

Electromagnetics:

The Smith Chart

(9-6)

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A Conformal Mapping (1)

Mapping between complex-valued variables:

$$\Gamma = f(z) \quad \leftarrow f : \text{Mapping function}$$

$$\rightarrow z = f^{-1}(\Gamma)$$

Suppose a mapping function: $f(z) = \frac{z-1}{z+1}$

$$z = z_r + jz_i = r + jx$$

$$\Gamma = \Gamma_r + j\Gamma_i = |\Gamma|e^{j\phi}$$

Cartesian coordinates

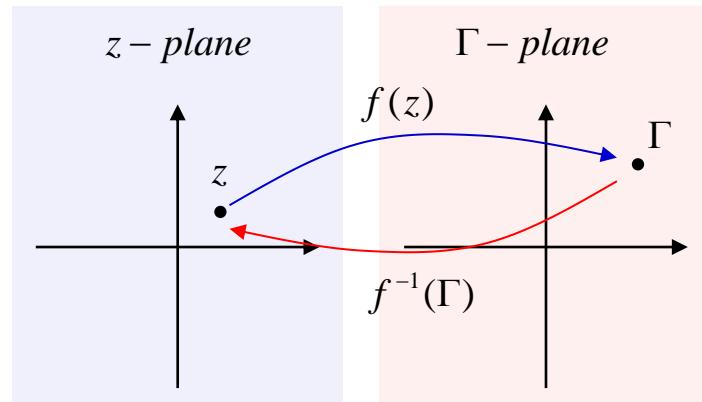
Polar coordinates

$$\rightarrow \Gamma = \frac{z-1}{z+1} \quad \rightarrow z = \frac{1+\Gamma}{1-\Gamma}$$

Consider: $z = f^{-1}(\Gamma) \rightarrow z = \frac{1+\Gamma}{1-\Gamma} \rightarrow r + jx = \frac{1+\Gamma_r + j\Gamma_i}{1-\Gamma_r - j\Gamma_i}$

$$\rightarrow r = \frac{1-\Gamma_r^2 - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2} \quad \rightarrow \left(\Gamma_r - \frac{r}{1+r} \right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r} \right)^2 \quad \leftarrow r\text{-circle}$$

$$\rightarrow x = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2} \quad \rightarrow \left(\Gamma_r - 1 \right)^2 + \left(\Gamma_i - \frac{1}{x} \right)^2 = \left(\frac{1}{x} \right)^2 \quad \leftarrow x\text{-circle}$$



A Conformal Mapping (2)

r-circle: $\left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r}\right)^2$

x-circle: $(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2$

Transformation: $r + jx \rightarrow \Gamma_r + j\Gamma_i$
 e.g. $r + jx = 1 + j0.5$

