Electromagnetics:

Cavity Resonators

(10-7)

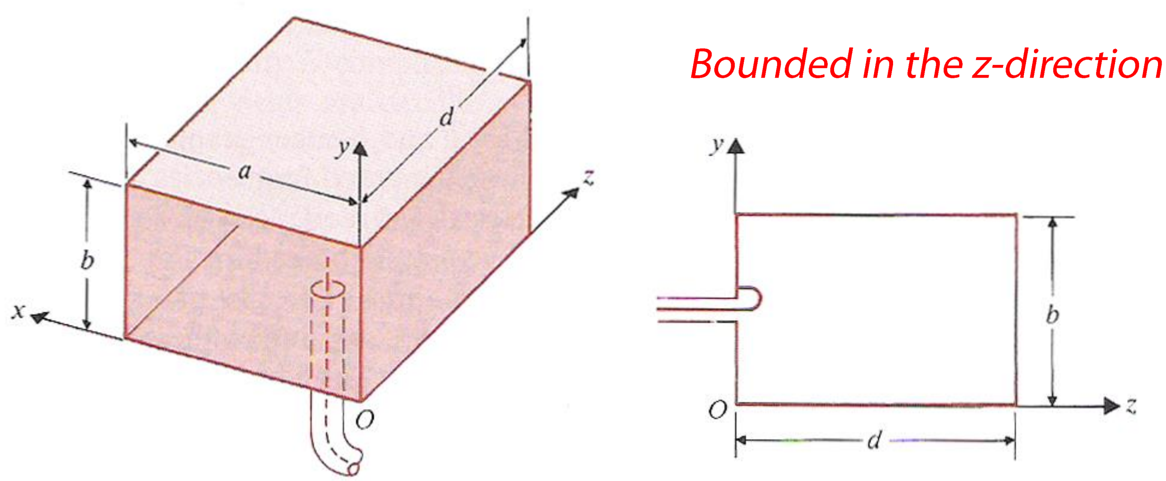
Yoonchan Jeong

School of Electrical Engineering, Seoul National University

Tel: +82 (0)2 880 1623, Fax: +82 (0)2 873 9953

Email: yoonchan@snu.ac.kr

Rectangular Cavity Resonator



(a) Probe excitation. (b) Loop excitation.

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Just recall:

$$\gamma = \alpha + j\beta \quad \rightarrow \text{Propagation constant}$$

$$\mathbf{E}(x,y,z;t) = \text{Re}[\underline{A_0 \mathbf{E}_+^0(x,y) e^{j\omega t - \gamma z}} + \underline{B_0 \mathbf{E}_-^0(x,y) e^{j\omega t + \gamma z}}]$$

TM mode: $H_z = 0 \quad \rightarrow \nabla_{xy}^2 E_{z\pm}^0 + (\gamma^2 + k^2) E_{z\pm}^0 = 0 \rightarrow \nabla_{xy}^2 E_{z\pm}^0 + h^2 E_{z\pm}^0 = 0$

TE mode: $E_z = 0 \quad \rightarrow \nabla_{xy}^2 H_{z\pm}^0 + (\gamma^2 + k^2) H_{z\pm}^0 = 0 \rightarrow \nabla_{xy}^2 H_{z\pm}^0 + h^2 H_{z\pm}^0 = 0$

$$\leftarrow h^2 = \gamma^2 + k^2$$

TM Modes in Rectangular Resonators (1)

Characteristic equation:

$$H_z = 0 \;\rightarrow \nabla^2_{xy} E^0_{z\pm} + (\gamma^2 + k^2) E^0_{z\pm} = 0 \;\rightarrow \nabla^2_{xy} E^0_{z\pm} + h^2 E^0_{z\pm} = 0$$

$$\rightarrow \frac{\partial^2 E^0_{z\pm}}{\partial x^2} + \frac{\partial^2 E^0_{z\pm}}{\partial y^2} + h^2 E^0_{z\pm} = 0$$

$$\rightarrow \gamma = j\beta \rightarrow h^2 + \beta^2 = k^2$$

Boundary conditions:

