# Coupled-mode theory for nonlinear optical interactions

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# Orthogonality of modes

- **Maxwell's equations**

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0, \quad \nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{J}$$

$$\nabla \cdot \boldsymbol{D} = \rho, \quad \nabla \cdot \boldsymbol{B} = 0$$

- **Constitutive relations**

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} = \varepsilon_o \boldsymbol{E} + \boldsymbol{P}$$

$$\boldsymbol{B} = \mu \boldsymbol{H} = \mu_o \boldsymbol{H} + \boldsymbol{M}$$

- **Lorentz reciprocity theorem**

$$\nabla \cdot (\boldsymbol{E}_q \times \boldsymbol{H}_p^* + \boldsymbol{E}_p^* \times \boldsymbol{H}_q) \quad (p, q = 1, 2, 3, \ldots) \Leftarrow \text{for eigenmodes}$$

$$= \boldsymbol{H}_p^* \cdot (\nabla \times \boldsymbol{E}_q) - \boldsymbol{E}_q \cdot (\nabla \times \boldsymbol{H}_p^*) + \boldsymbol{H}_q \cdot (\nabla \times \boldsymbol{E}_p^*) - \boldsymbol{E}_p^* \cdot (\nabla \times \boldsymbol{H}_q)$$

$$= -\boldsymbol{H}_p^* \cdot \frac{\partial \boldsymbol{B}_q}{\partial t} - \boldsymbol{E}_q \cdot \frac{\partial \boldsymbol{D}_p^*}{\partial t} - \boldsymbol{H}_q \cdot \frac{\partial \boldsymbol{B}_p^*}{\partial t} - \boldsymbol{E}_p^* \cdot \frac{\partial \boldsymbol{D}_q}{\partial t} \Leftarrow \text{for monochromatic radiation}$$

$$= -i\omega (H_{p,i}^* \mu_{ij} H_{q,j} - E_{q,i} \varepsilon_{ij}^* E_{p,j}^* - H_{q,i} \mu_{ij}^* H_{p,j}^* + E_{p,i}^* \varepsilon_{ij} E_{q,j})$$

$$= -i\omega (H_{p,i}^* \mu_{ij} H_{q,j} - E_{q,j} \varepsilon_{ji}^* E_{p,i}^* - H_{q,j} \mu_{ji}^* H_{p,i}^* + E_{p,i}^* \varepsilon_{ij} E_{q,j})$$

$$= 0$$

Note: Source free and Hermitian tensors of $\varepsilon$ and $\mu$

# Coupled-mode equations

- **Permittivity perturbation**

$\varepsilon' = \varepsilon + \Delta\varepsilon, \quad \mu' = \mu$ $\quad\Leftarrow$ perturbed medium

$\boldsymbol{E}', \boldsymbol{H}'$ $\quad\Leftarrow$ perturbed fields

- **Coupled-mode equations**

$$\nabla\cdot(\boldsymbol{E}'\times\boldsymbol{H}_p^* + \boldsymbol{E}_p^*\times\boldsymbol{H}') \quad (p = 1, 2, 3, \ldots)$$

$$= \boldsymbol{H}_p^*\cdot(\nabla\times\boldsymbol{E}') - \boldsymbol{E}'\cdot(\nabla\times\boldsymbol{H}_p^*) + \boldsymbol{H}'\cdot(\nabla\times\boldsymbol{E}_p^*) - \boldsymbol{E}_p^*\cdot(\nabla\times\boldsymbol{H}')$$

$$= -\boldsymbol{H}_p^*\cdot\frac{\partial\boldsymbol{B}'}{\partial t} - \boldsymbol{E}'\cdot\frac{\partial\boldsymbol{D}_p^*}{\partial t} - \boldsymbol{H}'\cdot\frac{\partial\boldsymbol{B}_p^*}{\partial t} - \boldsymbol{E}_p^*\cdot\frac{\partial\boldsymbol{D}'}{\partial t}$$

