Electromagnetics:
Brief Discussion on the Theory of Relativity

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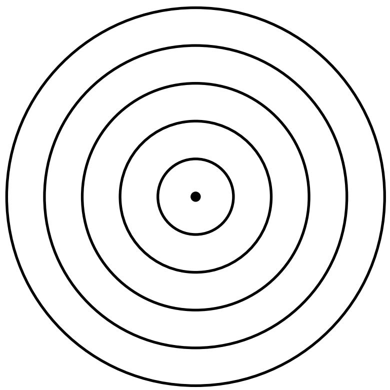
Doppler Effect

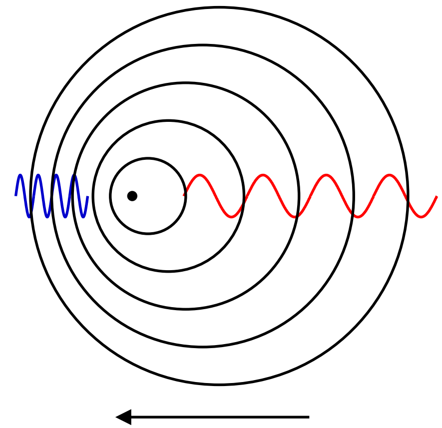
Stationary source:



$v = 0$

Moving source:

Blue-shift



Red-shift

$v \neq 0$

Recall: Speed of light

$$c = \frac{1}{\sqrt{\mu_o \varepsilon_o}} = 2.997930 \times 10^8 \text{ m/s}$$

← Is this frequency dependent?

→ *Relative to ether?*

Incoming wave (blue-shifted): $c'_{in} = c + v$

Outgoing wave (red shifted): $c'_{out} = c - v$

} Is this true?

Special & general relativity presented by A. Einstein!

The Situation Before 1900

Electromagnetic waves:

$$\nabla^2 \mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \;\; \nabla^2 \mathbf{H} - \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$$

(Homogeneous and no source)

$$\psi = e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}, \quad |\mathbf{k}| = \omega\sqrt{\mu\varepsilon}, \quad c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.997930 \times 10^8 \;\; \text{(m/s)}$$

Ether hypothesis:

- A hypothetical medium for light to propagate through
- Permeated all space
- Negligible density
- Negligible interaction with matter

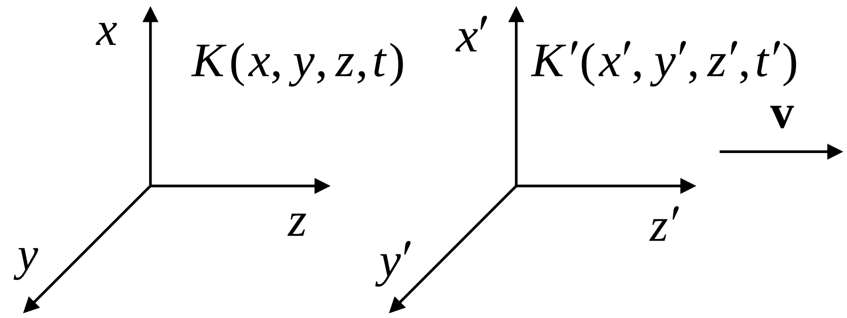
Invariance of physical laws:

- The laws of mechanics are the same in different coordinate systems moving uniformly relative to one another: The laws of mechanics are invariant under Galilean transformation.
- *The hypothesis of an ether set electromagnetic phenomena apart from the rest of physics.*

Galilean Relativity

Galilean transform:

$$\mathbf{x}' = \mathbf{x} - \mathbf{v}t$$
$$t' = t$$



Classical mechanics:

$$m_i \frac{d\mathbf{v}'_i}{dt'} = -\nabla'_i \sum_j V_{ij}(|\mathbf{x}'_i - \mathbf{x}'_j|) \leftarrow \begin{cases} \mathbf{v}'_i = \mathbf{v}_i - \mathbf{v} \\ \nabla'_i = \nabla_i \\ d\mathbf{v}'_i / dt' = d\mathbf{v}_i / dt \\ \mathbf{x}'_i - \mathbf{x}'_j = \mathbf{x}_i - \mathbf{x}_j \end{cases}$$

$$\rightarrow m_i \frac{d\mathbf{v}_i}{dt} = -\nabla_i \sum_j V_{ij}(|\mathbf{x}_i - \mathbf{x}_j|)$$

