

# Electromagnetics:

## Dielectric Waveguides

### (10-6)

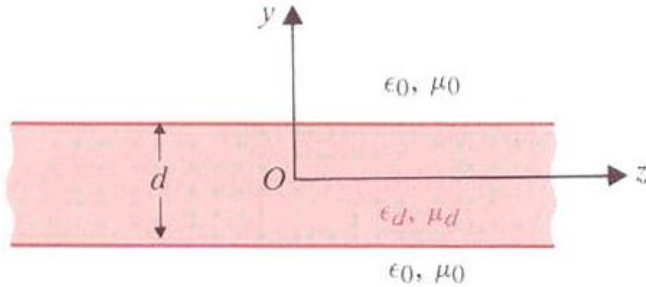
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# Dielectric Waveguides



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

*A dielectric slab waveguide: Lossless*

*→ No variation in the x-direction*  $\left(\frac{\partial}{\partial x} \rightarrow 0\right)$

Just recall:

$\gamma = j\beta$  *→ Propagation constant*

$$\mathbf{E}(y, z; t) = \text{Re}[\mathbf{E}^0(y)e^{(j\omega t - \gamma z)}]$$

TM mode:  $H_z = 0 \rightarrow \nabla_y^2 E_z^0 + (\gamma^2 + k^2)E_z^0 = 0 \rightarrow \nabla_y^2 E_z^0 + h^2 E_z^0 = 0$

TE mode:  $E_z = 0 \rightarrow \nabla_y^2 H_z^0 + (\gamma^2 + k^2)H_z^0 = 0 \rightarrow \nabla_y^2 H_z^0 + h^2 H_z^0 = 0$

$$\leftarrow h^2 = \gamma^2 + k^2 \quad 2$$

# TM Modes along a Dielectric Slab

Characteristic equation:

$$H_z = 0 \rightarrow \frac{\partial^2 E_z^0}{\partial y^2} + h^2 E_z^0 = 0$$

$$\leftarrow h^2 = \gamma^2 + k^2$$

$$\rightarrow \gamma = j\beta \rightarrow h^2 = k^2 - \beta^2 \quad \leftarrow k^2 = \omega^2 \mu \epsilon = \omega^2 \begin{cases} \mu_d \epsilon_d, & |y| \leq d/2 \\ \mu_0 \epsilon_0, & |y| > d/2 \end{cases}$$

Boundary condition:

*Tangential components of E and H fields: Continuous across the boundary*

$\rightarrow \beta$  must be the same in the core and cladding regions.

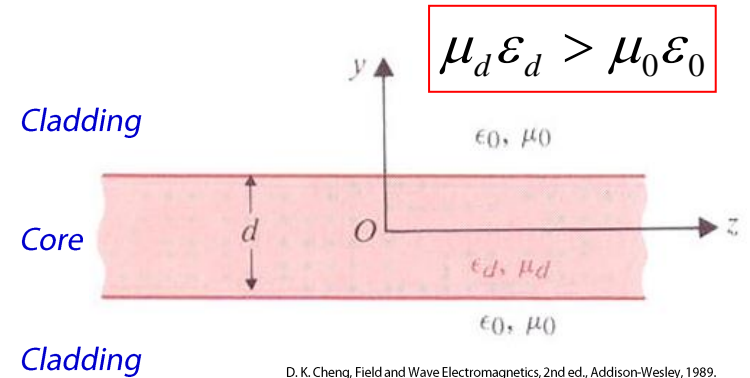
$\rightarrow h^2 \geq 0$  in the core: Propagating modes

$\rightarrow h^2 < 0$  in the cladding: Evanescent modes

Solution:

$$E_z^0(y) = E_o \sin k_y y + E_e \cos k_y y, \quad |y| \leq d/2 \quad \leftarrow k_y^2 = \omega^2 \mu_d \epsilon_d - \beta^2 = h_d^2$$

$$E_z^0(y) = \begin{cases} C_u e^{-\alpha(y-d/2)}, & y > d/2 \\ C_l e^{\alpha(y+d/2)}, & y < -d/2 \end{cases} \quad \leftarrow \alpha^2 = \beta^2 - \omega^2 \mu_0 \epsilon_0 = -h_0^2$$