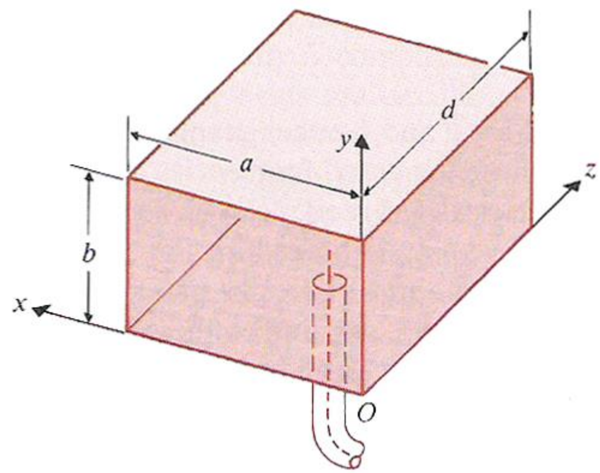
$$E_z = 0 \quad \leftarrow x = 0 \;\&\; x = a$$

$$E_z = 0 \quad \leftarrow y = 0 \;\&\; y = b$$

$$E_x = 0 \quad \leftarrow z = 0 \;\&\; z = d$$

or

$$E_y = 0 \quad \leftarrow z = 0 \;\&\; z = d$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Solution: $\mathbf{E}(x, y, z) = A_0 \mathbf{E}^0_+(x, y) e^{-j\beta z} + B_0 \mathbf{E}^0_-(x, y) e^{+j\beta z}$

We've already found the expressions for these.

TM Modes in Rectangular Resonators (2)

$\mathbf{E}_{+}^{0}(x,y)$

$$E_{z+}^{0}=E_0\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

$$E_{x+}^{0}=-\frac{j\beta}{h^2}\left(\frac{m\pi}{a}\right)E_0\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

$$E_{y+}^{0}=-\frac{j\beta}{h^2}\left(\frac{n\pi}{b}\right)E_0\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$$

$$H_{x+}^{0}=\frac{j\omega\varepsilon}{h^2}\left(\frac{n\pi}{b}\right)E_0\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$$

$$H_{y+}^{0}=-\frac{j\omega\varepsilon}{h^2}\left(\frac{m\pi}{a}\right)E_0\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

$\mathbf{E}_{-}^{0}(x,y)$

$$E_{z-}^{0}=E_0\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

$$E_{x-}^{0}=+\frac{j\beta}{h^2}\left(\frac{m\pi}{a}\right)E_0\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

$$E_{y-}^{0}=+\frac{j\beta}{h^2}\left(\frac{n\pi}{b}\right)E_0\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$$

$$H_{x-}^{0}=\frac{j\omega\varepsilon}{h^2}\left(\frac{n\pi}{b}\right)E_0\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$$

$$H_{y-}^{0}=-\frac{j\omega\varepsilon}{h^2}\left(\frac{m\pi}{a}\right)E_0\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

Recall: $\mathbf{E}(x,y,z)=A_0\mathbf{E}_{+}^{0}(x,y)e^{-j\beta z}+B_0\mathbf{E}_{-}^{0}(x,y)e^{+j\beta z}$

$$\rightarrow E_x(x,y,z)=E_{x+}^{0}(A_0e^{-j\beta z}-B_0e^{+j\beta z})$$

Boundary condition: $E_x=0 \quad \leftarrow z=0\ \&\ z=d$

$$\rightarrow A_0=B_0$$

$$\rightarrow \sin\beta d=0$$

$$\rightarrow \beta=\frac{p\pi}{d}$$

TM Modes in Rectangular Resonators (3)

$$\mathbf{E}(x,y,z) = A_0\mathbf{E}_+^0(x,y)e^{-j\beta z} + B_0\mathbf{E}_-^0(x,y)e^{+j\beta z}$$

$$E_z = E_0' \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\cos\left(\frac{p\pi}{d}z\right)$$

$$E_x = -\frac{1}{h^2}\left(\frac{m\pi}{a}\right)\left(\frac{p\pi}{d}\right)E_0' \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\sin\left(\frac{p\pi}{d}z\right)$$

$$E_y = -\frac{1}{h^2}\left(\frac{n\pi}{b}\right)\left(\frac{p\pi}{d}\right)E_0' \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\sin\left(\frac{p\pi}{d}z\right)$$

$$H_x = \frac{j\omega\varepsilon}{h^2}\left(\frac{n\pi}{b}\right)E_0' \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\cos\left(\frac{p\pi}{d}z\right)$$

$$H_y = -\frac{j\omega\varepsilon}{h^2}\left(\frac{m\pi}{a}\right)E_0' \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\cos\left(\frac{p\pi}{d}z\right)$$

