Electromagnetics:
General Transmission-Line Equations
(9-3)

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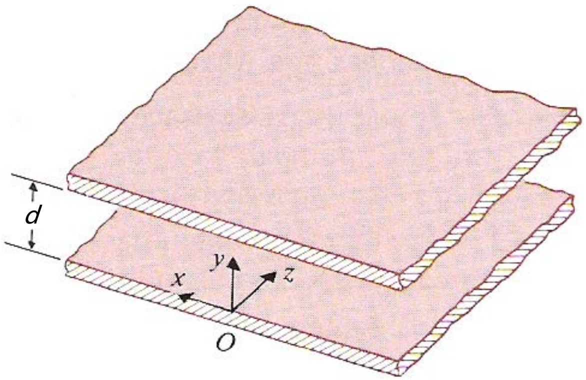
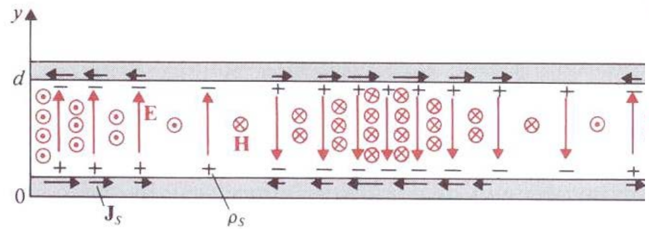
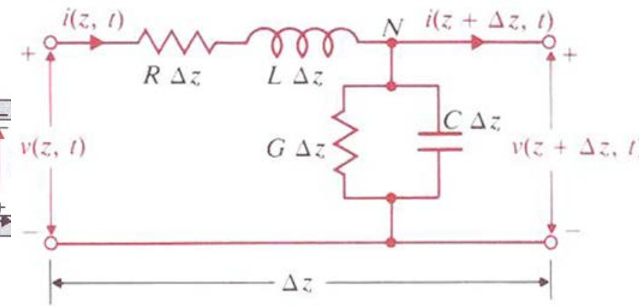
General Transmission Line Equations





→ *Equivalent circuit*



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

$$R = \frac{2}{w}\sqrt{\frac{\pi f \mu_c}{\sigma_c}} \qquad G = \sigma\frac{w}{d} \qquad L = \mu\frac{d}{w} \qquad C = \varepsilon\frac{w}{d}$$

Faraday's law:

→ *Kirchhoff's voltage law*

$$-\frac{\partial v(z,t)}{\partial z} = Ri(z,t) + L\frac{\partial i(z,t)}{\partial t} \quad \rightarrow v(z,t) - R\Delta z \cdot i(z,t) - L\Delta z\frac{\partial i(z,t)}{\partial t} - v(z+\Delta z,t) = 0$$

$$v(z,t) = \mathrm{Re}\left[V(z)e^{j\omega t}\right] \rightarrow -\frac{dV(z)}{dz} = (R + j\omega L)I(z)$$

Ampère's law:

$$-\frac{\partial i(z,t)}{\partial z} = Gv(z,t) + C\frac{\partial v(z,t)}{\partial t}$$

→ *Kirchhoff's current law*

$$\rightarrow i(z,t) - G\Delta z \cdot v(z+\Delta z,t) - C\Delta z\frac{\partial v(z+\Delta z,t)}{\partial t} - i(z+\Delta z,t) = 0$$

$$i(z,t) = \mathrm{Re}\left[I(z)e^{j\omega t}\right] \rightarrow -\frac{dI(z)}{dz} = (G + j\omega C)V(z)$$

Wave Characteristics on an Infinite Transmission Line

Time-harmonic transmission-line equations:

$$-\frac{dV(z)}{dz} = (R + j\omega L)I(z)$$
$$-\frac{dI(z)}{dz} = (G + j\omega C)V(z)$$

$\longrightarrow$

$$\frac{d^2V(z)}{dz^2} = \gamma^2 V(z)$$
$$\frac{d^2I(z)}{dz^2} = \gamma^2 I(z)$$

$$\leftarrow \gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$$

Propagation const.
(Attenuation const. & phase const.)

General solutions:

$$V(z) = V^+(z) + V^-(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

$$I(z) = I^+(z) + I^-(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z}$$

Characteristic impedance:

$$Z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \frac{R + j\omega L}{\gamma} = \frac{\gamma}{G + j\omega C} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Line Characteristics (1)

Lossless line: $\rightarrow R = 0, G = 0$

Propagation constant: $\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} = j\omega\sqrt{LC}$

Phase velocity: $u_p = \dfrac{\omega}{\beta} = \dfrac{1}{\sqrt{LC}}$

Characteristic impedance: $Z_0 = R_0 + jX_0 = \sqrt{\dfrac{R + j\omega L}{G + j\omega C}} = \sqrt{\dfrac{L}{C}}$

Low-loss line: $\rightarrow R << \omega L, G << \omega C$

Propagation constant: $\gamma = \alpha + j\beta = j\omega\sqrt{LC}\left(1 + \dfrac{R}{j\omega L}\right)^{1/2}\left(1 + \dfrac{G}{j\omega C}\right)^{1/2}$

$$\cong \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right) + j\omega\sqrt{LC}$$