# Odd TM Modes (1)

Odd-mode solution:

$$E_z^0(y) = E_o \sin k_y y + E_e \cos k_y y, \quad |y| \leq d/2$$

$$E_z^0(y) = \begin{cases} C_u e^{-\alpha(y-d/2)}, & y > d/2 \\ C_l e^{\alpha(y+d/2)}, & y < -d/2 \end{cases}$$

In the core:

$$E_z^0(y) = E_o \sin k_y y$$

$$E_y^0(y) = -\frac{j\beta}{k_y} E_o \cos k_y y$$

$$H_x^0(y) = \frac{j\omega\epsilon_d}{k_y} E_o \cos k_y y$$

In the cladding:

Upper cladding:

$$E_z^0(y) = C_u e^{-\alpha(y-d/2)}$$

$$E_y^0(y) = -\frac{j\beta}{\alpha} C_u e^{-\alpha(y-d/2)}$$

$$H_x^0(y) = \frac{j\omega\epsilon_0}{\alpha} C_u e^{-\alpha(y-d/2)}$$

Lower cladding:

$$E_z^0(y) = C_l e^{\alpha(y+d/2)}$$

$$E_y^0(y) = \frac{j\beta}{\alpha} C_l e^{\alpha(y+d/2)}$$

$$H_x^0(y) = -\frac{j\omega\epsilon_0}{\alpha} C_l e^{\alpha(y+d/2)}$$

Recall:

$$H_x^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial H_z^0}{\partial x} - j\omega\epsilon \frac{\partial E_z^0}{\partial y} \right)$$

$$H_y^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial H_z^0}{\partial y} + j\omega\epsilon \frac{\partial E_z^0}{\partial x} \right)$$

$$E_x^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial E_z^0}{\partial x} + j\omega\mu \frac{\partial H_z^0}{\partial y} \right)$$

$$E_y^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial E_z^0}{\partial y} - j\omega\mu \frac{\partial H_z^0}{\partial x} \right)$$

# Odd TM Modes (2)

$E_z^0$  &  $H_x^0$  must be continuous across the boundary!

Upper cl.:  $E_z^0(y) = C_u e^{-\alpha(y-d/2)}$

Core:  $E_z^0(y) = E_o \sin k_y y$

Lower cl.:  $E_z^0(y) = C_l e^{\alpha(y+d/2)}$

$E_y^0(y) = -\frac{j\beta}{\alpha} C_u e^{-\alpha(y-d/2)}$

$E_y^0(y) = -\frac{j\beta}{k_y} E_o \cos k_y y$

$E_y^0(y) = \frac{j\beta}{\alpha} C_l e^{\alpha(y+d/2)}$

$H_x^0(y) = \frac{j\omega\epsilon_0}{\alpha} C_u e^{-\alpha(y-d/2)}$

$H_x^0(y) = \frac{j\omega\epsilon_d}{k_y} E_o \cos k_y y$

$H_x^0(y) = -\frac{j\omega\epsilon_0}{\alpha} C_l e^{\alpha(y+d/2)}$

$\rightarrow C_u = E_o \sin \frac{k_y d}{2}$

$\rightarrow C_l = -E_o \sin \frac{k_y d}{2}$

$\rightarrow \frac{j\omega\epsilon_0}{\alpha} \left( E_o \sin \frac{k_y d}{2} \right) = \frac{j\omega\epsilon_d}{k_y} E_o \cos \frac{k_y d}{2}$

$\rightarrow \alpha = \frac{\epsilon_0}{\epsilon_d} k_y \tan \frac{k_y d}{2}$

Recall:  $\alpha^2 = \beta^2 - \omega^2 \mu_0 \epsilon_0$ ,  $\beta^2 = \omega^2 \mu_d \epsilon_d - k_y^2$

