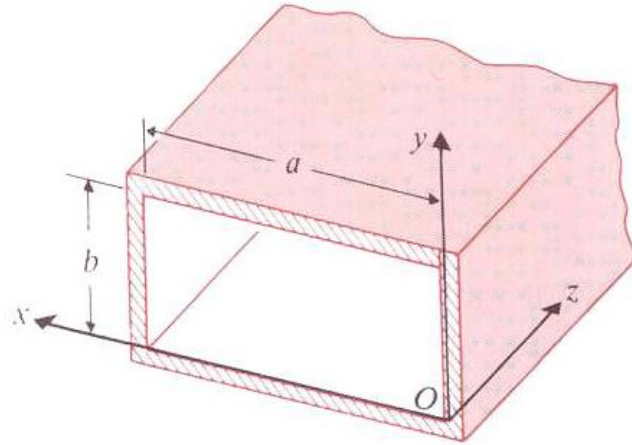
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Rectangular Waveguides



Very similar to parallel-plate waveguides

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Just recall:

$$\gamma = \alpha + j\beta \quad \rightarrow \text{Propagation constant}$$

$$\mathbf{E}(x, y, z; t) = \text{Re}[\mathbf{E}^0(x, y)e^{(j\omega t - \gamma z)}] \quad \leftarrow h^2 = \gamma^2 + k^2$$

$$H_z = 0 \quad \rightarrow \nabla_{xy}^2 E_z^0 + (\gamma^2 + k^2)E_z^0 = 0 \quad \rightarrow \nabla_{xy}^2 E_z^0 + h^2 E_z^0 = 0$$

$$E_z = 0 \quad \rightarrow \nabla_{xy}^2 H_z^0 + (\gamma^2 + k^2)H_z^0 = 0 \quad \rightarrow \nabla_{xy}^2 H_z^0 + h^2 H_z^0 = 0$$

TM Waves in Rectangular Waveguides (1)

Characteristic equation:

$$\begin{aligned} H_z = 0 &\rightarrow \nabla_{xy}^2 E_z^0 + (\gamma^2 + k^2) E_z^0 = 0 \rightarrow \nabla_{xy}^2 E_z^0 + h^2 E_z^0 = 0 \\ &\leftarrow h^2 = \gamma^2 + k^2 \\ &\rightarrow \frac{\partial^2 E_z^0}{\partial x^2} + \frac{\partial^2 E_z^0}{\partial y^2} + h^2 E_z^0 = 0 \end{aligned}$$

Boundary conditions: *What about E_x and E_y or H_x and H_y ?*

$$E_z^0 = 0 \quad \leftarrow x = 0 \text{ \& } x = a$$

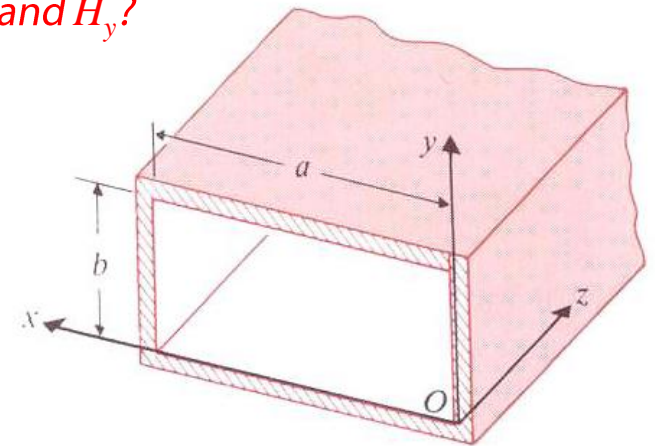
$$E_z^0 = 0 \quad \leftarrow y = 0 \text{ \& } y = b$$

Solution:

$$E_z^0 = E_0 \sin k_x x \sin k_y y$$

$$\rightarrow k_x^2 + k_y^2 = h^2$$

$$\rightarrow k_x = \frac{m\pi}{a}, \quad k_y = \frac{n\pi}{b} \rightarrow m \text{ \& } n: \text{ Integer numbers}$$



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TM Waves in Rectangular Waveguides (2)