$$= -i\omega(\boldsymbol{H}_p^*\cdot\boldsymbol{B}' - \boldsymbol{E}'\cdot\boldsymbol{D}_p^* - \boldsymbol{H}'\cdot\boldsymbol{B}_p^* + \boldsymbol{E}_p^*\cdot\boldsymbol{D}')$$

$$= -i\omega\boldsymbol{E}_p^*\Delta\varepsilon\boldsymbol{E}' = -i\omega\boldsymbol{E}_p^*\Delta\boldsymbol{P}'$$

where $\boldsymbol{D}' = \varepsilon'\boldsymbol{E}' = (\varepsilon + \Delta\varepsilon)\boldsymbol{E}' = \varepsilon\boldsymbol{E}' + \Delta\boldsymbol{P}'$,

$\boldsymbol{B}' = \mu\boldsymbol{H}'$

In result,

$$\nabla\cdot(\boldsymbol{E}'\times\boldsymbol{H}_p^* + \boldsymbol{E}_p^*\times\boldsymbol{H}') = -i\omega\boldsymbol{E}_p^*\Delta\varepsilon\boldsymbol{E}' \quad (p = 1, 2, 3, \ldots)$$

# Coupled-mode equations

- **Derivation of 1st-order differential equations**

Let us take the perturbed fields in the forms

$$\begin{pmatrix} \boldsymbol{E}' \\ \boldsymbol{H}' \end{pmatrix} \equiv \sum_q a_q(z) \begin{pmatrix} \boldsymbol{E}_q \\ \boldsymbol{H}_q \end{pmatrix} = \sum_q a_q(z) \begin{pmatrix} \boldsymbol{e}_q(x,y)\exp(-i\beta_q z) \\ \boldsymbol{h}_q(x,y)\exp(-i\beta_q z) \end{pmatrix}$$

It follows

$$\sum_q a_q(z) \nabla \cdot (\boldsymbol{E}_q \times \boldsymbol{H}_p^* + \boldsymbol{E}_p^* \times \boldsymbol{H}_q) + \sum_q \frac{da_q(z)}{dz} (\boldsymbol{E}_q \times \boldsymbol{H}_p^* + \boldsymbol{E}_p^* \times \boldsymbol{H}_q) \cdot \hat{z}$$

$$= -i\omega \sum_q a_q(z) \boldsymbol{E}_p^* \cdot \Delta\varepsilon(x,y,z) \boldsymbol{E}_q$$

where

$$\frac{1}{4} \int (\boldsymbol{E}_q \times \boldsymbol{H}_p^* + \boldsymbol{E}_p^* \times \boldsymbol{H}_q) \cdot \hat{z}\, dxdy = \begin{cases} 1, & p = q \\ 0, & p \neq q \end{cases}$$

In result,

$$\frac{da_p(z)}{dz} \exp(-i\beta_p z) = -i\frac{\omega}{4} \sum_q a_q(z) \exp(-i\beta_q z) \int \boldsymbol{e}_p^*(x,y) \cdot \Delta\varepsilon(x,y,z) \boldsymbol{e}_q(x,y) dxdy$$

# Mode coupling in periodic media

## Periodic permittivity perturbation

Let us assume the perturbed permittivity in the form

$$\varepsilon'(x,y,z) = \varepsilon_o(x,y) + \Delta\varepsilon(x,y,z)$$

In isotropic medium

$$\Delta\varepsilon(x,y,z) = \varepsilon_o[n'^2(x,y,z) - n^2(x,y)]$$
$$\cong 2\varepsilon_o n(x,y)\Delta n(x,y,z), \quad if\ \Delta n(x,y,z) << n(x,y),$$

$$\Delta n(x,y,z) = \Delta n_{av}(x,y) + \Delta n_{gr}(x,y)\cos(\frac{2\pi}{\Lambda}z + \phi_{gr})$$
$$= \Delta n_{av}(x,y) + \frac{\Delta n_{gr}(x,y)}{2}\left\{\exp[-i(\frac{2\pi}{\Lambda}z + \phi_{gr})] + \exp[i(\frac{2\pi}{\Lambda}z + \phi_{gr})]\right\},$$