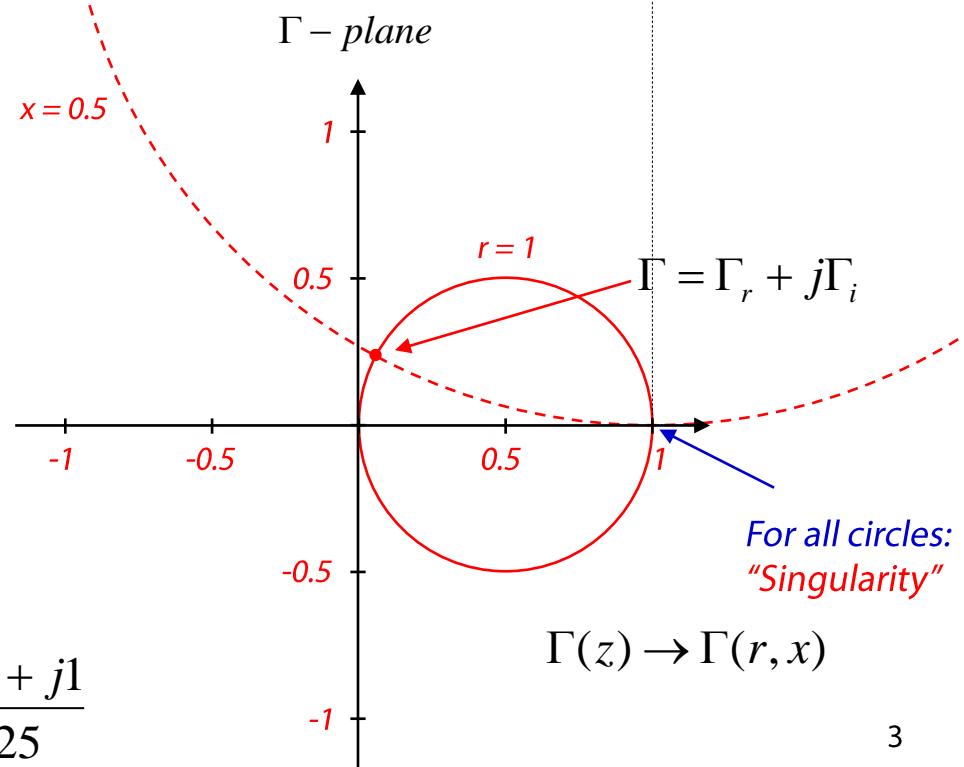
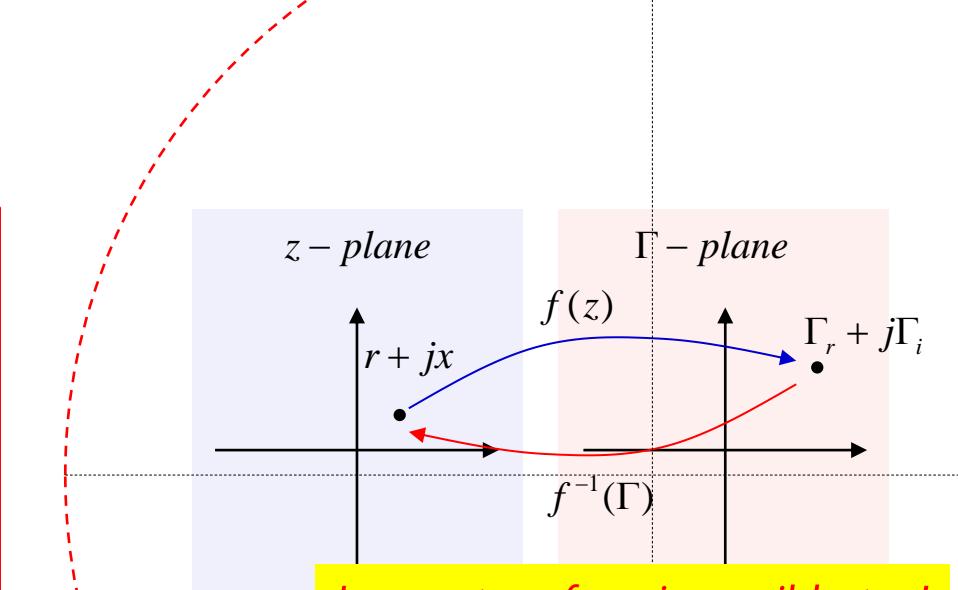
r-circle:

$$(\Gamma_r - 0.5)^2 + \Gamma_i^2 = 0.5^2$$

x-circle:

$$(\Gamma_r - 1)^2 + (\Gamma_i - 2)^2 = 2^2$$

Recall: $\Gamma = \frac{z-1}{z+1} = \frac{1+j0.5-1}{1+j0.5+1} = \frac{0.25+j1}{4.25}$



The Smith Chart: A Conformal Mapping (1)

Voltage reflection coefficient at the load for a lossless transmission line:

$$\Gamma = \frac{Z_L - R_0}{Z_L + R_0} = \Gamma_r + j\Gamma_i = |\Gamma| e^{j\theta_\Gamma}$$

Polar coordinates

Cartesian coordinates

Normalized load impedance:

$$z_L = \frac{Z_L}{R_0} = \frac{R_L}{R_0} + j \frac{X_L}{R_0} = r + jx$$

$$\rightarrow \Gamma = \frac{Z_L - R_0}{Z_L + R_0} = \frac{z_L - 1}{z_L + 1}$$

$$\rightarrow z_L = \frac{1 + \Gamma}{1 - \Gamma} = \frac{1 + |\Gamma| e^{j\theta_\Gamma}}{1 - |\Gamma| e^{j\theta_\Gamma}}$$

$$\rightarrow r + jx = \frac{(1 + \Gamma_r) + j\Gamma_i}{(1 - \Gamma_r) - j\Gamma_i}$$

$$\rightarrow r = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1 - \Gamma_r)^2 + \Gamma_i^2}$$

$$\rightarrow x = \frac{2\Gamma_i}{(1 - \Gamma_r)^2 + \Gamma_i^2}$$

$$\textcolor{red}{r\text{-circle:}} \quad \left(\Gamma_r - \frac{r}{1+r} \right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r} \right)^2$$

$$\textcolor{red}{x\text{-circle:}} \quad (\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x} \right)^2 = \left(\frac{1}{x} \right)^2$$

The Smith Chart: A Conformal Mapping (2)

r-circle: $\left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r}\right)^2$

x-circle: $(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2$

e.g.