With: $E_z^0 = E_0 \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right)$

Recall:

$$H_x^0 = -\frac{1}{h^2} \left(\gamma \frac{\partial H_z^0}{\partial x} - j\omega\epsilon \frac{\partial E_z^0}{\partial y} \right)$$

$$H_y^0 = -\frac{1}{h^2} \left(\gamma \frac{\partial H_z^0}{\partial y} + j\omega\epsilon \frac{\partial E_z^0}{\partial x} \right)$$

$$E_x^0 = -\frac{1}{h^2} \left(\gamma \frac{\partial E_z^0}{\partial x} + j\omega\mu \frac{\partial H_z^0}{\partial y} \right)$$

$$E_y^0 = -\frac{1}{h^2} \left(\gamma \frac{\partial E_z^0}{\partial y} - j\omega\mu \frac{\partial H_z^0}{\partial x} \right)$$

$$E_x^0 = -\frac{\gamma}{h^2} \left(\frac{m\pi}{a} \right) E_0 \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right)$$

$$E_y^0 = -\frac{\gamma}{h^2} \left(\frac{n\pi}{b} \right) E_0 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right)$$

$$H_x^0 = \frac{j\omega\epsilon}{h^2} \left(\frac{n\pi}{b} \right) E_0 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right)$$

$$H_y^0 = -\frac{j\omega\epsilon}{h^2} \left(\frac{m\pi}{a} \right) E_0 \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right)$$

$TM_{11} \rightarrow$ Lowest-order mode

$$\leftarrow \gamma = j\beta = j\sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

Cutoff frequency and wavelength:

\rightarrow Neither m nor n can be zero!

$$\rightarrow f_{c,mn} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \quad \rightarrow \lambda_{c,mn} = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

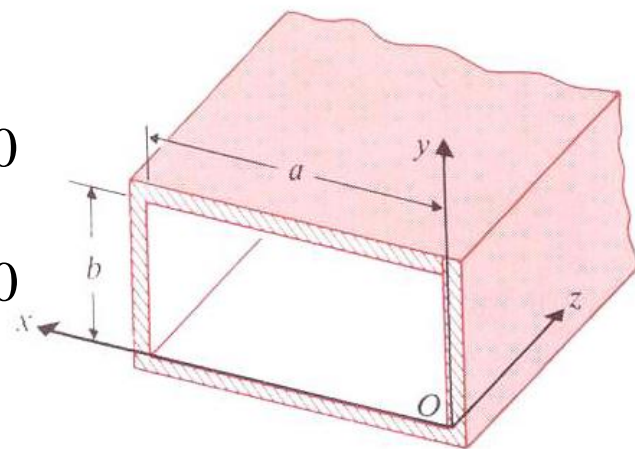
TE Waves in Rectangular Waveguides (1)

Characteristic equation:

$$E_z = 0 \rightarrow \nabla_{xy}^2 H_z^0 + (\gamma^2 + k^2) H_z^0 = 0 \rightarrow \nabla_{xy}^2 H_z^0 + h^2 H_z^0 = 0$$
$$\leftarrow h^2 = \gamma^2 + k^2$$
$$\rightarrow \frac{\partial^2 H_z^0}{\partial x^2} + \frac{\partial^2 H_z^0}{\partial y^2} + h^2 H_z^0 = 0$$

Boundary conditions:

$$E_y^0 = 0 \leftarrow x = 0 \text{ \& } x = a \rightarrow \frac{\partial H_z^0}{\partial x} = 0$$
$$E_x^0 = 0 \leftarrow y = 0 \text{ \& } y = b \rightarrow \frac{\partial H_z^0}{\partial y} = 0$$



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Solution:

$$H_z^0 = H_0 \cos k_x x \cos k_y y$$

$$\leftarrow k_x^2 + k_y^2 = h^2$$

$$\rightarrow k_x = \frac{m\pi}{a}, k_y = \frac{n\pi}{b} \rightarrow m \text{ \& } n: \text{ Integer numbers}$$

TE Waves in Rectangular Waveguides (2)