# Two-mode coupling approximation (TMCA)

- **Periodic permittivity perturbation**

Let us assume the perturbed permittivity in the form

$$\frac{da_p(z)}{dz} = -i\frac{|\beta_p|}{\beta_p}\sum_q \kappa_{pq}(z)a_q(z)\exp[-i(\beta_q-\beta_p)z], \quad p=1,2,3,\ldots$$

where

$$\kappa_{pq}(z) = \frac{\pi c}{2\lambda}\iint \boldsymbol{e}_p^*(x,y)\cdot\Delta\varepsilon(x,y,z)\boldsymbol{e}_q(x,y)\,dxdy$$

- **Two-mode coupling approximation**

Let us take dominant two modes and their coupling constants on the condition of the longitudinal phase matching

- Codirectional coupling: $\beta_d/\beta_i > 0$
- Contradirectional coupling: $\beta_d/\beta_i < 0$

# Coupling coefficients for TMCA

- **Coupling coefficients**

Let us take the coupling coefficients in the forms:

for the incident and diffracted modes $E_i$ and $E_d$

self- and cross-coupling coefficients

$$\kappa_{\sigma,i} = \frac{\pi\varepsilon_o c}{\lambda}\iint \boldsymbol{e}_i^*(x,y)\cdot n(x,y)\Delta n_{av}(x,y)\boldsymbol{e}_i(x,y)\,dxdy$$

$$\kappa_{\sigma,d} = \frac{\pi\varepsilon_o c}{\lambda}\iint \boldsymbol{e}_d^*(x,y)\cdot n(x,y)\Delta n_{av}(x,y)\boldsymbol{e}_d(x,y)\,dxdy$$

$$\kappa_{\xi} = \frac{\pi\varepsilon_o c}{2\lambda}\iint \boldsymbol{e}_i^*(x,y)\cdot n(x,y)\Delta n_{gr}(x,y)\boldsymbol{e}_d(x,y)\,dxdy$$

# Codirectional coupling

- **Coupled-mode equations**

$$\frac{da_i(z)}{dz} = -i\kappa_{\sigma,i} a_i(z) - i\kappa_\xi a_d(z) \exp(i\Delta\beta z - i\phi_{gr})$$

$$\frac{da_d(z)}{dz} = -i\kappa_{\sigma,d} a_d(z) - i\kappa_\xi^* a_i(z) \exp(-i\Delta\beta z + i\phi_{gr})$$

where

$$\Delta\beta = \beta_i - \beta_d - \frac{2\pi}{\Lambda}, \ (\beta_i > \beta_d)$$

- **Boundary conditions**

$$E_i(z)\big|_{z=0} = E_i(0), \quad E_d(z)\big|_{z=0} = E_d(0)$$

# Codirectional coupling

- **Solution**

$$E_i(z) = a_i(z)\exp(-i\beta_i z)$$
$$= \left\{ \left(\cos\sigma_f z - i\frac{\Delta\beta'}{2\sigma_f}\sin\sigma_f z\right)E_i(0) - i\frac{\kappa_\xi}{\sigma_f}\sin\sigma_f z\exp(-i\phi_{gr})E_d(0) \right\}\exp[-i(\gamma_f + \pi/\Lambda)z]$$

$$E_d(z) = a_d(z)\exp(-i\beta_d z)$$
$$= \left\{ -i\frac{\kappa_\xi^*}{\sigma_f}\sin\sigma_f z\exp(+i\phi_{gr})E_i(0) + \left(\cos\sigma_f z + i\frac{\Delta\beta'}{2\sigma_f}\sin\sigma_f z\right)E_d(0) \right\}\exp[-i(\gamma_f - \pi/\Lambda)z]$$

where

$$\Delta\beta' = \Delta\beta + \kappa_{\sigma,i} - \kappa_{\sigma,d}, \quad \gamma_f = \frac{\beta_i + \beta_d + \kappa_{\sigma,i} + \kappa_{\sigma,d}}{2}, \quad \sigma_f^2 = \kappa_\xi^*\kappa_\xi + \left(\frac{\Delta\beta'}{2}\right)^2$$