$\rightarrow$ Invariant under Galilean transformations

Electromagnetics waves:

$$\left(\sum_i \frac{\partial^2}{\partial x_i'^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t'^2} \right) \psi = 0 \quad \leftarrow \quad \begin{cases} \nabla'^2 = \nabla^2 \\ \dfrac{\partial}{\partial t'} = \mathbf{v} \cdot \nabla + \dfrac{\partial}{\partial t} \end{cases}$$

$$\rightarrow \left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{2}{c^2} \mathbf{v} \cdot \nabla \frac{\partial}{\partial t} - \frac{1}{c^2} \mathbf{v} \cdot \nabla \mathbf{v} \cdot \nabla \right) \psi = 0$$

$\rightarrow$ Not invariant under Galilean transformations $\rightarrow$ Why?

Possibilities that Einstein Began to Think About:

1. The Maxwell equations were incorrect. The proper theory of electromagnetism was invariant under Galilean transformation.

2. Galilean relativity applied to classical mechanics, but electromagnetism had a preferred reference frame, the frame in which the luminiferous ether was at rest.

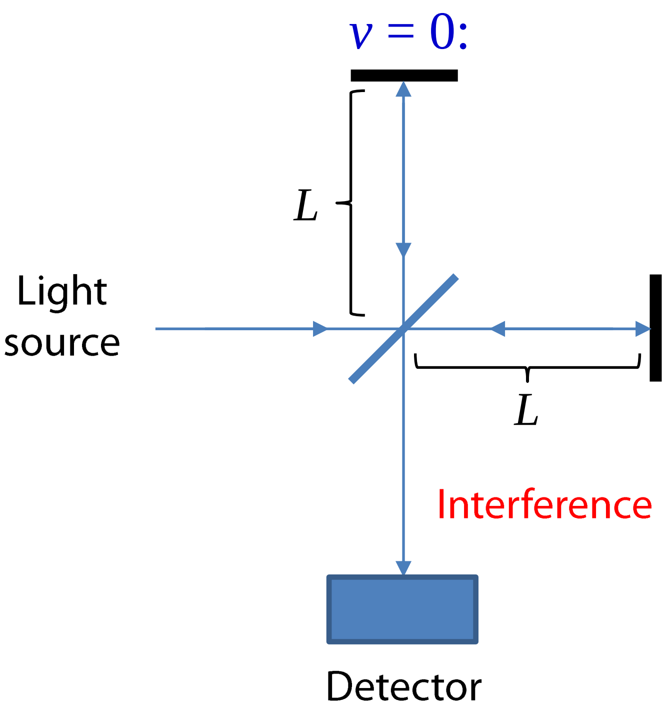
3. There existed a relativity principle for both classical mechanics and electromagnetism, but it was not Galilean relativity. This would imply that the laws of mechanics were in need of modification.