With: $H_z^0 = H_0 \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right)$

Recall:

$$H_x^0 = -\frac{1}{h^2} \left(\gamma \frac{\partial H_z^0}{\partial x} - j\omega\epsilon \frac{\partial E_z^0}{\partial y} \right)$$

$$H_y^0 = -\frac{1}{h^2} \left(\gamma \frac{\partial H_z^0}{\partial y} + j\omega\epsilon \frac{\partial E_z^0}{\partial x} \right)$$

$$E_x^0 = -\frac{1}{h^2} \left(\gamma \frac{\partial E_z^0}{\partial x} + j\omega\mu \frac{\partial H_z^0}{\partial y} \right)$$

$$E_y^0 = -\frac{1}{h^2} \left(\gamma \frac{\partial E_z^0}{\partial y} - j\omega\mu \frac{\partial H_z^0}{\partial x} \right)$$

$$E_x^0 = \frac{j\omega\mu}{h^2} \left(\frac{n\pi}{b} \right) H_0 \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right)$$

$$E_y^0 = -\frac{j\omega\mu}{h^2} \left(\frac{m\pi}{a} \right) H_0 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right)$$

$$H_x^0 = \frac{\gamma}{h^2} \left(\frac{m\pi}{a} \right) H_0 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right)$$

$$H_y^0 = \frac{\gamma}{h^2} \left(\frac{n\pi}{b} \right) H_0 \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right)$$

TE_{10} or $TE_{01} \rightarrow$ Lowest-order mode

$$\leftarrow \gamma = j\beta = j\sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

Cutoff frequency or wavelength:

\rightarrow Either m or n can be zero but not both!

$$f_{c,mm} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$$\lambda_{c,mm} = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

TM/TE Waves in Rectangular Waveguides (1)

TM modes: $\rightarrow m \neq 0$ and $n \neq 0$

$$E_x^0 = -\frac{\gamma}{h^2} \left(\frac{m\pi}{a} \right) E_0 \cos \left(\frac{m\pi}{a} x \right) \sin \left(\frac{n\pi}{b} y \right)$$

$$E_y^0 = -\frac{\gamma}{h^2} \left(\frac{n\pi}{b} \right) E_0 \sin \left(\frac{m\pi}{a} x \right) \cos \left(\frac{n\pi}{b} y \right)$$

$$H_x^0 = \frac{j\omega\epsilon}{h^2} \left(\frac{n\pi}{b} \right) E_0 \sin \left(\frac{m\pi}{a} x \right) \cos \left(\frac{n\pi}{b} y \right)$$

$$H_y^0 = -\frac{j\omega\epsilon}{h^2} \left(\frac{m\pi}{a} \right) E_0 \cos \left(\frac{m\pi}{a} x \right) \sin \left(\frac{n\pi}{b} y \right)$$

TE modes: $\rightarrow m \neq 0$ or $n \neq 0$

$$E_x^0 = \frac{j\omega\mu}{h^2} \left(\frac{n\pi}{b} \right) H_0 \cos \left(\frac{m\pi}{a} x \right) \sin \left(\frac{n\pi}{b} y \right)$$

$$E_y^0 = -\frac{j\omega\mu}{h^2} \left(\frac{m\pi}{a} \right) H_0 \sin \left(\frac{m\pi}{a} x \right) \cos \left(\frac{n\pi}{b} y \right)$$

$$H_x^0 = \frac{\gamma}{h^2} \left(\frac{m\pi}{a} \right) H_0 \sin \left(\frac{m\pi}{a} x \right) \cos \left(\frac{n\pi}{b} y \right)$$

$$H_y^0 = \frac{\gamma}{h^2} \left(\frac{n\pi}{b} \right) H_0 \cos \left(\frac{m\pi}{a} x \right) \sin \left(\frac{n\pi}{b} y \right)$$

Lowest-order mode: TM_{11}

$$f_{c,11} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}$$

$$\lambda_{c,mn} = 2 / \sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}$$

Lowest-order mode: TE_{10} ($a > b$)

$$f_{c,10} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{0}{b}\right)^2} = \frac{1}{2a\sqrt{\mu\epsilon}}$$

$$\lambda_{c,mn} = 2a \rightarrow \text{Dominant mode}$$

What if: $f_{c,TE_{10}} < f < f_{c,TM_{11}} \rightarrow TE\text{-mode excitation!}$

TM/TE Waves in Rectangular Waveguides (2)

TM₁₁ mode:

$$E_x^0 = -\frac{\gamma}{h^2} \left(\frac{\pi}{a} \right) E_0 \cos \left(\frac{\pi}{a} x \right) \sin \left(\frac{\pi}{b} y \right)$$