# Codirectional coupling

- **Transmissivity**

Boundary conditions: $E_i(z)\big|_{z=0} = E_i(0), \quad E_d(z)\big|_{z=0} = 0$

$$T_i \equiv \left|\frac{E_i(L)}{E_i(0)}\right|^2 = \cos^2 \sigma_f L + \left(\frac{\Delta\beta'}{2\sigma_f}\right)^2 \sin^2 \sigma_f L$$

$$T_d \equiv \left|\frac{E_d(L)}{E_i(0)}\right|^2 = \frac{\kappa_\xi^* \kappa_\xi}{\sigma_f^2} \sin^2 \sigma_f L$$

When $\beta' = 0$

$$T_{i,\min} = \cos^2 |\kappa_\xi| L$$

$$\lambda_{\min} = [\Delta n_{eff} + \delta n_{eff,av}(1-u_\sigma)]\Lambda = \left[1 + \frac{\delta n_{eff,av}(1-u_\sigma)}{\Delta n_{eff}}\right]\lambda_{F,0}$$

# Contradirectional coupling

- **Coupled-mode equations**

$$\frac{da_i(z)}{dz} = -i\kappa_{\sigma,i} a_i(z) - i\kappa_{\xi} a_d(z) \exp(i\Delta\beta z - i\phi_{gr})$$

$$\frac{da_d(z)}{dz} = i\kappa_{\sigma,d} a_d(z) + i\kappa_{\xi}^{*} a_i(z) \exp(-i\Delta\beta z + i\phi_{gr})$$

where

$$\Delta\beta = \beta_i + \beta_d - \frac{2\pi}{\Lambda} \quad (\beta_i > 0, \beta_d > 0)$$

- **Boundary conditions**

$$E_i(z)\big|_{z=0} = E_i(0), \quad E_d(z)\big|_{z=L} = E_d(L)$$

# Contradirectional coupling

- **Solution**

$$E_i(z) = a_i(z)\exp(-i\beta_i z)$$

$$= \left\{ \frac{\sigma_b \cosh \sigma_b (L-z) + i(\Delta\beta'/2)\sinh \sigma_b (L-z)}{\sigma_b \cosh \sigma_b L + i(\Delta\beta'/2)\sinh \sigma_b L} E_i(0) + \frac{-i\kappa_\xi \sinh \sigma_b z \exp[i(\gamma_b - \pi/\Lambda)L - i\phi_{gr}]}{\sigma_b \cosh \sigma_b L + i(\Delta\beta'/2)\sinh \sigma_b L} E_d(L) \right\} \exp[-i(\gamma_b + \pi/\Lambda)z],$$

$$E_d(z) = a_d(z)\exp(i\beta_d z)$$

$$= \left\{ \frac{-i\kappa_\xi^* \sinh \sigma_b (L-z)\exp(i\phi_{gr})}{\sigma_b \cosh \sigma_b L + i(\Delta\beta'/2)\sinh \sigma_b L} E_i(0) + \frac{\sigma_b \cosh \sigma_b z + i(\Delta\beta'/2)\sinh \sigma_b z}{\sigma_b \cosh \sigma_b L + i(\Delta\beta'/2)\sinh \sigma_b L} \exp[i(\gamma_b - \pi/\Lambda)L] E_d(L) \right\} \exp[-i(\gamma_b - \pi/\Lambda)z],$$

where

$$\Delta\beta' = \Delta\beta + \kappa_{\sigma,i} + \kappa_{\sigma,d}, \quad \gamma_b = \frac{\beta_i - \beta_d + \kappa_{\sigma,i} - \kappa_{\sigma,d}}{2}, \quad \sigma_b^2 = \kappa_\xi^* \kappa_\xi - \left(\frac{\Delta\beta'}{2}\right)^2$$

# Contradirectional coupling

- **Reflectivity**

  Boundary conditions: $E_i(z)\big|_{z=0} = E_i(0), \quad E_d(z)\big|_{z=L} = 0$

$$R \equiv \left|\frac{E_d(0)}{E_i(0)}\right|^2 = \left|\frac{\kappa_\xi^* \kappa_\xi \sinh^2 \sigma_b L}{\sigma_b^2 \cosh^2 \sigma_b L + (\Delta\beta'/2)^2 \sinh^2 \sigma_b L}\right|$$