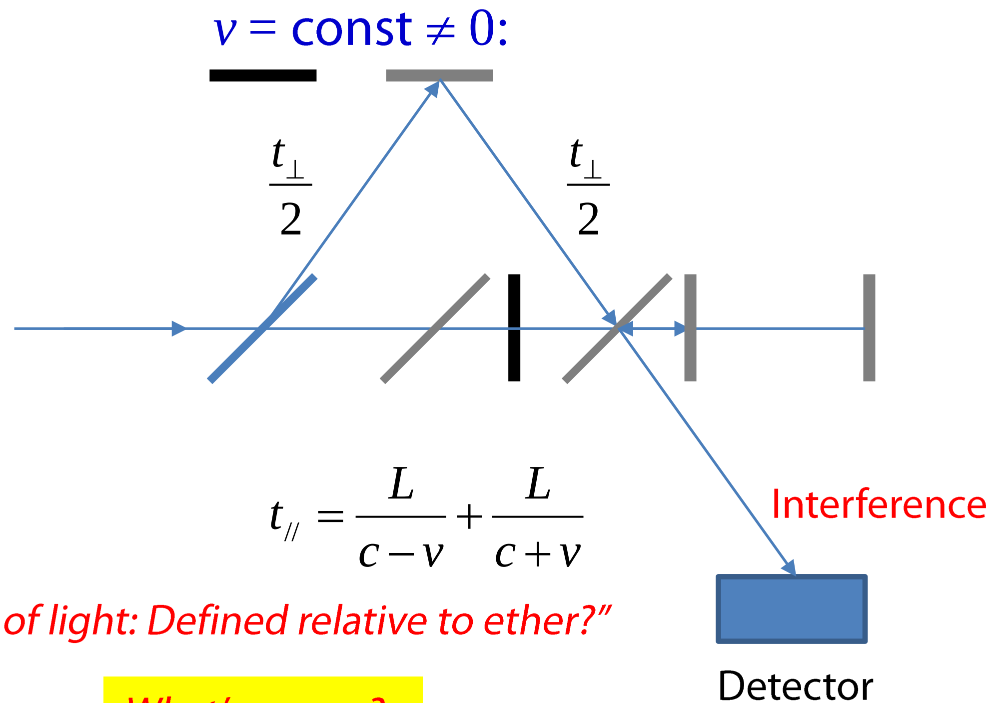
The 2nd alternative was accepted by the time, however:

1. Efforts to observe motion of the earth and its laboratories relative to the rest frame of the ether, e.g., the Michelson-Morley experiment (1887), had failed.
2. FitzGerald-Lorentz contraction hypothesis (1892): $L(v) = L_0\sqrt{1-\frac{v^2}{c^2}}$
3. Lorentz transformations (1887 - 1905): Maxwell's eqs. invariant

→ Special theory of relativity by Einstein in 1905

Michelson-Morley Experiment





$$t_{//} = \frac{L}{c-v} + \frac{L}{c+v}$$

"Speed of light: Defined relative to ether?"

What's wrong?

Round trip time for the vertical arm:

$$t_\perp = \frac{2}{c}\sqrt{L^2 + \left(v\frac{t_\perp}{2}\right)^2} = \frac{2L}{c}\frac{1}{\sqrt{1-v^2/c^2}}$$

Round trip time for the horizontal arm:

$$t_{//} = \frac{L}{c-v} + \frac{L}{c+v} = \frac{2L}{c}\frac{1}{1-v^2/c^2}$$

$c = 3\times10^8$ (m/s)

$v_{earth} = 3\times10^4$ (m/s)

Alteration in the interference while rotating the MMI?

→ Null result!

Einstein's Two Postulates

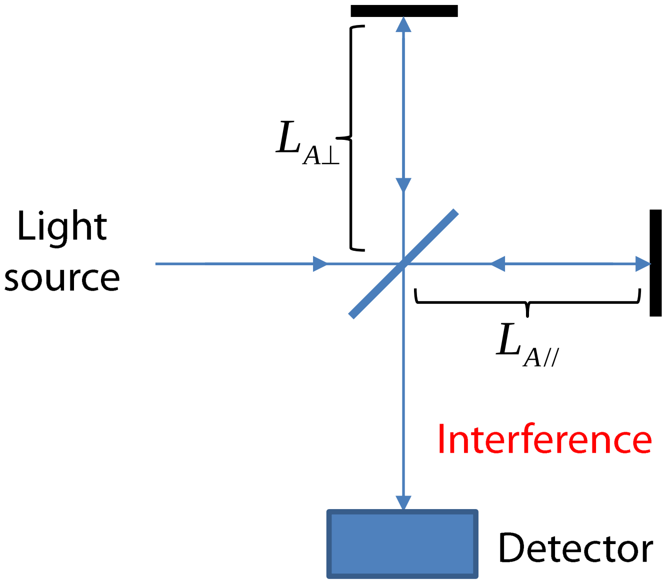
1. Postulate of relativity:
   The laws of nature and the results of all experiments performed in a given frame of reference are independent of the translational motion of the system as a whole. ← Inertial reference frames

2. Postulate of the constancy of the speed of light:
   The speed of light is finite and independent of the motion of its source.