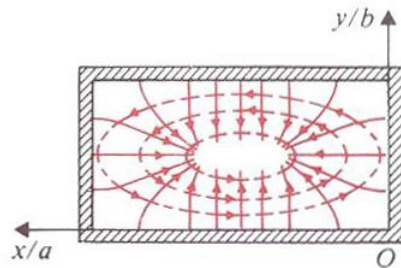
$$E_y^0 = -\frac{\gamma}{h^2} \left(\frac{\pi}{b} \right) E_0 \sin \left(\frac{\pi}{a} x \right) \cos \left(\frac{\pi}{b} y \right)$$

$$E_z^0 = E_0 \sin \left(\frac{\pi}{a} x \right) \sin \left(\frac{\pi}{b} y \right)$$

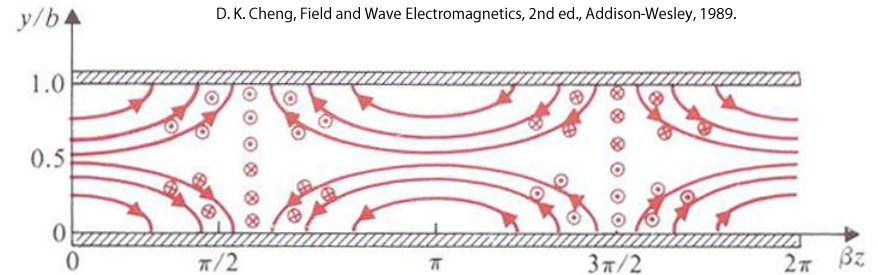
$$H_x^0 = \frac{j\omega\epsilon}{h^2} \left(\frac{\pi}{b} \right) E_0 \sin \left(\frac{\pi}{a} x \right) \cos \left(\frac{\pi}{b} y \right)$$

$$H_y^0 = -\frac{j\omega\epsilon}{h^2} \left(\frac{\pi}{a} \right) E_0 \cos \left(\frac{\pi}{a} x \right) \sin \left(\frac{\pi}{b} y \right)$$

$$H_z^0 = 0$$



(a)



(b)

— Electric field lines - - - - - Magnetic field lines

TE₁₀ mode:

$$E_x^0 = 0$$

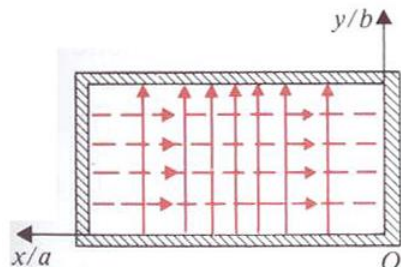
$$E_y^0 = -\frac{j\omega\mu}{h^2} \left(\frac{\pi}{a} \right) H_0 \sin \left(\frac{\pi}{a} x \right)$$

$$E_z^0 = 0$$

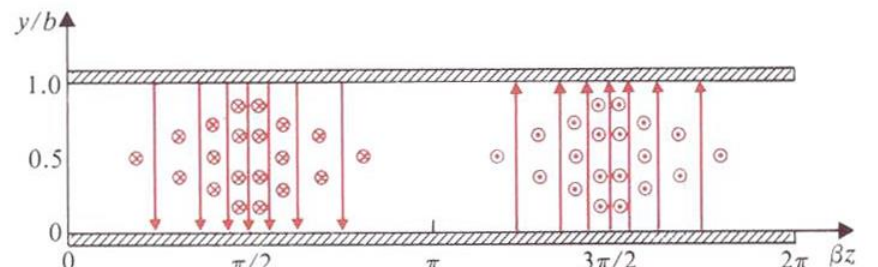
$$H_x^0 = \frac{\gamma}{h^2} \left(\frac{\pi}{a} \right) H_0 \sin \left(\frac{\pi}{a} x \right)$$