  When $\beta' = 0$

$$R_{\max} = \tanh^2 |\kappa_\xi| L, \qquad \lambda_{\max} = 2(n_{eff} + \delta n_{eff,av})\Lambda = \left(1 + \delta n_{eff,av}/n_{eff}\right)\lambda_{B,0},$$

- **Bandwidth**

$$R_{edge} = \frac{\kappa_\xi^* \kappa_\xi L^2}{1 + \kappa_\xi^* \kappa_\xi L^2}, \qquad \lambda_{edge} = \lambda_{\max} \pm \frac{\delta n_{eff,gr}}{2 n_{eff}} \lambda_{B,0}$$

$$\frac{\Delta\lambda_{edge}}{\lambda_{B,0}} = \frac{\delta n_{eff,gr}}{n_{eff}}$$

# Transfer matrix method (TMM)

- **Aperiodic Media**

  Piecewise segmentation

- **Formulation**

$$\begin{pmatrix} {}_{r}E_{i,N} \\ {}_{r}E_{d,N} \end{pmatrix} = \boldsymbol{T}_N \cdots \boldsymbol{T}_{k+1} \cdot \boldsymbol{T}_k \cdots \boldsymbol{T}_1 \cdot \begin{pmatrix} {}_{f}E_{i,1} \\ {}_{f}E_{d,1} \end{pmatrix}$$

# Transfer matrix method

- **Codirectional coupling**

$$T_{k,(1,1)} = (\cos\sigma_{f,k}L_k - i\frac{\Delta\beta'_k}{2\sigma_{f,k}}\sin\sigma_{f,k}L_k)\exp[-i(\gamma_{f,k} + \pi/\Lambda_k)L_k]$$

$$T_{k,(1,2)} = -i\frac{\kappa_{\xi,k}}{\sigma_{f,k}}\sin\sigma_{f,k}L_k\exp[-i(\gamma_{f,k} + \pi/\Lambda_k)L_k - i\phi_{gr,k}]$$

$$T_{k,(2,1)} = -i\frac{\kappa^*_{\xi,k}}{\sigma_{f,k}}\sin\sigma_{f,k}L_k\exp[-i(\gamma_{f,k} - \pi/\Lambda_k)L_k + i\phi_{gr,k}]$$

$$T_{k,(2,2)} = (\cos\sigma_{f,k}L_k + i\frac{\Delta\beta'_k}{2\sigma_{f,k}}\sin\sigma_{f,k}L_k)\exp[-i(\gamma_{f,k} - \pi/\Lambda_k)L_k]$$

- **Contradirectional coupling**

$$T_{k,(1,1)} = (\cosh\sigma_{b,k}L_k - i\frac{\Delta\beta'_k}{2\sigma_{b,k}}\sinh\sigma_{b,k}L_k)\exp[-i(\gamma_{b,k} + \pi/\Lambda_k)L_k]$$

$$T_{k,(1,2)} = -i\frac{\kappa_{\xi,k}}{\sigma_{b,k}}\sinh\sigma_{b,k}L_k\exp[-i(\gamma_{b,k} + \pi/\Lambda_k)L_k - i\phi_{gr,k}]$$

$$T_{k,(2,1)} = +i\frac{\kappa^*_{\xi,k}}{\sigma_{b,k}}\sinh\sigma_{b,k}L_k\exp[-i(\gamma_{b,k} - \pi/\Lambda_k)L_k + i\phi_{gr,k}]$$

$$T_{k,(2,2)} = (\cosh\sigma_{b,k}L_k + i\frac{\Delta\beta'_k}{2\sigma_{b,k}}\sinh\sigma_{b,k}L_k)\exp[-i(\gamma_{b,k} - \pi/\Lambda_k)L_k]$$