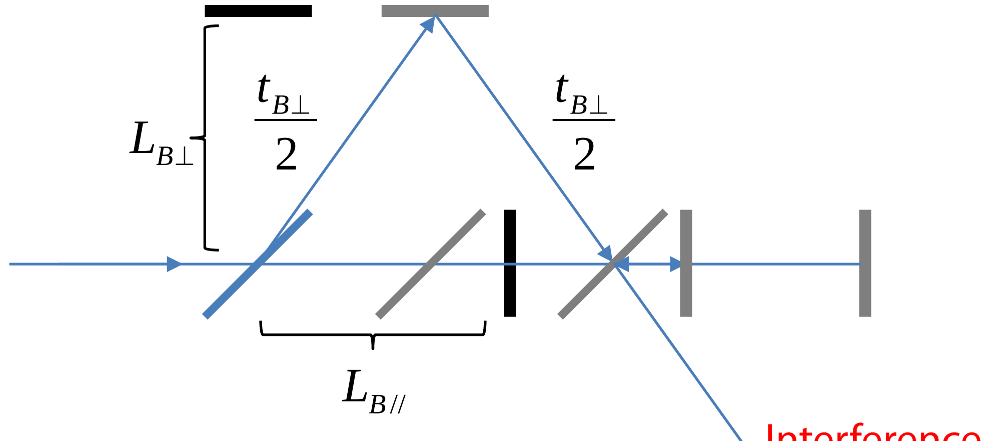
2′. Postulate of a universal limiting speed:
   In every inertial frame, there is a finite universal limiting speed C for physical entities. ← Experimentally, the limiting speed C is equal to the speed c of light in vacuum.

Michelson-Morley Experiment Explained by STR

Observed in the moving frame: K'



Observed in the rest frame: K



$$t_{B//} = \frac{L_{B//}}{c-v} + \frac{L_{B//}}{c+v}$$

For the vertical arm:

$$\rightarrow t_{A\perp} = \frac{2L_{A\perp}}{c} \rightarrow t_{B\perp} = \frac{2}{c}\sqrt{L_{B\perp}^2 + \left(v\frac{t_{B\perp}}{2}\right)^2} = \frac{2L_{B\perp}}{c}\frac{1}{\sqrt{1-v^2/c^2}} = \frac{2L_{B\perp}}{c}\frac{1}{\sqrt{1-\beta^2}} = \frac{2L_{B\perp}}{c}\gamma$$

$\leftarrow \gamma \geq 1$

$$\rightarrow L_{A\perp} = L_{B\perp} \rightarrow t_{A\perp} = \frac{t_{B\perp}}{\gamma}$$ ← *"Time dilation"*

For the horizontal arm:

$$\rightarrow t_{B//} = \frac{L_{B//}}{c-v} + \frac{L_{B//}}{c+v} = \frac{2L_{B//}}{c}\frac{1}{1-v^2/c^2} = \frac{2L_{B//}}{c}\gamma^2 = \gamma t_{A//} = \gamma\frac{2L_{A//}}{c}$$

$$\rightarrow L_{B//} = \frac{L_{A//}}{\gamma}$$ ← *"Length contraction"*

Lorentz Transformation of Coordinates

Lorentz transformation:

$$c^2t^2 - (x^2 + y^2 + z^2) = c^2t'^2 - (x'^2 + y'^2 + z'^2)$$

$$\left.\begin{aligned} x'_0 &= \gamma(x_0 - \beta x_1) \\ x'_1 &= \gamma(x_1 - \beta x_0) \\ x'_2 &= x_2 \\ x'_3 &= x_3 \end{aligned}\right\} \leftarrow \; x_0 = ct,\;\; x_1 = z,\;\; x_2 = x,\;\; x_3 = y,$$

$$\boldsymbol{\beta} = \frac{\mathbf{v}}{c},\;\; \beta = |\boldsymbol{\beta}|,\;\; \gamma = (1-\beta^2)^{-1/2}$$