$\rightarrow \alpha = [\omega^2 (\mu_d \epsilon_d - \mu_0 \epsilon_0) - k_y^2]^{1/2}$

Transcendental equation:

$$[\omega^2 (\mu_d \epsilon_d - \mu_0 \epsilon_0) - k_y^2]^{1/2} = \frac{\epsilon_0}{\epsilon_d} k_y \tan \frac{k_y d}{2}$$

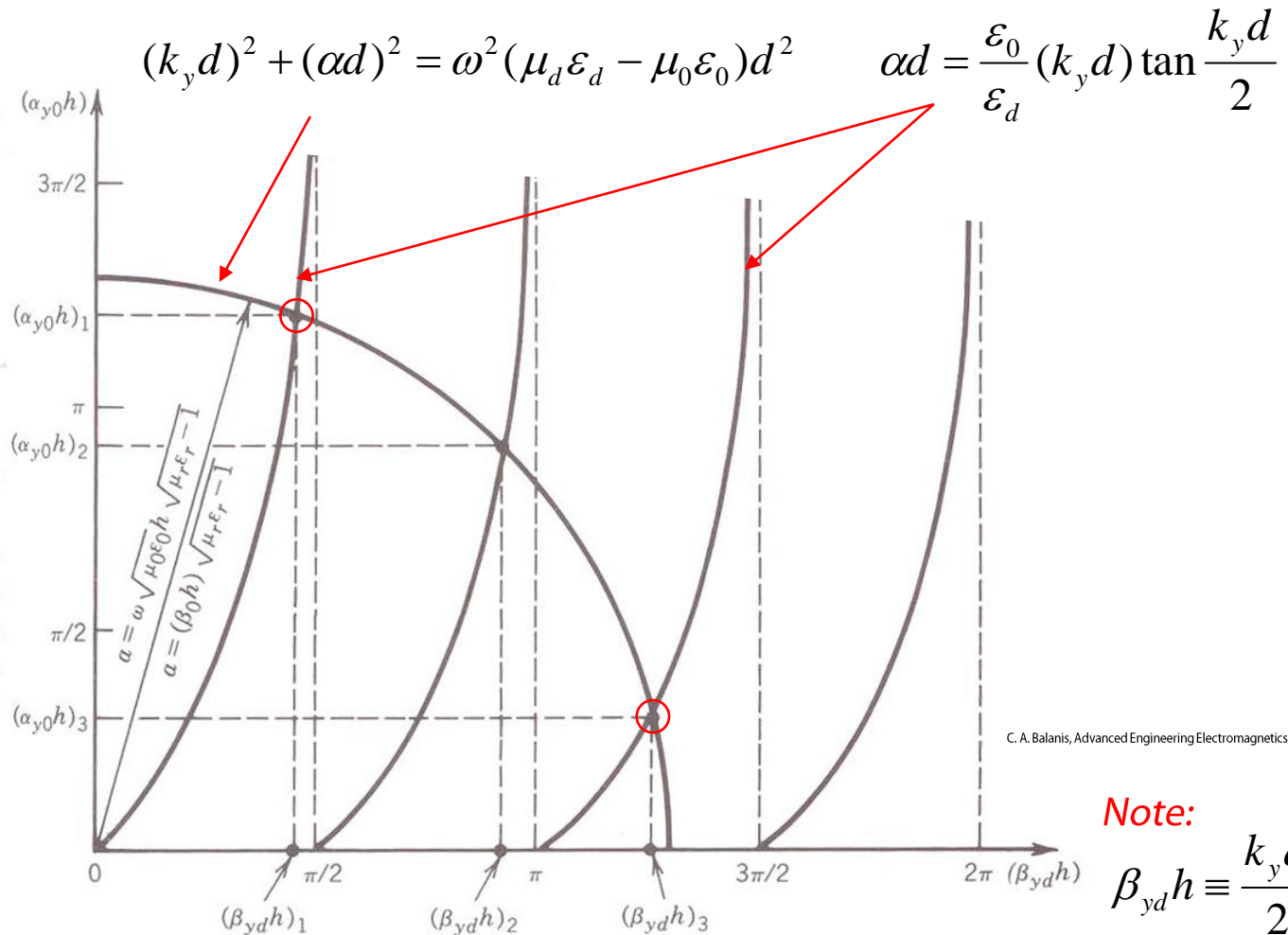
To be solved numerically

# Odd TM Modes (3)

Graphical method to determine:  $k_y$

Transcendental equation:

$$[\omega^2(\mu_d \epsilon_d - \mu_0 \epsilon_0) - k_y^2]^{1/2} = \frac{\epsilon_0}{\epsilon_d} k_y \tan \frac{k_y d}{2}$$



Note:

$$\beta_{yd} h \equiv \frac{k_y d}{2} \quad 6$$

# Even TM Modes

$E_z^0$  &  $H_x^0$  must be continuous across the boundary!

Upper cl.:  $E_z^0(y) = C_u e^{-\alpha(y-d/2)}$

Core:  $E_z^0(y) = E_e \cos k_y y$

Lower cl.:  $E_z^0(y) = C_l e^{\alpha(y+d/2)}$

$E_y^0(y) = -\frac{j\beta}{\alpha} C_u e^{-\alpha(y-d/2)}$

$E_y^0(y) = \frac{j\beta}{k_y} E_e \sin k_y y$

$E_y^0(y) = \frac{j\beta}{\alpha} C_l e^{\alpha(y+d/2)}$

$H_x^0(y) = \frac{j\omega\epsilon_0}{\alpha} C_u e^{-\alpha(y-d/2)}$

$H_x^0(y) = -\frac{j\omega\epsilon_d}{k_y} E_e \sin k_y y$

$H_x^0(y) = -\frac{j\omega\epsilon_0}{\alpha} C_l e^{\alpha(y+d/2)}$

$\rightarrow C_u = E_e \cos \frac{k_y d}{2}$

$\rightarrow C_l = E_e \cos \frac{k_y d}{2}$

$\rightarrow \frac{j\omega\epsilon_0}{\alpha} \left( E_e \cos \frac{k_y d}{2} \right) = -\frac{j\omega\epsilon_d}{k_y} E_e \sin \frac{k_y d}{2}$

$\rightarrow \alpha = -\frac{\epsilon_0}{\epsilon_d} k_y \cot \frac{k_y d}{2}$

Recall:  $\alpha^2 = \beta^2 - \omega^2 \mu_0 \epsilon_0$ ,  $\beta^2 = \omega^2 \mu_d \epsilon_d - k_y^2$

$\rightarrow \alpha = [\omega^2 (\mu_d \epsilon_d - \mu_0 \epsilon_0) - k_y^2]^{1/2}$

Transcendental equation:

$$[\omega^2 (\mu_d \epsilon_d - \mu_0 \epsilon_0) - k_y^2]^{1/2} = -\frac{\epsilon_0}{\epsilon_d} k_y \cot \frac{k_y d}{2}$$

To be solved numerically

# Cutoff Frequencies of TM modes

Guiding condition:

$$k_y^2 = \omega^2 \mu_d \varepsilon_d - \beta^2 \rightarrow \beta^2 < \omega^2 \mu_d \varepsilon_d$$

$$\alpha^2 = \beta^2 - \omega^2 \mu_0 \varepsilon_0 \rightarrow \beta^2 > \omega^2 \mu_0 \varepsilon_0$$

$$\rightarrow \omega^2 \mu_0 \varepsilon_0 < \beta^2 < \omega^2 \mu_d \varepsilon_d$$

Lowest-order mode?