1. Let us assume that the voltage reflection coefficient is given by: $\Gamma_1 = \Gamma_{1r} + j\Gamma_{1i}$

2. Find *r*- and *x*-circles that pass through Γ_1

3. Read the *r*- and *x*-values for the circles:

$$\rightarrow r = 1, x = 0.5$$

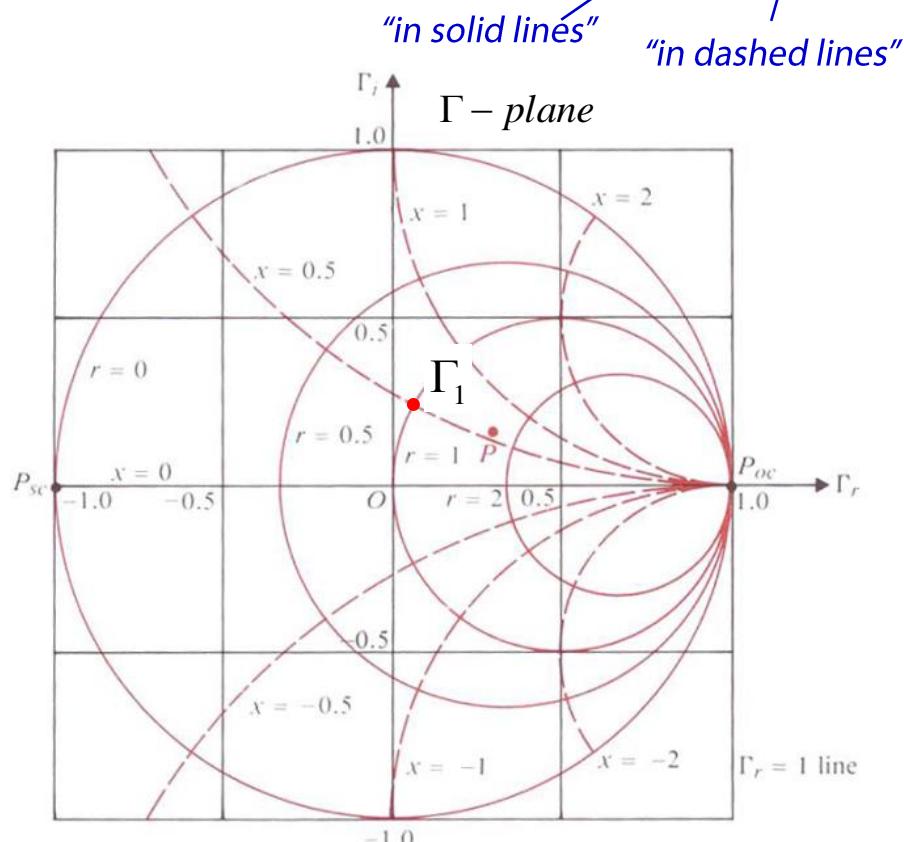
In result, when the voltage reflection coefficient is Γ_1 , the normalized load impedance is given by: $z_L = r + jx = 1 + j0.5$

This can also be verified:

$$\Gamma_1 = \frac{z_L - 1}{z_L + 1} = \frac{1 + j0.5 - 1}{1 + j0.5 + 1} = \frac{0.25 + j1}{4.25}$$

$$\rightarrow |\Gamma|^2 \leq 1 \quad \leftarrow \text{Reflection coefficient}$$

In the Smith chart, there are many circles already drawn for various values of "r" and "x"!



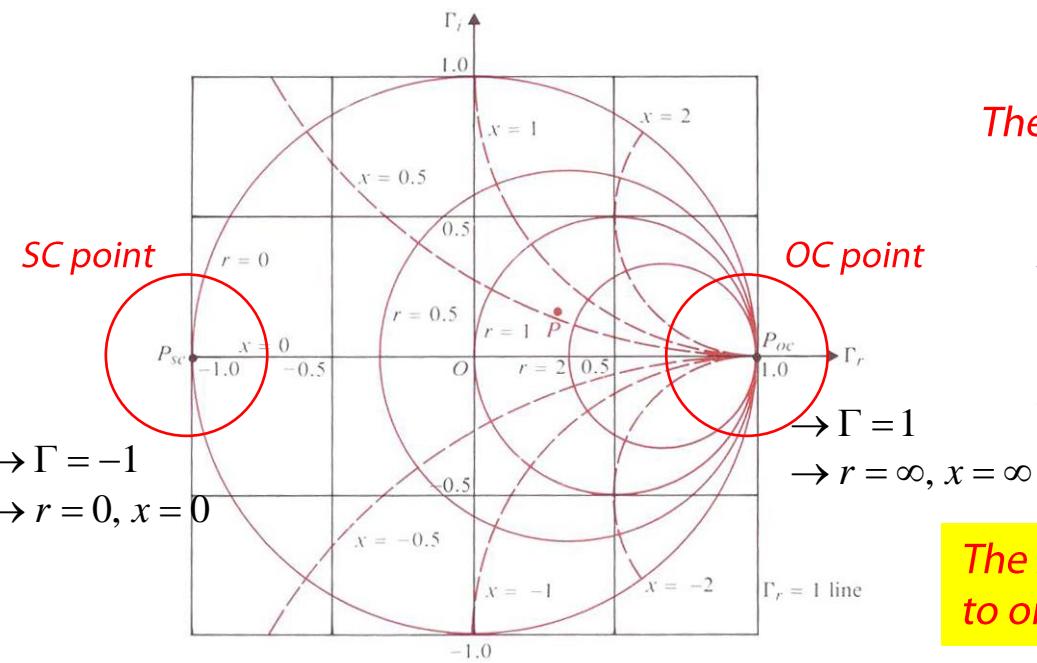
The Smith Chart: A Conformal Mapping (3)