Recall: $h^2 + \beta^2 = k^2$

Resonant frequency: $\rightarrow \omega_{mnp} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2}$

Lowest-order mode: TM_{110}

TE Modes in Rectangular Resonators (1)

Characteristic equation:

$$E_z = 0 \;\rightarrow \nabla_{xy}^2 H_{z\pm}^0 + (\gamma^2 + k^2) H_{z\pm}^0 = 0 \rightarrow \nabla_{xy}^2 H_{z\pm}^0 + h^2 H_{z\pm}^0 = 0$$

$$\rightarrow \frac{\partial^2 H_{z\pm}^0}{\partial x^2} + \frac{\partial^2 H_{z\pm}^0}{\partial y^2} + h^2 H_{z\pm}^0 = 0$$

$$\rightarrow \gamma = j\beta \rightarrow h^2 + \beta^2 = k^2$$

Boundary conditions:

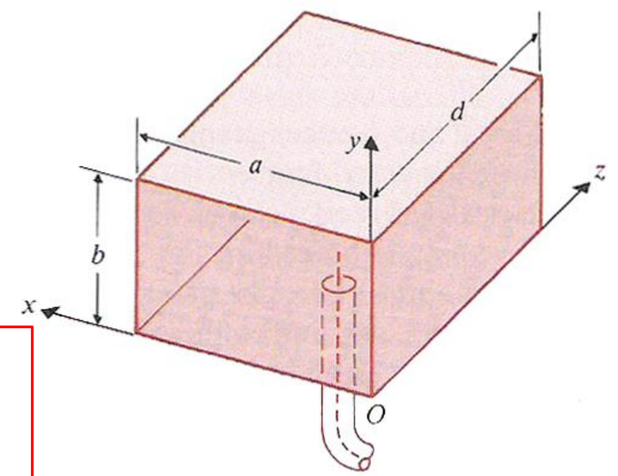
$$E_y^0 = 0 \;\leftarrow x = 0 \;\&\; x = a \;\rightarrow \frac{\partial H_z^0}{\partial x} = 0$$

$$E_x^0 = 0 \;\leftarrow y = 0 \;\&\; y = b \;\rightarrow \frac{\partial H_z^0}{\partial y} = 0$$

$$E_x = 0 \;\leftarrow z = 0 \;\&\; z = d$$

or

$$E_y = 0 \;\leftarrow z = 0 \;\&\; z = d$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Solution: $\mathbf{H}(x,y,z) = A_0 \mathbf{H}_+^0(x,y) e^{-j\beta z} + B_0 \mathbf{H}_-^0(x,y) e^{+j\beta z}$

We've already found the expressions for these.

TE Modes in Rectangular Resonators (2)

$$\mathbf{H}(x,y,z) = A_0\mathbf{H}_+^0(x,y)e^{-j\beta z} + B_0\mathbf{H}_-^0(x,y)e^{+j\beta z}$$

$$H_z = H_0' \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\sin\left(\frac{p\pi}{d}z\right)$$

$$E_x = \frac{j\omega\mu}{h^2}\left(\frac{n\pi}{b}\right)H_0' \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\sin\left(\frac{p\pi}{d}z\right)$$

$$E_y = -\frac{j\omega\mu}{h^2}\left(\frac{m\pi}{a}\right)H_0' \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\sin\left(\frac{p\pi}{d}z\right)$$

$$H_x = -\frac{1}{h^2}\left(\frac{m\pi}{a}\right)\left(\frac{p\pi}{d}\right)H_0' \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\cos\left(\frac{p\pi}{d}z\right)$$