Odd TM<sub>1</sub>

Cutoff condition:  $\alpha = 0 \rightarrow k_y = \omega_{co} \sqrt{\mu_d \varepsilon_d - \mu_0 \varepsilon_0}$

Odd TM modes:

$$\alpha = [\omega^2 (\mu_d \varepsilon_d - \mu_0 \varepsilon_0) - k_y^2]^{1/2}$$

$$= \frac{\varepsilon_0}{\varepsilon_d} k_y \tan \frac{k_y d}{2} = 0$$

$$\rightarrow \pi f_{co} d \sqrt{\mu_d \varepsilon_d - \mu_0 \varepsilon_0} = (n-1)\pi$$

$$\rightarrow f_{co} = \frac{(n-1)}{d \sqrt{\mu_d \varepsilon_d - \mu_0 \varepsilon_0}}$$

Even TM modes:

$$\alpha = [\omega^2 (\mu_d \varepsilon_d - \mu_0 \varepsilon_0) - k_y^2]^{1/2}$$

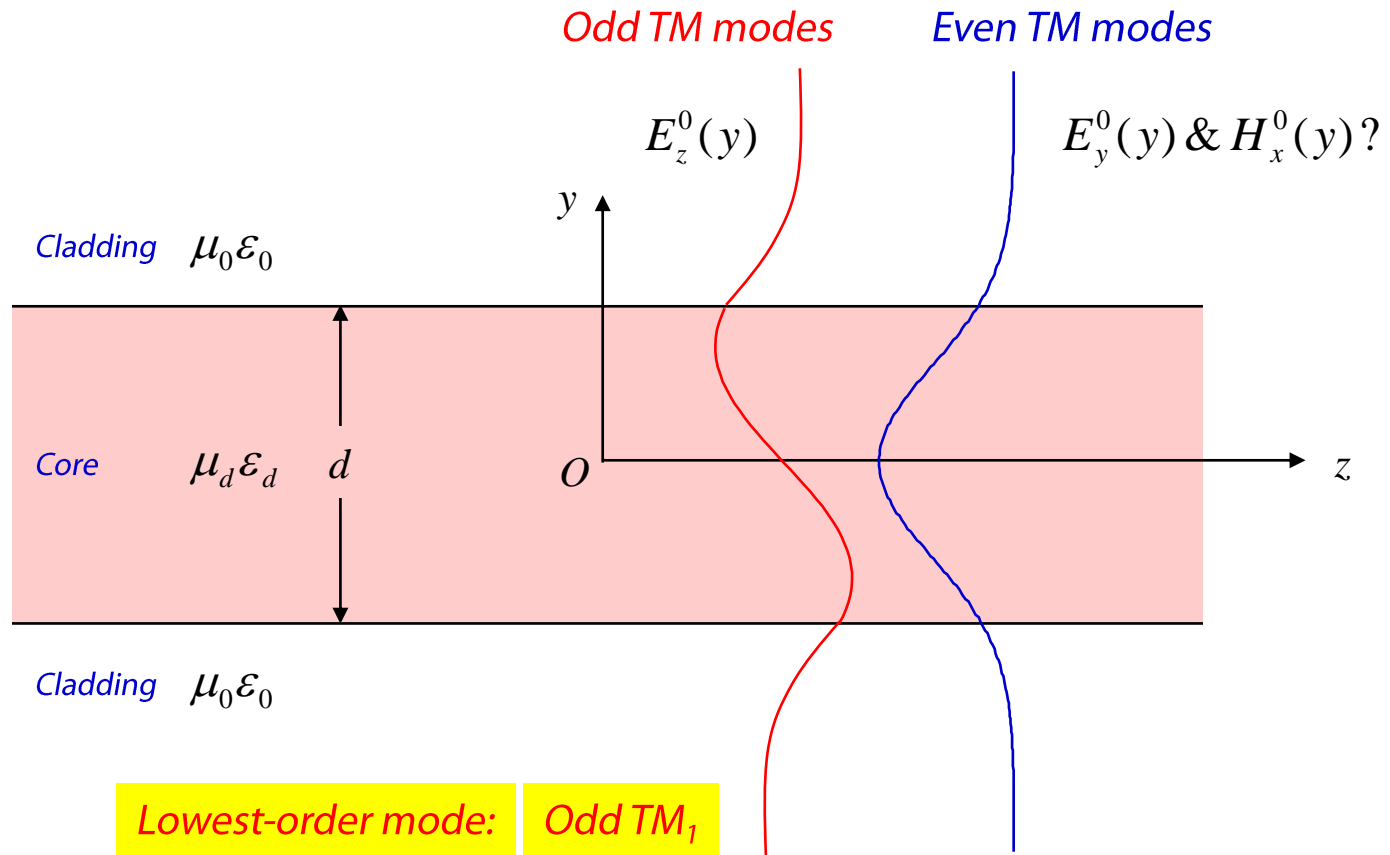
$$= -\frac{\varepsilon_0}{\varepsilon_d} k_y \cot \frac{k_y d}{2} = 0$$