$$H_y^0 = 0$$

$$H_z^0 = H_0 \cos \left(\frac{\pi}{a} x \right)$$

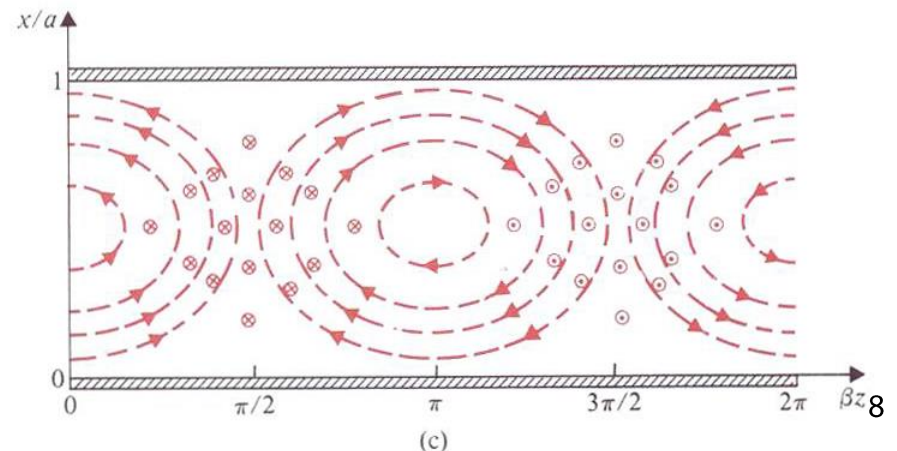


(a)



(b)

— Electric field lines - - - - - Magnetic field lines



(c)

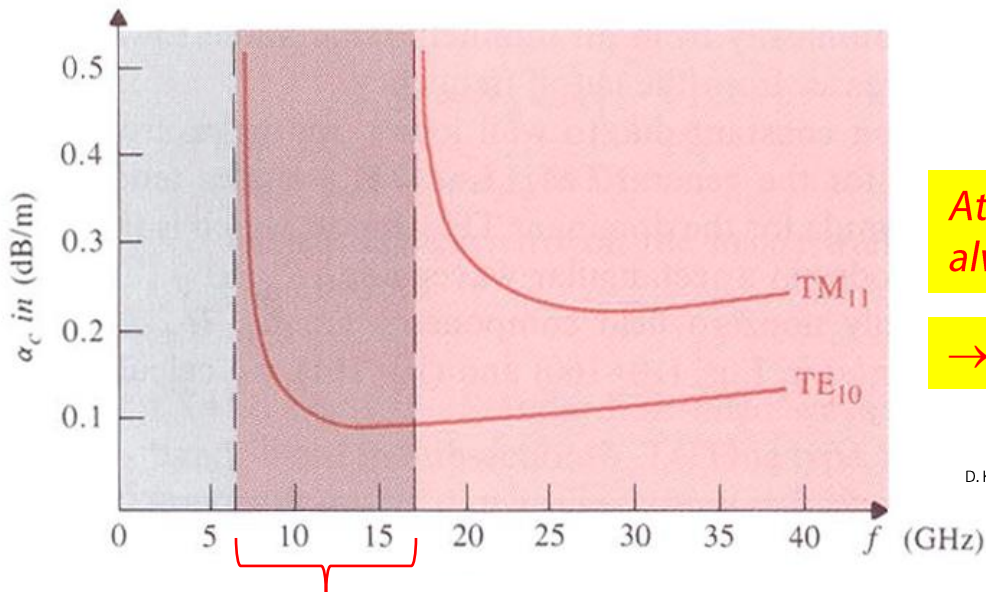
Attenuation in Rectangular Waveguides

Attenuation constant:

$$\rightarrow \alpha = \alpha_d + \alpha_c$$

\swarrow Loss in the dielectric
 \searrow Loss in the conducting wall

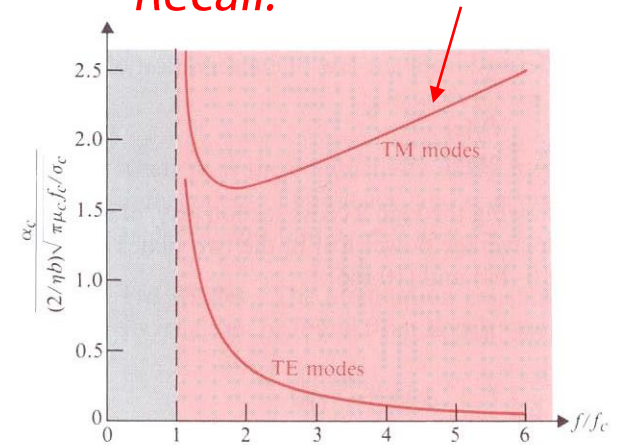
Attenuation due to wall losses:



TE-mode operation

$$\rightarrow \eta_c = \sqrt{\frac{\mu}{\epsilon_c}} \cong \frac{1+j}{\sqrt{2}} \sqrt{\frac{\omega\mu}{\sigma}}$$

Recall:



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

PP waveguide

Attenuation of TE₁₀ mode is always lower than that of TM₁₁.

→ HW

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.