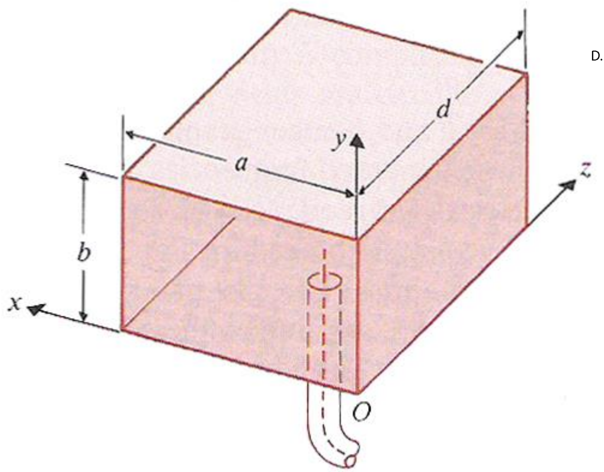
$$H_y = -\frac{1}{h^2}\left(\frac{n\pi}{b}\right)\left(\frac{p\pi}{d}\right)H_0' \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\cos\left(\frac{p\pi}{d}z\right)$$

Recall: $h^2 + \beta^2 = k^2$

Resonant frequency: $\rightarrow \omega_{mnp} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2}$

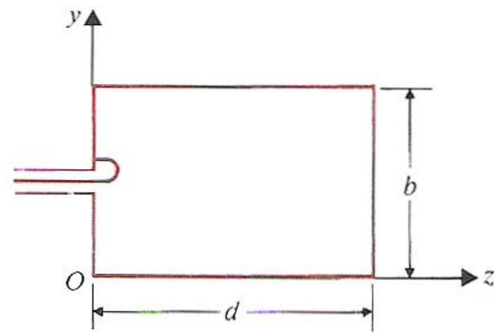
Lowest-order mode: TE_{011} or TE_{101}

Mode Excitation & Quality Factor of Cavity Resonator



(a) Probe excitation.

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.



(b) Loop excitation.

e.g. TE_{101} mode:

$$E_y^0 = -\frac{j\omega\mu a}{\pi} H_0 \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{d}z\right)$$

$$H_x^0 = -\frac{a}{d} H_0 \sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{d}z\right)$$

$$H_z^0 = H_0 \cos\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{d}z\right)$$

→ *The choice of the direction and position of the excitation is important!*

Nonzero surface resistance results in a decay of the stored energy.

$$Q = 2\pi \frac{\text{Time-average energy stored at a resonant frequency}}{\text{Energy dissipated in one period of this frequency}}$$ (Dimensionless)

$$= \frac{\omega W}{P_L}$$

$\leftarrow W = W_e + W_m$

← *Time-average power dissipated in the cavity*

Circular Cavity Resonator

Just recall:

$$\gamma = \alpha + j\beta \qquad \rightarrow \textit{Propagation constant}$$

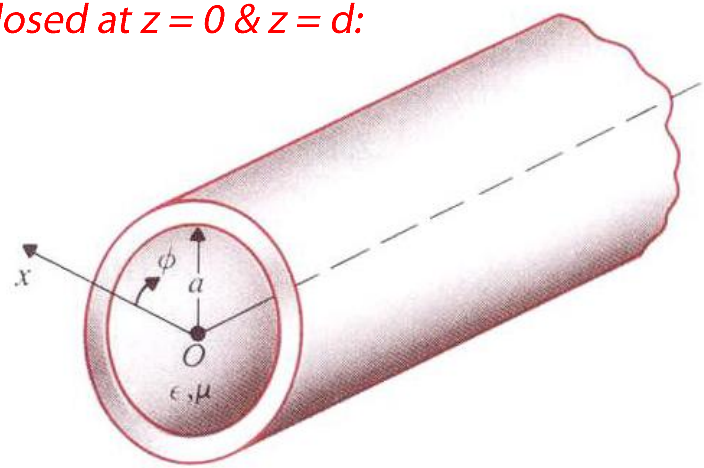
$$\mathbf{E}(x,y,z;t) = \mathrm{Re}[A_0 \mathbf{E}_+^0(x,y) e^{j\omega t - \gamma z} + B_0 \mathbf{E}_-^0(x,y) e^{j\omega t + \gamma z}]$$