$$\rightarrow \pi f_{co} d \sqrt{\mu_d \varepsilon_d - \mu_0 \varepsilon_0} = (n - \frac{1}{2})\pi$$

$$\rightarrow f_{co} = \frac{(n - \frac{1}{2})}{d \sqrt{\mu_d \varepsilon_d - \mu_0 \varepsilon_0}}$$



# TM Modes along a Dielectric Slab



$$\rightarrow f_{co} = \frac{(n-1)}{d \sqrt{\mu_d \epsilon_d - \mu_0 \epsilon_0}} = 0 \quad \text{Why?}$$

# TE Modes along a Dielectric Slab

Characteristic equation:

$$E_z = 0 \rightarrow \frac{\partial^2 H_z^0}{\partial y^2} + h^2 H_z^0 = 0$$

$$\leftarrow h^2 = \gamma^2 + k^2$$

$$\rightarrow \gamma = j\beta \rightarrow h^2 = k^2 - \beta^2 \quad \leftarrow k^2 = \omega^2 \mu \epsilon = \omega^2 \begin{cases} \mu_d \epsilon_d, & |y| \leq d/2 \\ \mu_0 \epsilon_0, & |y| > d/2 \end{cases}$$

Boundary condition:

*Tangential components of E and H fields: Continuous across the boundary*

$\rightarrow \beta$  must be the same in the core and cladding regions.

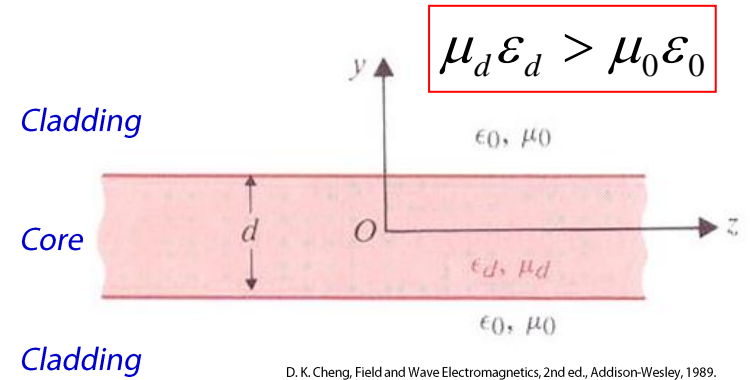
$\rightarrow h^2 \geq 0$  in the core: Propagating modes

$\rightarrow h^2 < 0$  in the cladding: Evanescent modes

Solution:

$$H_z^0(y) = H_o \sin k_y y + H_e \cos k_y y, \quad |y| \leq d/2 \quad \leftarrow k_y^2 = \omega^2 \mu_d \epsilon_d - \beta^2 = h_d^2$$

$$H_z^0(y) = \begin{cases} C_u e^{-\alpha(y-d/2)}, & y > d/2 \\ C_l e^{\alpha(y+d/2)}, & y < -d/2 \end{cases} \quad \leftarrow \alpha^2 = \beta^2 - \omega^2 \mu_0 \epsilon_0 = -h_0^2$$



# Odd TE Modes (1)

Odd-mode solution:

$$H_z^0(y) = H_o \sin k_y y + H_e \cos k_y y, \quad |y| \leq d/2$$

$$H_z^0(y) = \begin{cases} C_u e^{-\alpha(y-d/2)}, & y > d/2 \\ C_l e^{\alpha(y+d/2)}, & y < -d/2 \end{cases}$$