Inverse Lorentz transformation:

$$\left.\begin{aligned} x_0 &= \gamma(x'_0 + \beta x'_1) \\ x_1 &= \gamma(x'_1 + \beta x'_0) \\ x_2 &= x'_2 \\ x_3 &= x'_3 \end{aligned}\right\}$$

$\leftarrow$ Einstein's 1st postulate

For the velocity $\mathbf{v}$ in arbitrary direction:

Separate treatment for the parallel and perpendicular components to $\mathbf{v}$*!*

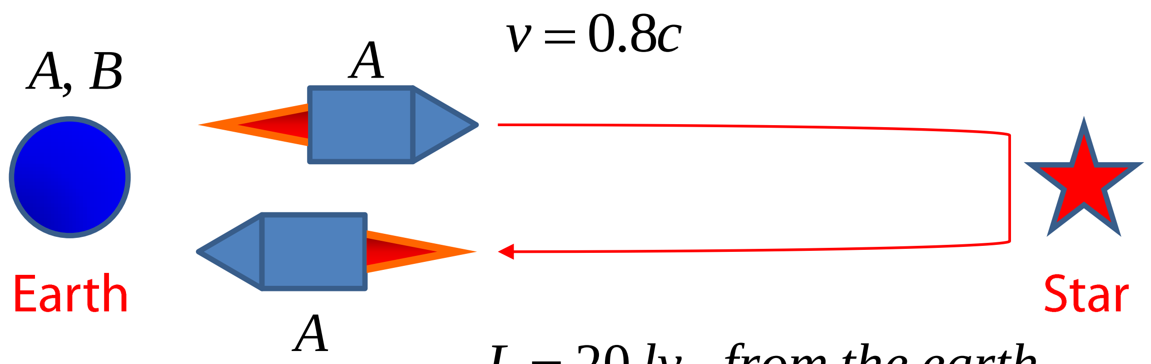
$$x'_0 = \gamma(x_0 - \boldsymbol{\beta}\cdot\mathbf{x})$$

$$\mathbf{x}' = \mathbf{x} + \frac{(\gamma-1)}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{x})\boldsymbol{\beta} - \gamma\boldsymbol{\beta}x_0$$

$$\begin{cases} \mathbf{x}'_{/\!/} = \gamma(\mathbf{x}_{/\!/} - \boldsymbol{\beta}x_0) = \dfrac{\gamma}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{x})\boldsymbol{\beta} - \gamma\boldsymbol{\beta}x_0 \\ \mathbf{x}'_{\perp} = \mathbf{x}_{\perp} = \mathbf{x} - \dfrac{1}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{x})\boldsymbol{\beta} \end{cases}$$

Twin Paradox?

Time dilation or length contraction for the space traveller:



$$L = 20\ ly \quad \textit{from the earth}$$

Time measured by *B*:

$$T_B = 2 \times \frac{20}{0.8} = 50\ yrs$$

Time measured by *A*:

$$L' = \frac{L}{\gamma} = L\sqrt{1-\beta^2} = 20 \times 0.6 = 12\ ly$$

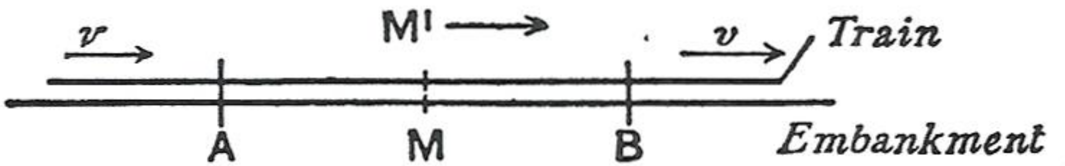
$$T_A = 2 \times \frac{12}{0.8} = 30\ yrs$$

What if A thinks that the earth and star are in motion?