Phase velocity: $u_p = \dfrac{\omega}{\beta} \cong \dfrac{1}{\sqrt{LC}}$

Characteristic impedance: $Z_0 = R_0 + jX_0 = \sqrt{\dfrac{L}{C}}\left(1 + \dfrac{R}{j\omega L}\right)^{1/2}\left(1 + \dfrac{G}{j\omega C}\right)^{-1/2}$

$$\cong \sqrt{\frac{L}{C}}\left[1 - \frac{j}{2\omega}\left(\frac{R}{L} - \frac{G}{C}\right)\right]$$

Distortionless line: $\dfrac{R}{L} = \dfrac{G}{C} \rightarrow \gamma = \sqrt{\dfrac{C}{L}}(R + j\omega L) \rightarrow u_p = \dfrac{1}{\sqrt{LC}}$

Transmission-Line Parameters (1)

Propagation constant: $\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$

For low loss conductors: $R << \omega L \;\; \rightarrow \gamma = j\omega\sqrt{LC}\left(1 + \frac{G}{j\omega C}\right)^{1/2}$

Propagation constant for a TEM wave in a lossy dielectric medium:

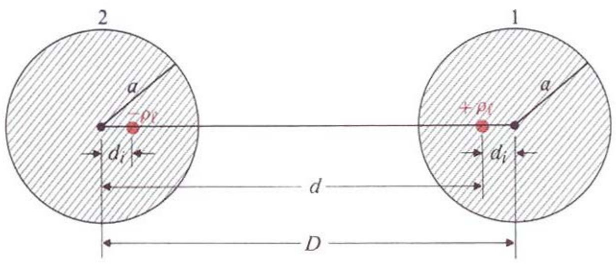
$$\gamma = j\omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)^{1/2}$$

$$\rightarrow \frac{G}{C} = \frac{\sigma}{\varepsilon}$$

$$\rightarrow LC = \mu\varepsilon$$

Useful relations to determine transmission-line parameters if any single of those is known!

Transmission-Line Parameters (2)



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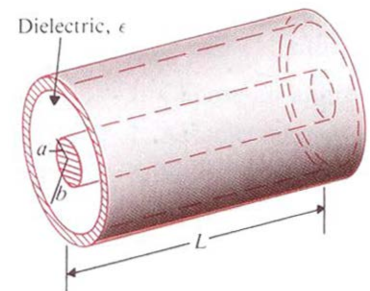

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Two-wire transmission line:

$$C = \frac{\pi\varepsilon}{\cosh^{-1}(D/2a)}$$ *(See Eq. 4-47)*

$$\rightarrow L = \frac{\mu\varepsilon}{C} = \frac{\mu}{\pi}\cosh^{-1}\left(\frac{D}{2a}\right)$$

$$\rightarrow G = \frac{\sigma}{\varepsilon}C = \frac{\pi\sigma}{\cosh^{-1}(D/2a)}$$

$$R = 2\frac{R_s}{w_a} = \frac{1}{\pi a}\sqrt{\frac{\pi f\mu_c}{\sigma_c}}$$

Coaxial transmission line:

$$C = \frac{2\pi\varepsilon}{\ln(b/a)}$$ *(See Eq. 3-139)*

$$\rightarrow L = \frac{\mu}{2\pi}\ln\frac{b}{a}$$

$$\rightarrow G = \frac{2\pi\sigma}{\ln(b/a)}$$

$$R = R_s\left(\frac{1}{w_a} + \frac{1}{w_b}\right) = \frac{1}{2\pi}\sqrt{\frac{\pi f\mu_c}{\sigma_c}}\left(\frac{1}{a} + \frac{1}{b}\right)$$

Attenuation Constant from Power Relations

Propagation constant: $\gamma = \alpha + j\beta$

$$= \sqrt{(R + j\omega L)(G + j\omega C)}$$

Attenuation constant: $\alpha = \mathrm{Re}[\sqrt{(R + j\omega L)(G + j\omega C)}]$

Time-averaged power loss:

$$V(z) = V_0 e^{-(\alpha + j\beta)z}$$

$$I(z) = I_0 e^{-(\alpha + j\beta)z} = \frac{V_0}{Z_0} e^{-(\alpha + j\beta)z} \quad \leftarrow Z_0 = R_0 + jX_0$$

$$\rightarrow P(z) = \frac{1}{2}\mathrm{Re}\left[V(z)I^*(z)\right] = \frac{V_0^2}{2|Z_0|^2} R_0 e^{-2\alpha z}$$

Loss by conducting walls

Loss by dielectric

$$\rightarrow -\frac{\partial P(z)}{\partial z} = P_L(z) = \frac{1}{2}\left[|I(z)|^2 R + |V(z)|^2 G\right] = \frac{V_0^2}{2|Z_0|^2}(R + G|Z_0|^2)e^{-2\alpha z}$$

$$= 2\alpha P(z)$$

Time-averaged power loss per unit length

$$\rightarrow \alpha = \frac{P_L(z)}{2P(z)} = \frac{1}{2R_0}(R + G|Z_0|^2)$$

For a low-loss line: $Z_0 \cong R_0 = \sqrt{L/C}$

$$\rightarrow \alpha \cong \frac{1}{2R_0}\left(R + GR_0^2\right) = \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right)$$