In the core:

$$H_z^0(y) = H_o \sin k_y y$$

$$H_y^0(y) = -\frac{j\beta}{k_y} H_o \cos k_y y$$

$$E_x^0(y) = -\frac{j\omega\mu_d}{k_y} H_o \cos k_y y$$

Recall:

$$H_x^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial H_z^0}{\partial x} - j\omega\epsilon \frac{\partial E_z^0}{\partial y} \right)$$

$$H_y^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial H_z^0}{\partial y} + j\omega\epsilon \frac{\partial E_z^0}{\partial x} \right)$$

$$E_x^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial E_z^0}{\partial x} + j\omega\mu \frac{\partial H_z^0}{\partial y} \right)$$

$$E_y^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial E_z^0}{\partial y} - j\omega\mu \frac{\partial H_z^0}{\partial x} \right)$$

In the cladding:

Upper cladding:

$$H_z^0(y) = C_u e^{-\alpha(y-d/2)}$$

$$H_y^0(y) = -\frac{j\beta}{\alpha} C_u e^{-\alpha(y-d/2)}$$

Lower cladding:

$$H_z^0(y) = C_l e^{\alpha(y+d/2)}$$

$$H_y^0(y) = \frac{j\beta}{\alpha} C_l e^{\alpha(y+d/2)}$$

$$E_x^0(y) = -\frac{j\omega\mu_0}{\alpha} C_u E_o e^{-\alpha(y-d/2)}$$

$$E_x^0(y) = \frac{j\omega\mu_0}{\alpha} C_l E_o e^{\alpha(y+d/2)}$$

# Odd TE Modes (2)

$H_z^0$  &  $E_x^0$  must be continuous across the boundary!

<i>Upper cl.:</i>	$H_z^0(y) = C_u e^{-\alpha(y-d/2)}$	$H_y^0(y) = -\frac{j\beta}{\alpha} C_u e^{-\alpha(y-d/2)}$	$E_x^0(y) = -\frac{j\omega\mu_0}{\alpha} C_u e^{-\alpha(y-d/2)}$
<i>Core:</i>	$H_z^0(y) = H_o \sin k_y y$	$H_y^0(y) = -\frac{j\beta}{k_y} H_o \cos k_y y$	$E_x^0(y) = -\frac{j\omega\mu_d}{k_y} H_o \cos k_y y$
<i>Lower cl.:</i>	$H_z^0(y) = C_l e^{\alpha(y+d/2)}$	$H_y^0(y) = \frac{j\beta}{\alpha} C_l e^{\alpha(y+d/2)}$	$E_x^0(y) = \frac{j\omega\mu_0}{\alpha} C_l e^{\alpha(y+d/2)}$

$$\rightarrow C_u = H_o \sin \frac{k_y d}{2}$$

$$\rightarrow C_l = -H_o \sin \frac{k_y d}{2}$$

$$\rightarrow -\frac{j\omega\mu_0}{\alpha} \left( H_o \sin \frac{k_y d}{2} \right) = -\frac{j\omega\mu_d}{k_y} H_o \cos \frac{k_y d}{2}$$

$$\rightarrow \alpha = \frac{\mu_0}{\mu_d} k_y \tan \frac{k_y d}{2}$$

*Recall:*  $\alpha^2 = \beta^2 - \omega^2 \mu_0 \epsilon_0$ ,  $\beta^2 = \omega^2 \mu_d \epsilon_d - k_y^2$

$$\rightarrow \alpha = [\omega^2 (\mu_d \epsilon_d - \mu_0 \epsilon_0) - k_y^2]^{1/2}$$

*Transcendental equation:*

$$[\omega^2 (\mu_d \epsilon_d - \mu_0 \epsilon_0) - k_y^2]^{1/2} = \frac{\mu_0}{\mu_d} k_y \tan \frac{k_y d}{2}$$

To be solved numerically.

# Even TE Modes

$H_z^0$  &  $E_x^0$  must be continuous across the boundary!