Is this symmetric? $\rightarrow$ General theory of relativity

Special and General Principle of Relativity

Relative motion:



A. Einstein, *Relativity: The Special and General Theory*, Crown Publishers, 1931.

(a) The carriage is in motion relative to the embankment:
→ The embankment as reference-body

(b) The embankment is in motion relative to the carriage:
→ The carriage as reference-body

Which is correct? → *Only "experience" can decide!*

Special theory of relativity: → Only applicable to a uniform motion

For a non-uniform motion: Linked to acceleration

→ General theory of relativity

Gravitational Field

"If we pick up an apple and then let it go, why does it fall to the ground?"

$\rightarrow$ "Because it is attracted by earth: Newton's law of universal gravitation."

$$F = G\frac{m_1 m_2}{r^2}$$

What about electric charges? $\rightarrow$ Coulomb's law

$$F = k\frac{q_1 q_2}{r^2} \quad \rightarrow \mathbf{F}_{12} = q_1 \mathbf{E}_2$$ $\leftarrow$ Electric "field" involved

Einstein's thought: Why not "gravitational field"?

Newton's law of motion:

$$(\text{Force}) = (\text{intertial mass}) \times (\text{acceleration})$$

Under gravitational field:

$$(\text{Force}) = (\text{gravitational mass}) \times (\text{intensity of gravitational field})$$

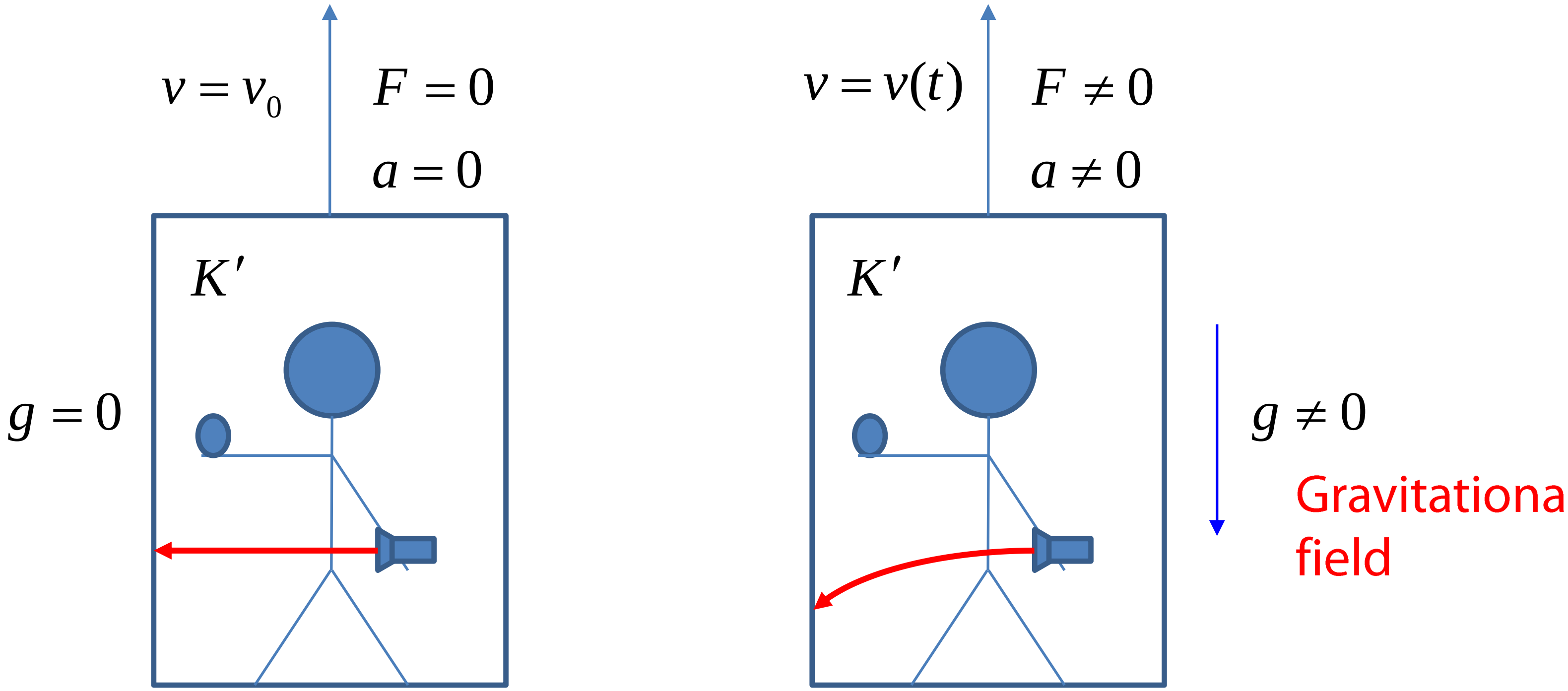
$$\rightarrow (\text{acceleration}) = \frac{(\text{gravitational mass})}{(\text{inertial mass})} \times (\text{intensity of gravitational field})$$