TM mode: $H_z = 0 \rightarrow \nabla_{xy}^2 E_{z\pm}^0 + (\gamma^2 + k^2) E_{z\pm}^0 = 0 \quad \rightarrow \nabla_{xy}^2 E_{z\pm}^0 + h^2 E_{z\pm}^0 = 0$

TE mode: $E_z = 0 \rightarrow \nabla_{xy}^2 H_{z\pm}^0 + (\gamma^2 + k^2) H_{z\pm}^0 = 0 \quad \rightarrow \nabla_{xy}^2 H_{z\pm}^0 + h^2 H_{z\pm}^0 = 0$

$$\leftarrow h^2 = \gamma^2 + k^2$$

Enclosed at z = 0 & z = d:



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

→ Choice of the coordinate system:

$$\rightarrow \nabla_{r\phi}^2 E_{z\pm}^0 + h^2 E_{z\pm}^0 = 0$$

$$\rightarrow \nabla_{r\phi}^2 H_{z\pm}^0 + h^2 H_{z\pm}^0 = 0$$

TM Modes in Circular Cavity Resonators (1)

Let: $E_z(r,\phi,z) = A_0 E^0_{z+}(r,\phi)e^{-\gamma z} + B_0 E^0_{z-}(r,\phi)e^{+\gamma z} \quad \leftarrow \gamma = j\beta$

Recall:

$$R(r) = \begin{cases} CJ_n(hr) + DN_n(hr) & \text{for } h^2 > 0 \\ CI_n(hr) + DK_n(hr) & \text{for } h^2 < 0 \end{cases}$$

$$\rightarrow E^0_{z\pm} = C_n J_n(hr)\cos n\phi$$

Also recall: $\mathbf{E}^0_{T\pm} = -\frac{\pm\gamma}{h^2}\nabla_T E^0_{z\pm} \quad \leftarrow \nabla_T E^0_{z\pm} = \left(\mathbf{a}_r \frac{\partial}{\partial r} + \mathbf{a}_\phi \frac{\partial}{r\partial\phi}\right)E^0_{z\pm}$

Solution:

$$E^0_{r\pm} = -\frac{\pm j\beta}{h} C_n J'_n(hr)\cos n\phi$$

$$E^0_{\phi\pm} = \frac{\pm j\beta n}{h^2 r} C_n J_n(hr)\sin n\phi$$

$$H^0_{r\pm} = -\frac{j\omega\varepsilon n}{h^2 r} C_n J_n(hr)\sin n\phi$$

$$H^0_{\phi\pm} = -\frac{j\omega\varepsilon}{h} C_n J'_n(hr)\cos n\phi$$

$$\rightarrow \mathbf{H}_\pm = \frac{\pm 1}{Z_{TM\pm}}\mathbf{a}_z \times \mathbf{E}_\pm$$

TM Modes in Circular Cavity Resonators (2)

$$\mathbf{E}(x,y,z;t) = \text{Re}[A_0\mathbf{E}_+^0(x,y)e^{j\omega t-\gamma z} + B_0\mathbf{E}_-^0(x,y)e^{j\omega t+\gamma z}]$$

Solution:

$$E_{r\pm}^0 = -\frac{\pm j\beta}{h}C_n J_n'(hr)\cos n\phi$$
$$E_{\phi\pm}^0 = \frac{\pm j\beta n}{h^2 r}C_n J_n(hr)\sin n\phi$$
$$E_{z\pm}^0 = C_n J_n(hr)\cos n\phi$$

$$H_{r\pm}^0 = -\frac{j\omega\varepsilon n}{h^2 r}C_n J_n(hr)\sin n\phi$$
$$H_{\phi\pm}^0 = -\frac{j\omega\varepsilon}{h}C_n J_n'(hr)\cos n\phi$$
$$H_{z\pm}^0 = 0$$

Boundary condition:

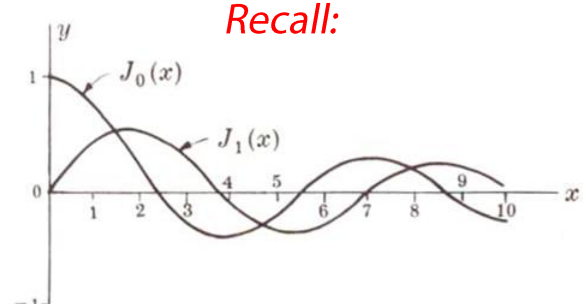
TABLE 10–2
Zeros of $J_n(x)$, x_{np}

p \ n	n = 0	n = 1	n = 2
1	2.405	3.832	5.136
2	5.520	7.016	8.417

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

$$E_{z\pm}^0 = E_{\phi\pm}^0 = 0 \quad at \;\; r = a$$
$$\rightarrow J_n(ha) = 0 \quad at \;\; r = a$$
$$\rightarrow ha = x_{nm}$$

Recall:



M. R. Spiegel, Mathematical Handbook of Formulas and Tables, McGraw-Hill, 1990.

where x_{nm} represents the m^th zero of $J_n(x)$.

$$E_r = 0 \quad at \;\; z = 0 \;\&\; z = d$$
$$\rightarrow E_r(r,\phi,z) = E_{r+}^0(A_0 e^{-j\beta z} - B_0 e^{+j\beta z})$$
$$\rightarrow A_0 = B_0$$
$$\rightarrow \sin\beta d = 0$$
$$\rightarrow \beta = \frac{p\pi}{d}$$

TM Modes in Circular Cavity Resonators (3)

$$\mathbf{E}(r,\phi,z) = A_0\mathbf{E}_+^0(r,\phi)e^{-j\beta z} + B_0\mathbf{E}_-^0(r,\phi)e^{+j\beta z}$$

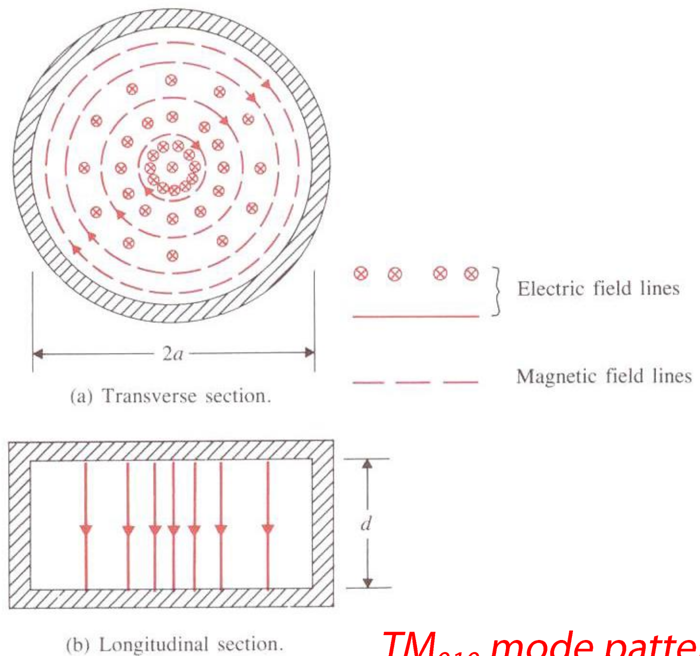
$$E_z = E_0' J_n(hr)\cos(n\phi)\cos\left(\frac{p\pi}{d}z\right)$$

$$E_r = -\frac{\beta}{h}E_0' J_n'(hr)\cos n\phi \sin\left(\frac{p\pi}{d}z\right)$$

$$E_\phi = \frac{\beta n}{h^2 r}E_0' J_n(hr)\sin n\phi \sin\left(\frac{p\pi}{d}z\right)$$

$$H_r = -\frac{j\omega\varepsilon n}{h^2 r}E_0' J_n(hr)\sin n\phi \cos\left(\frac{p\pi}{d}z\right)$$

$$H_\phi = -\frac{j\omega\varepsilon}{h^2}E_0' J_n'(hr)\cos n\phi \cos\left(\frac{p\pi}{d}z\right)$$



TM_{010} mode pattern

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Recall: $h^2 + \beta^2 = k^2$

Resonant frequency: $\rightarrow \omega_{nmp} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{x_{nm}}{a}\right)^2 + \left(\frac{p\pi}{d}\right)^2}$

Lowest-order mode: TM_{010}

TE Modes in Circular Cavity Resonators

This is your Homework, EMF2_Ex_06: Derive TE modes in a circular cavity!