$$\Gamma = \frac{z_L - 1}{z_L + 1}$$

$$z_L = \frac{1 + \Gamma}{1 - \Gamma}$$

$$\left(\Gamma_r - \frac{r}{1+r} \right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r} \right)^2$$

$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x} \right)^2 = \left(\frac{1}{x} \right)^2$$



The r -circles:

1. Centred at $(\Gamma_r = r/(r+1), \Gamma_i = 0)$: On the Γ_r -axis
2. $r = 0$: Centred at the origin with a unity radius
3. $r \rightarrow \infty$: Centred at $(\Gamma_r = 1, \Gamma_i = 0)$
4. All r -circles: Passing through $(\Gamma_r = 1, \Gamma_i = 0)$.

The x -circles:

1. Centred at $(\Gamma_r = 1, \Gamma_i = 1/x)$: On the $\Gamma_r = 1$ line
2. $x = 0$: $\Gamma_i = 0$ line, i.e., Γ_r -axis
3. $x \rightarrow \infty$: Centred at $(\Gamma_r = 1, \Gamma_i = 0)$
4. All x -circles: Passing through $(\Gamma_r = 1, \Gamma_i = 0)$.

The r - and x -circles are everywhere orthogonal to one another!

Smith Chart with Polar Coordinates (1)

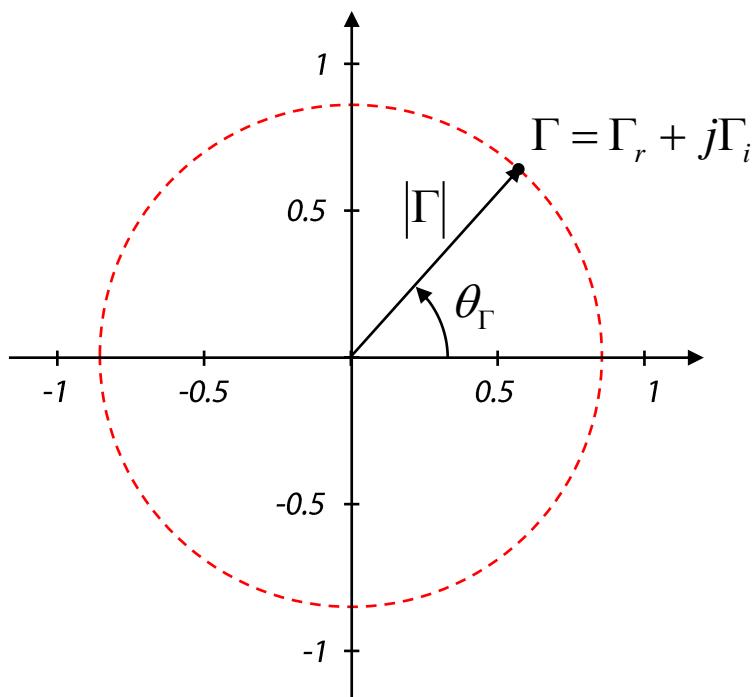
Voltage reflection coefficient at the load for a lossless transmission line:

$$\Gamma = \frac{Z_L - R_0}{Z_L + R_0} = \frac{z_L - 1}{z_L + 1} = \Gamma_r + j\Gamma_i = |\Gamma| e^{j\theta_\Gamma}$$

Polar coordinates

Cartesian coordinates

Γ -plane



Properties to note:

1. All $|\Gamma|$ -circles: Radii from 0 to 1
2. θ_Γ : Measured from the positive real axis

Smith Chart with Polar Coordinates (2)

Input impedance:

$$Z_i(z') = \frac{V(z')}{I(z')} = Z_0 \left[\frac{1 + \Gamma e^{-j2\beta z'}}{1 - \Gamma e^{-j2\beta z'}} \right]$$

Normalized "input" impedance:

$$z_i = \frac{Z_i}{Z_0} = \frac{1 + \Gamma e^{-j2\beta z'}}{1 - \Gamma e^{-j2\beta z'}} = \frac{1 + |\Gamma| e^{j\phi}}{1 - |\Gamma| e^{j\phi}}$$

$$\leftarrow \phi = \theta_\Gamma - 2\beta z'$$

Normalized "load" impedance:

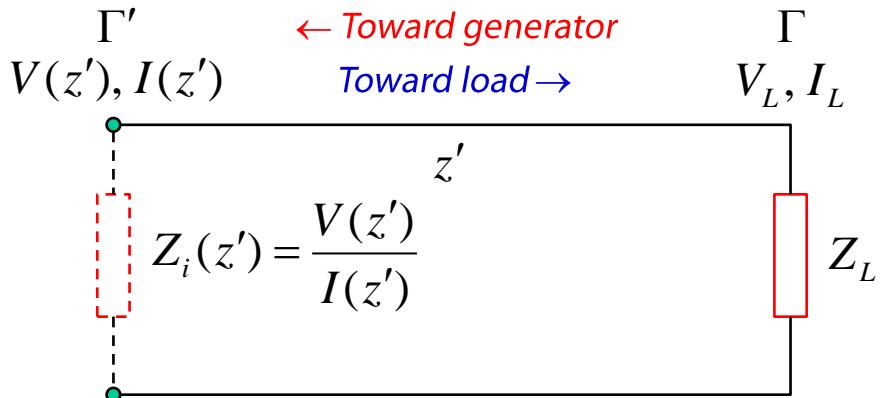
$$\rightarrow z_L = \frac{1 + \Gamma}{1 - \Gamma} = \frac{1 + |\Gamma| e^{j\theta_\Gamma}}{1 - |\Gamma| e^{j\theta_\Gamma}}$$

Transformation: $\Gamma(z_L) \rightarrow \Gamma'(z_i)$

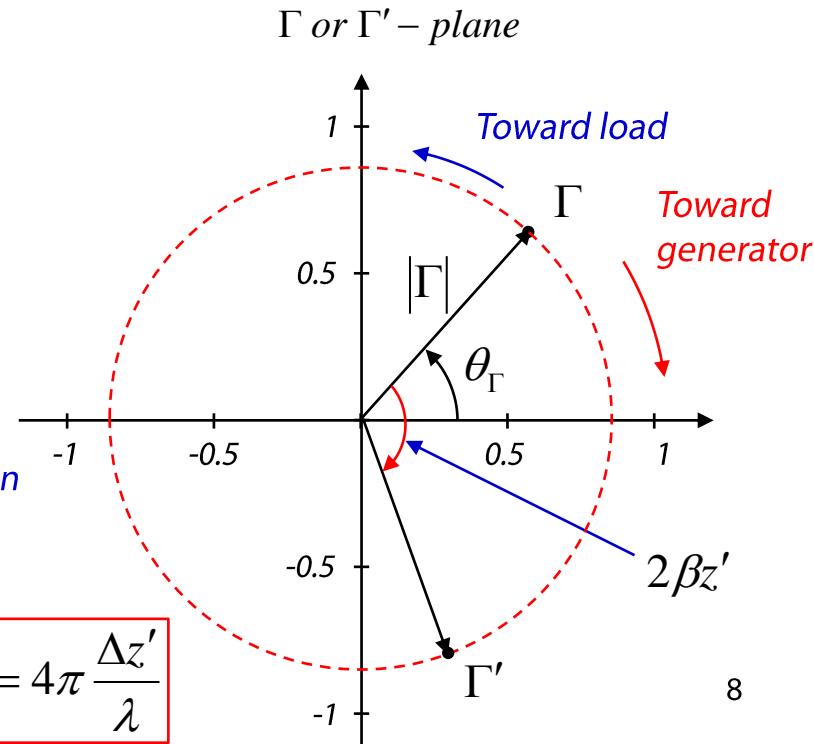
$$z_i = \frac{1 + |\Gamma| e^{j\phi}}{1 - |\Gamma| e^{j\phi}} = \frac{1 + \Gamma'}{1 - \Gamma'} \quad \text{"Effective" reflection coefficient at } z'$$

$$\leftarrow |\Gamma'| = |\Gamma|$$

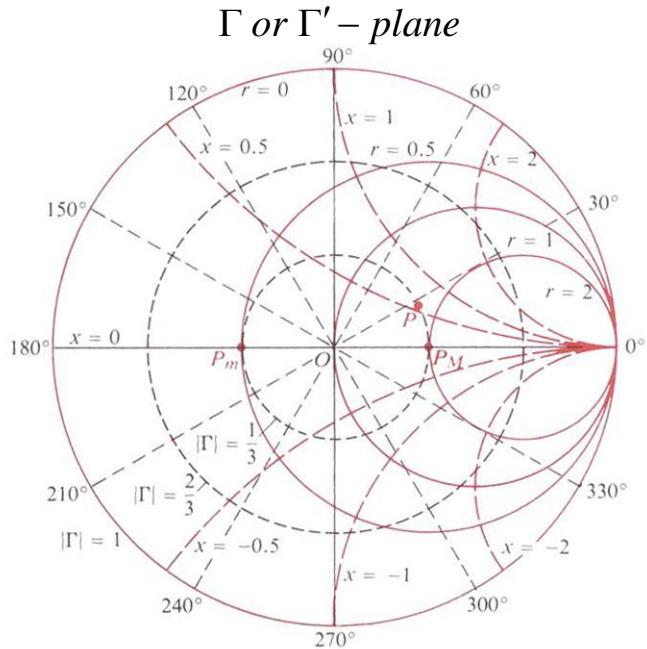
$$\phi = \theta_\Gamma - 2\beta z' \rightarrow \Delta\phi = 2\beta\Delta z' = 4\pi \frac{\Delta z'}{\lambda}$$



Of exactly same form!!