# Discretization method

- **Coupled-mode theory by discretization method**

$$\nabla\cdot(E'\times H_p^* + E_p^*\times H') = -i\omega E_p^*\cdot\Delta\varepsilon_L(m)E', \quad (p=1,2,3,\ldots)$$

where

$$\boldsymbol{E}_p = \hat{e}_p(r,\phi)\exp(-i\beta_p z) : \textit{eigen mode},$$

$$\boldsymbol{E}'(m) = \sum_q a_q(z;m)\hat{e}_q(r,\phi)\exp(-i\beta_q z) : \textit{perturbed mode}.$$

In result,

$$\frac{d\boldsymbol{F}_L(z;m)}{dz} = -i\left(\boldsymbol{B}+\boldsymbol{K}(m)\right)\cdot\boldsymbol{F}_L(z;m) \equiv \boldsymbol{D}(m)\cdot\boldsymbol{F}_L(z;m),$$

where

$$F_{L,(p)}(z;m) = a_p(z;m)\exp(-i\beta_p z),$$

$$B_{(p,q)} = \begin{cases}\beta_p & (p=q),\\ 0 & (p\neq q),\end{cases} \qquad K_{(p,q)} = \frac{\omega}{4}\int \hat{e}_p^*\cdot\varepsilon(m)\hat{e}_q\, dS.$$

# Discretization method

- **Solution**

$$\boldsymbol{F}_L(z_f;m)=\boldsymbol{V}(m)\cdot\exp[\boldsymbol{U}(m)\cdot l(m)]\cdot\boldsymbol{V}(m)^{-1}\cdot\boldsymbol{F}_L(z_i;m)\equiv\boldsymbol{T}(m)\cdot\boldsymbol{F}_L(z_i;m),$$

where '$z_i$' and '$z_f$' denote the left-end and the right-end positions of the *m*-th section whose length is $l(m)$, respectively, $\boldsymbol{U}(m)$ and $\boldsymbol{V}(m)$ denote the diagonal eigenvalue matrix of $\boldsymbol{D}(m)$ and the corresponding eigenvector matrix, respectively.

$$\boldsymbol{F}_L^{right}=\cdots\boldsymbol{T}(m+1)\cdot\boldsymbol{T}(m)\cdot\boldsymbol{T}(m-1)\cdots\boldsymbol{T}(1)\cdot\boldsymbol{F}_L^{left}.$$

Therefore, the overall spectral response of the fiber grating can be obtained by cascading each transfer relation of discretized section with initial incident conditions.

# Nonlinear perturbation

- **Coupled-mode theory with nonlinear perturbation terms**

With second-order nonlinearity

$$\nabla\cdot(\boldsymbol{E}'_{\omega}\times\boldsymbol{H}^{*}_{\omega,p}+\boldsymbol{E}^{*}_{\omega,p}\times\boldsymbol{H}'_{\omega})=-i\omega\boldsymbol{E}^{*}_{\omega,p}\cdot\Delta\boldsymbol{P}'_{\omega},\quad(p=1,2,\ldots),$$

$$\begin{aligned}\Delta\boldsymbol{P}'_{\omega_1}&=\Delta\varepsilon_{ij}(\omega_1)E'_{\omega_1,j}+2d_{ijk}(-\omega_1,\omega_2,-\omega_1)E'_{\omega_2,j}E'^{*}_{\omega_1,k}\\&+\left\{3\chi_{ijkl}(-\omega_1,\omega_1,-\omega_1,\omega_1)E'_{\omega_1,j}E'^{*}_{\omega_1,k}+6\chi_{ijkl}(-\omega_1,\omega_2,-\omega_2,\omega_1)E'_{\omega_2,j}E'^{*}_{\omega_2,k}\right\}E'_{\omega_1,l},\end{aligned}$$

$$\begin{aligned}\Delta\boldsymbol{P}'_{\omega_2}&=\Delta\varepsilon_{ij}(\omega_2)E'_{\omega_2,j}+d_{ijk}(-\omega_2,\omega_1,\omega_1)E'_{\omega_1,k}E'_{\omega_1,k}\\&+\left\{3\chi_{ijkl}(-\omega_2,\omega_2,-\omega_2,\omega_2)E'_{\omega_2,j}E'^{*}_{\omega_2,k}+6\chi_{ijkl}(-\omega_2,\omega_1,-\omega_1,\omega_2)E'_{\omega_1,j}E'^{*}_{\omega_1,k}\right\}E'_{\omega_2,l}.'\end{aligned}$$