Upper cl.:  $H_z^0(y) = C_u e^{-\alpha(y-d/2)}$

$$H_y^0(y) = -\frac{j\beta}{\alpha} C_u e^{-\alpha(y-d/2)}$$

$$E_x^0(y) = -\frac{j\omega\mu_0}{\alpha} C_u e^{-\alpha(y-d/2)}$$

Core:  $H_z^0(y) = H_e \cos k_y y$

$$H_y^0(y) = \frac{j\beta}{k_y} H_e \sin k_y y$$

$$E_x^0(y) = \frac{j\omega\mu_d}{k_y} H_e \sin k_y y$$

Lower cl.:  $H_z^0(y) = C_l e^{\alpha(y+d/2)}$

$$H_y^0(y) = \frac{j\beta}{\alpha} C_l e^{\alpha(y+d/2)}$$

$$E_x^0(y) = \frac{j\omega\mu_0}{\alpha} C_l e^{\alpha(y+d/2)}$$

$$\rightarrow C_u = H_e \cos \frac{k_y d}{2}$$

$$\rightarrow -\frac{j\omega\mu_0}{\alpha} \left( H_e \cos \frac{k_y d}{2} \right) = \frac{j\omega\mu_d}{k_y} H_e \sin \frac{k_y d}{2}$$

$$\rightarrow C_l = H_e \cos \frac{k_y d}{2}$$

$$\rightarrow \alpha = -\frac{\mu_0}{\mu_d} k_y \cot \frac{k_y d}{2}$$

Recall:  $\alpha^2 = \beta^2 - \omega^2 \mu_0 \epsilon_0$ ,  $\beta^2 = \omega^2 \mu_d \epsilon_d - k_y^2$

$$\rightarrow \alpha = [\omega^2 (\mu_d \epsilon_d - \mu_0 \epsilon_0) - k_y^2]^{1/2}$$

Transcendental equation:

$$[\omega^2 (\mu_d \epsilon_d - \mu_0 \epsilon_0) - k_y^2]^{1/2} = -\frac{\mu_0}{\mu_d} k_y \cot \frac{k_y d}{2}$$

To be solved numerically

# Cutoff Frequencies of TE modes

Guiding condition:

$$k_y^2 = \omega^2 \mu_d \epsilon_d - \beta^2 \rightarrow \beta^2 < \omega^2 \mu_d \epsilon_d$$

$$\alpha^2 = \beta^2 - \omega^2 \mu_0 \epsilon_0 \rightarrow \beta^2 > \omega^2 \mu_0 \epsilon_0$$

$$\rightarrow \omega^2 \mu_0 \epsilon_0 < \beta^2 < \omega^2 \mu_d \epsilon_d$$

Cutoff condition:  $\alpha = 0 \rightarrow k_y = \omega_{co} \sqrt{\mu_d \epsilon_d - \mu_0 \epsilon_0}$

*Odd TE modes:*

$$\alpha = [\omega^2 (\mu_d \epsilon_d - \mu_0 \epsilon_0) - k_y^2]^{1/2}$$

$$= \frac{\mu_0}{\mu_d} k_y \tan \frac{k_y d}{2} = 0$$

$$\rightarrow \pi f_{co} d \sqrt{\mu_d \epsilon_d - \mu_0 \epsilon_0} = (n-1)\pi$$

$$\rightarrow f_{co} = \frac{(n-1)}{d \sqrt{\mu_d \epsilon_d - \mu_0 \epsilon_0}}$$

*Even TE modes:*

$$\alpha = [\omega^2 (\mu_d \epsilon_d - \mu_0 \epsilon_0) - k_y^2]^{1/2}$$

$$= -\frac{\mu_0}{\mu_d} k_y \cot \frac{k_y d}{2} = 0$$

$$\rightarrow \pi f_{co} d \sqrt{\mu_d \epsilon_d - \mu_0 \epsilon_0} = (n - \frac{1}{2})\pi$$

$$\rightarrow f_{co} = \frac{(n - \frac{1}{2})}{d \sqrt{\mu_d \epsilon_d - \mu_0 \epsilon_0}}$$