Smith Chart with Polar Coordinates (3)



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Recall:

Normalized load impedance:

$$z_L = \frac{1+\Gamma}{1-\Gamma} = \frac{1+|\Gamma|e^{j\theta_\Gamma}}{1-|\Gamma|e^{j\theta_\Gamma}}$$

Normalized input impedance:

$$z_i = \frac{1+\Gamma'}{1-\Gamma'} = \frac{1+|\Gamma|e^{j\phi}}{1-|\Gamma|e^{j\phi}} \quad \leftarrow \phi = \theta_\Gamma - 2\beta z$$

Additional properties to note:

1. P_M : The r -circle value $\rightarrow S$ (SWR)

2. P_m : The r -circle value $\rightarrow 1/S$

$$\boxed{S = \left| \frac{V_{\max}}{V_{\min}} \right| = \left| \frac{I_{\max}}{I_{\min}} \right| = \frac{1+|\Gamma|}{1-|\Gamma|} \rightarrow |\Gamma| = \frac{S-1}{S+1}$$

$$\rightarrow z_i = \frac{1+|\Gamma|e^{j\phi}}{1-|\Gamma|e^{j\phi}} \rightarrow r_M = \frac{1+|\Gamma|}{1-|\Gamma|} \rightarrow r_m = \frac{1-|\Gamma|}{1+|\Gamma|}}$$

Example 9-13:

Given:

$$R_0 = 50 \text{ } (\Omega)$$

$$z_L = 0$$

$$z' = 0.1\lambda$$

Solution:

1. Point P_{sc} : $r = x = 0$

2. Azimuthal shift in the CWD by 0.1 from P_{sc} to P_1

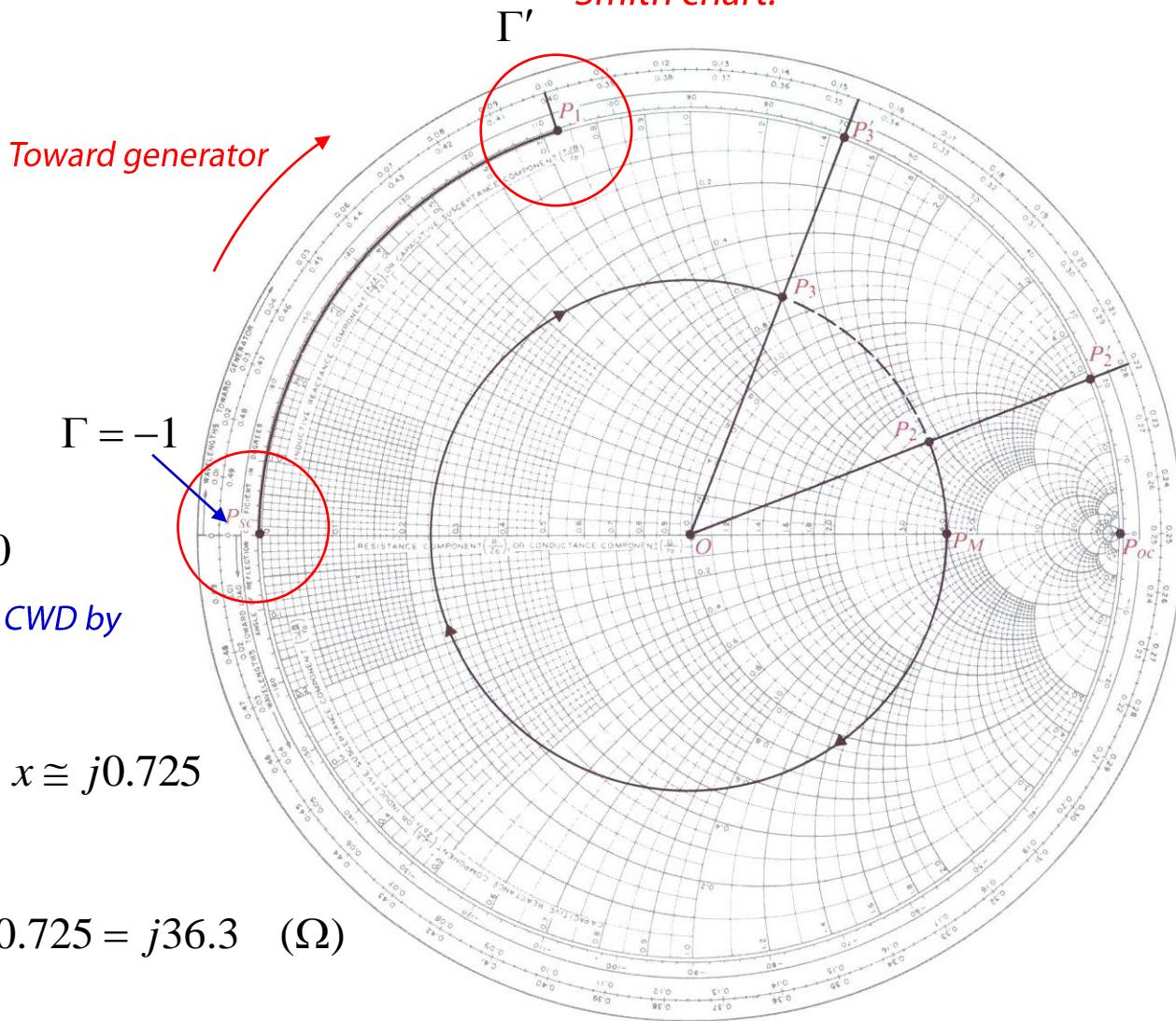
3. At P_1 , read r & x

$$r = 0, \quad x \cong j0.725$$

In result:

$$Z_i = R_0 z_i = 50 \times j0.725 = j36.3 \text{ } (\Omega)$$

Smith chart:

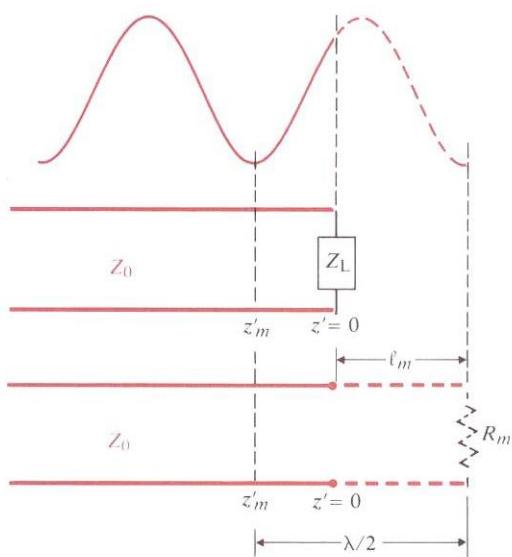


D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Example 9-15:

Smith chart:

Recall:



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Given:

$$R_0 = 50 \text{ } (\Omega)$$

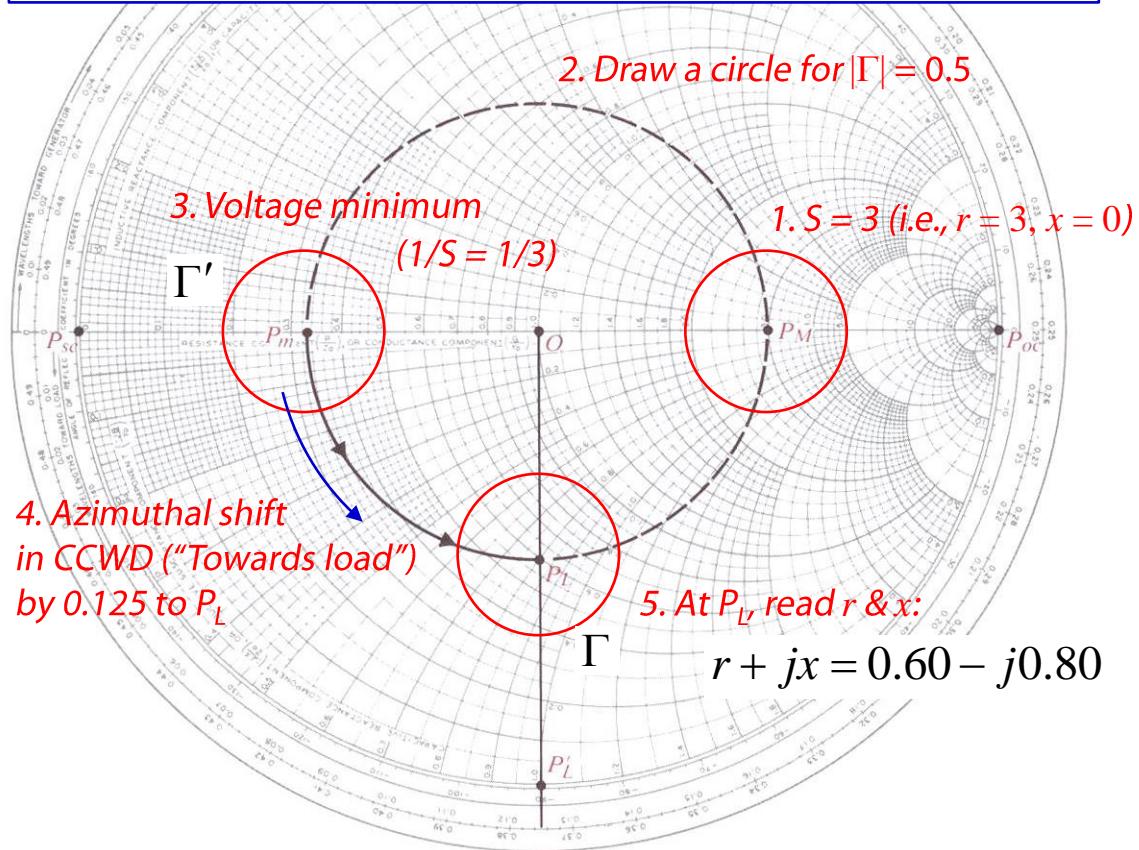
$$S = 3.0$$

$$\lambda = 2 \times 0.2 = 0.4 \text{ } (m)$$

$$z'_m = 0.05 \text{ } (m)$$

(First voltage minimum)

$$S = \left| \frac{V_{\max}}{V_{\min}} \right| = \left| \frac{I_{\max}}{I_{\min}} \right| = \frac{1 + |\Gamma|}{1 - |\Gamma|} \rightarrow |\Gamma| = \frac{S-1}{S+1} \rightarrow z_i = \frac{1 + |\Gamma| e^{j\phi}}{1 - |\Gamma| e^{j\phi}}$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Note: $z'_m / \lambda = 0.05 / 0.4 = 0.125$

$$Z_L = 50(0.60 - j0.80) = 30 - j40 \text{ } (\Omega)$$

$$\Gamma = |\Gamma| e^{-j\pi/2} = -j0.5$$

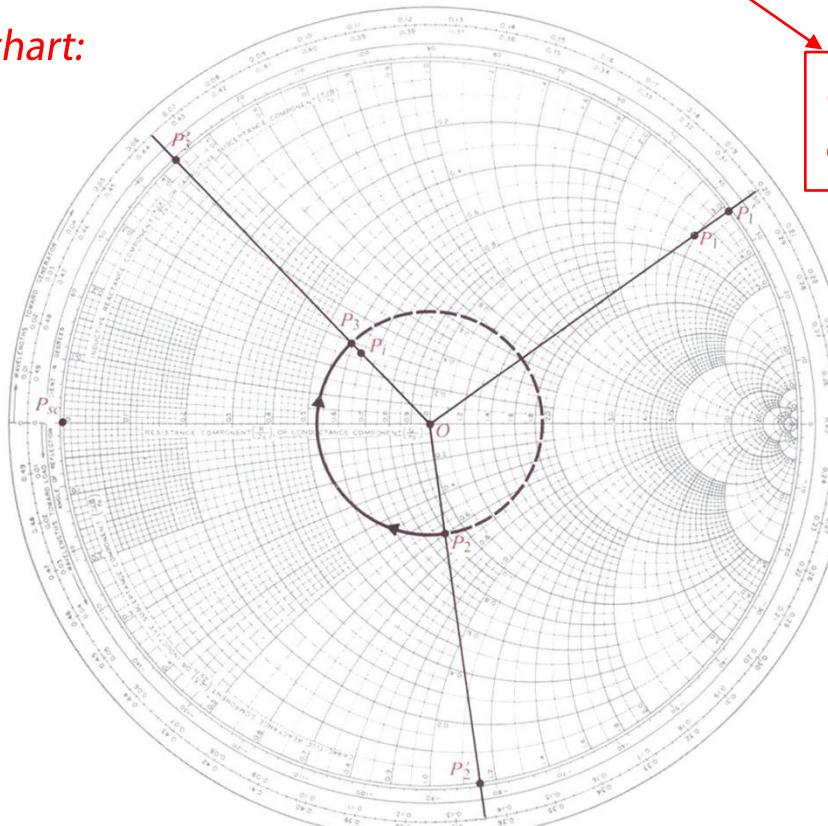
Smith-Chart Calculations for Lossy Lines

Normalized input impedance for a lossy line:

$$z_i = \frac{1 + \Gamma e^{-2\alpha z'} e^{-j2\beta z'}}{1 - \Gamma e^{-2\alpha z'} e^{-j2\beta z'}} = \frac{1 + |\Gamma| e^{-2\alpha z} e^{j\phi}}{1 - |\Gamma| e^{-2\alpha z} e^{j\phi}}$$

$\leftarrow \phi = \theta_\Gamma - 2\beta z'$

Smith chart:



Consider shrinkage
due to attenuation: $e^{-2\alpha z'}$

$$|\Gamma'| = |\Gamma| e^{-2\alpha z'}$$