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Without second-order nonlinearity

$$\nabla\cdot(E'\times H^{*}_{p}+E^{*}_{p}\times H')=-i\omega E^{*}_{p}\cdot(\Delta\varepsilon_{L}+\Delta\varepsilon_{NL})E',\quad(p=1,2,3,\ldots)$$

$$\Delta\varepsilon_{NL\,(q)}=\varepsilon_{o}\frac{3}{4}\chi^{(3)}(r,\phi,z)\cdot\sum_{s}\alpha_{(q,s)}\left|E_{s}(t,z)\right|^{2}\left|\hat{e}_{s}\right|^{2},\ \alpha_{(q,s)}=\begin{cases}1 & (q=s),\\ 2 & (q\neq s),\end{cases}$$

# Nonlinear pulse propagation

- **Wave packet representation**

$$\frac{d\boldsymbol{E}(t,z',m)}{dz'} = \frac{d}{dz'}\int \boldsymbol{F}(\Omega,z';m)\exp(i\Omega t)d\Omega$$
$$= \int \boldsymbol{D}(\Omega,m)\cdot\boldsymbol{F}(\Omega,z';m)\exp(i\Omega t)d\Omega - i\boldsymbol{X}(t,z';m)\cdot\boldsymbol{E}(t,z',m),$$

where

$$X_{(p,q)}(t,z';m) = \frac{\omega_c}{4}\int \hat{e}_p^* \cdot \Delta\varepsilon_{NL(q)}(t,z';m)\ \hat{e}_q\ dS.$$

# Nonlinear pulse propagation

- **Solution by Fourier transform method**

$$\boldsymbol{F}(\Omega, z'; m) = \boldsymbol{T}(\Omega, z'; m) \cdot \boldsymbol{\Phi}(\Omega, z'; m),$$

where

$$\boldsymbol{\Phi}(\Omega, z' = 0; m) = \frac{1}{2\pi} \int \boldsymbol{E}(t, z' = 0; m) \exp(-i\Omega t) dt.$$

$$\boldsymbol{G}(\Omega, z'; m) = \frac{1}{2\pi} \int \boldsymbol{T}^{-1}(\Omega, z'; m) \cdot \boldsymbol{X}(t, z'; m) \cdot \boldsymbol{E}(t, z'; m) \exp(-i\Omega t) dt.$$

In result,

$$\frac{d\boldsymbol{\Phi}(\Omega, z'; m)}{dz'} = -i\boldsymbol{G}(\Omega, z'; m),$$

which can be solved by fast Fourier transform method with a predictor-corrector scheme.

# Volume holographic gratings

- **Born and paraxial approximation $\Rightarrow$ Single diffraction**
- **Coupled-mode theory for volume holographic gratings**

- Applicable to arbitrary cases of multiplexed holograms
- Continuous spectrum $\Rightarrow$ discrete spectrum (FFT)

$$\nabla \cdot (E' \times H_p^* + E_p^* \times H') = -i\omega E_p^* \cdot \Delta\varepsilon E', \quad (p = 1, 2, ...)$$

$$\boldsymbol{E}'(z) = \sum_q a_q(z) \boldsymbol{e}_q(x, y) \exp(-i\beta_q z)$$

$$\Rightarrow \qquad \boldsymbol{E}'(z) = \int a(k_x', k_y', k_z'; z) \exp[-i(k_x' x + k_y' y + k_z' z)] dk_x' dk_y'$$

$$\frac{da(k_x, k_y, k_z'; z)}{dz} = -i \frac{\omega^2 \mu}{2k_z} \int a(k_x', k_y', k_z'; z) G(k_x', k_y', k_x, k_y; z) \exp[-i(k_z' - k_z) z] dk_x' dk_y'$$