"Equivalent"

Equivalence Principle

Imagine a chest with an observer inside that is being pulled upwards:

→ Without external gravitation



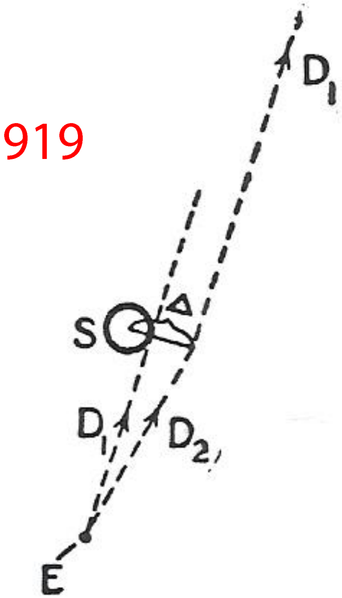
→ *The acceleration of the object towards the floor of the chest is always of the same magnitude, whatever kind of body the person may happen to use for the experiment.*

→ *Light bends toward the direction of the gravitational field.*

General Principle of Relativity

Deflection of light in gravitational field:

→ Measured during the solar eclipse of 29th May, 1919

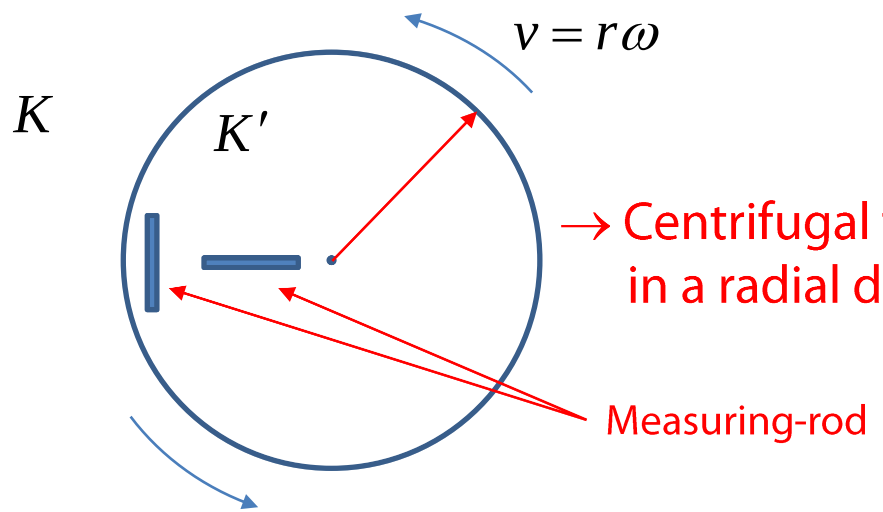


A. Einstein, *Relativity: The Special and General Theory*, Crown Publishers, 1931.

Time dilation by gravitational field:

→ Time is slowed down in gravitational field.

Rotating body of reference:



→ Centrifugal force (acceleration) outwards in a radial direction:

$$a = \frac{v^2}{r}$$

$$\rightarrow \; C' \neq 2\pi R$$

→ Non-Euclidean geometry: → Curved

→ Finite and yet unbounded universe: → In doubt ← Hubble's law